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A NOTE ON COMPACT REPRESENTATIONS AND ALMOST PERIODICITY IN TOPOLOGICAL GROUPS

ERIK M. ALFSEN and PER HOLM

Introduction.

In the present paper we offer an approach to the theory of almost periodicity in topological groups which is new as far as we know. It is based on an explicit construction of the Bohr compactification in terms of the group topology, without application of complex valued functions or linear representations. Further explanation requires some terminology.

A continuous representation ξ of a topological group G onto a dense subgroup of a (separated) compact group H will briefly be termed a *compact representation* of G, and noted (ξ, H). A compact representation $(\varrho, \hat{G})$ of G will be termed *maximal* if it has the *universal factorization property* for G; that is, if every compact representation (ξ, H) of G admits a unique decomposition $\xi = \xi' \circ \varrho$, where ξ' is some continuous representation of $\hat{G}$ onto H. Evidently a maximal compact representation is determined up to an algebraic and topologic isomorphism.

By well-known properties of uniform structures and their separated completions (cf. the remark of [1, p. 101]), we may conclude:

A topological group G admits a maximal compact representation if and only if there exists a finest uniform structure $\mathcal{U}$ on G with the following properties:

(a) *$\mathcal{U}$ is totally bounded (precompact).*

(b) *$\mathcal{U}$ is compatible with the group structure in the sense that the mapping $x \to x^{-1}$ is uniformly continuous and the mapping $(x,y) \to xy$ is jointly uniformly continuous.*

(c) *$\mathcal{U}$ defines a topology coarser than the initial topology of G.*

If such a structure exists, then the relation of "non-separability" (i.e. common membership of all entourages) *will be a congruence on G. Hence the group operations on $\varrho(G) \subset \hat{G}$ are well defined, $(\varrho, \hat{G})$ being the separated completion of G provided with the structure $\mathcal{U}$. By the uniform continuity*

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these operations may be extended to the entire complete and compact set $\hat{G}$, *thus making* $(\varrho, \hat{G})$ *a maximal compact representation of* G.

In section 1 of the present paper we shall determine explicitly a finest uniform structure $\mathscr{U}_B$ with the properties (a), (b), (c) above. The corresponding maximal compact representation will of course be the well-known Bohr compactification, thus constructed without reference to almost periodic functions or to finite dimensional unitary representations.

In section 2 we shall give a direct proof that almost periodicity is a necessary and sufficient condition for a complex valued function to be $\mathscr{U}_B$-uniformly continuous, and hence extendable by continuity to the Bohr compactification. Thus the almost periodic functions may be defined in terms of $\mathscr{U}_B$-uniform continuity. By this procedure the relations to continuous functions on compact groups becomes evident from the starting point.

The application of the Bohr compactification to the study of almost periodic functions is of course not new. However, in all presentations known to the authors, its construction requires an independent study of almost periodic functions or of the closely related finite dimensional unitary representations (cf. e.g. [5, p. 165] and [6, p. 125]).

1. The uniform structure of the maximal representation.

It is well known [3, p. 31 ex. 3] that the left and right uniform structures, $\mathscr{U}_l$ and $\mathscr{U}_r$, on a topological group $(G, \mathscr{T})$ coalesce if and only if every neighbourhood V of the identity element e admits a neighbourhood W of e such that

$$xWx^{-1} \subset V \qquad \text{for all } x \in G \tag{1.1}$$

or, what is equivalent, if there exists a fundamental system of neighbourhoods of e whose members V are all *invariant* in the sense that

$$xVx^{-1} = V \qquad \text{for all } x \in G\,. \tag{1.2}$$

In this case, and only then, the left (right) uniform structure will be compatible with the group structure. (The requirement (1.1) may be considered as the "uniform version" of the fourth axiom GV_{IV} of [3, p. 4]).

PROPOSITION 1. *Let* G *be a group,* $\mathscr{U}$ *a uniform structure on* G *compatible with the group structure, and* $\mathscr{T}$ *the topology derived from* $\mathscr{U}$. *Then* $(G, \mathscr{T})$ *is a topological group whose left and right uniform structures both are equal to* $\mathscr{U}$.

In the sequel $(G, \mathscr{T})$ will be termed *the topological group associated with the uniform structure* $\mathscr{U}$ *on* G.

PROOF. $\mathscr{T}$ is compatible with the group structure in virtue of the (even uniform) continuity of the group operations. From the uniform continuity of the operation $(x,y) \to xy$, we easily deduce that $\mathscr{U}$ admits a fundamental system of entourages S which are *left invariant* in the sense that:

$$(1.3) \qquad (x,y) \in S, \ a \in G \ \Rightarrow \ (ax,ay) \in S \ .$$

This entails $\mathscr{U} = \mathscr{U}_l$ [3, p. 31 ex. 1]. Similarly we prove $\mathscr{U} = \mathscr{U}_r$.

REMARK. It follows from Proposition 1 that a uniform structure $\mathscr{U}$ compatible with the group structure on G admits a fundamental system of symmetric left and right invariant entourages. The reverse statement is also valid since any such entourage S can be expressed by $S = V_l = V_r$ where V is the S-neighbourhood of e, and these neighbourhoods are seen to satisfy GV_{I}–GV_{III} of [3] (cf. also [3] for the definition of V_l, V_r). Hence, *a uniform structure on a group is compatible with the group if and only if it admits a fundamental system of symmetric left and right invariant entourages.*

A subset A of a group is termed *left* (*right*) *relatively dense* if there exists a finite sequence $(a_1, \ldots, a_n)$ of elements of G such that $G = \bigcup_{i=1}^n a_i A$ (respectively, $G = \bigcup_{i=1}^n A\, a_i$). Clearly, the left (right) uniform structure of a topological group is totally bounded if and only if every neighbourhood V of e is left (right) relatively dense.

PROPOSITION 2. *Let G be a group and $\mathscr{U}$ a uniform structure on G with a fundamental system $\mathscr{V}$ of symmetric left invariant entourages. Let $\tilde{\mathscr{V}}$ denote the collection of sets*

$$\tilde{S} = \{(a,b)\colon \ (ax,bx) \in S \ \text{for all} \ x \in G\} ,$$

where $S \in \mathscr{V}$.

Then $\tilde{\mathscr{V}}$ is a fundamental system of entourages for a uniform structure $\tilde{\mathscr{U}}$ on G which is the coarsest uniform structure finer than $\mathscr{U}$ and compatible with the group structure. $\tilde{\mathscr{U}}$ is totally bounded whenever $\mathscr{U}$ is.

PROOF. 1) It is easily verified that $\tilde{\mathscr{U}}$ satisfies the axioms of uniform structures, and so the first statement of the proposition follows from the left and right invariance of the sets $\tilde{S}$ in virtue of the above remark.

2) We assume $\mathscr{U}$ totally bounded. Let $S \in \mathscr{V}$ be arbitrary and assume $T \in \mathscr{V}$, $\overset{3}{T} \subset S$. We determine a finite covering $\{A_i\}_{i=1,\ldots,n}$ of G such that $A_i \times A_i \subset T$ for $i = 1, \ldots, n$, and we pick out n points $a_1, \ldots, a_n$ such that $a_i \in A_i$ for $i = 1, \ldots, n$.

Let J be the finite set of all mappings j of $\{1, \ldots, n\}$ into itself for which there exists an element b of G such that $ba_i \in A_{j(i)}$ for $i = 1, \ldots, n$, and let us assign one such element b_j to each $j \in J$. We shall prove that $G = \bigcup_{j \in J} b_j \tilde{V}$, where $\tilde{V}$ is the neighbourhood of e determined by the entourage $\tilde{S}$ of $\tilde{\mathscr{V}}$.

Let $b \in G$ be arbitrary, and assign to every index i, $i = 1, \ldots, n$, another index $j(i)$ such that $ba_i \in A_{j(i)}$ for $i = 1, \ldots, n$. Then $j \in J$, and hence $(b_j a_i, ba_i) \in A_{j(i)} \times A_{j(i)} \subset T$. Now we consider an arbitrary element x of G, and assume $x \in A_i$. Then $(x, a_i) \in A_i \times A_i \subset T$, and by left invariance $(b_j x, b_j a_i) \in T$, $(bx, ba_i) \in T$. Hence:

$$(b_j x, bx) \in \overset{3}{T} \subset S .$$

Since x was arbitrary, this means that $(b_j, b) \in \tilde{S}$, and so $b \in b_j \tilde{V}$, q.e.d.

Similarly, if G has a fundamental system of symmetric right invariant entourages T we establish the result of Proposition 2 by means of the collection of sets

$$\tilde{T} = \{(a,b) \colon \ (xa, xb) \in T \text{ for all } x \in G\} .$$

Proposition 3. *If U and V are left relatively dense subsets of a group G, then the set $W = (U^{-1}U) \cap (V^{-1}V)$ is also left relatively dense.*

Proof. We assume $G = \bigcup_{i=1}^{m} a_i U = \bigcup_{j=1}^{n} b_j V$. For those pairs of indices i, j for which $(a_i U) \cap (b_j V) \neq \emptyset$, we pick out one $c_{ij} \in (a_i U) \cap (b_j V)$ and we are to prove that $\bigcup_{i,j} c_{ij} W = G$. Let $x \in G$ be arbitrary and determine h and k such that $x \in a_h U$ and $x \in b_k V$. Then

$$c_{hk}^{-1} x = (c_{hk}^{-1} a_h)(a_h^{-1} x) \in U^{-1} U .$$

Thus $x \in c_{hk} U^{-1} U$. Similarly we have $x \in c_{hk} V^{-1} V$ and therefore $x \in c_{hk} W$.

Theorem 1. *Every topological group $(G, \mathscr{T})$ admits a finest uniform structure $\mathscr{U}$ with the properties*

(a) *$\mathscr{U}$ is totally bounded.*
(b) *$\mathscr{U}$ is compatible with the group structure.*
(c) *$\mathscr{U}$ defines a topology coarser than the initial topology on G.*

The system of neighbourhoods of e for the topological group associated with $\mathscr{U}$ consists of those subsets V of G which admit a sequence $\{V_n\}_{n=1,2,\ldots}$ of sets such that

(d) *$V_1^2 \subset V$ and $V_{n+1}^2 \subset V_n$, for $n = 1, 2, \ldots$.*
(e) *Every V_n is a symmetric, invariant and left relatively dense $\mathscr{T}$-neighbourhood of e.*

Proof. We first prove that the collection $\mathscr{V}$ of sets V satisfying (d) and (e) is a filter. Obviously it is sufficient to show that $\mathscr{V}$ is closed with respect to finite intersections. Let U, $V \in \mathscr{V}$, and let $\{U_n\}$, $\{V_n\}$ denote the corresponding sequences described in (d), (c). We have for every natural number n

$$(U_{n+1}{}^{-1} \cdot U_{n+1}) \cap (V_{n+1}{}^{-1} \cdot V_{n+1}) = U_{n+1}{}^2 \cap V_{n+1}{}^2 \subset U_n \cap V_n .$$

By proposition 3, $U_n \cap V_n$ will be left relatively dense and so the sequence $\{U_n \cap V_n\}$ will have the required properties relatively to $U \cap V$. Hence $U \cap V \in \mathscr{V}$.

To see that $\mathscr{V}$ satisfies the axioms for the neighbourhood filter of e in a topological group, and the fourth axiom even in the uniform version (1.1), we only have to observe that for any $V \in \mathscr{V}$ with corresponding sequence $\{V_n\}_{n=1,2,\ldots}$, the sets V_1, V_2, ... also belong to $\mathscr{V}$, and we have

$$V_1^{-1} \cdot V_1 = V_1^2 \subset V , \tag{1.4}$$

$$x V_1 x^{-1} = V_1 \subset V \qquad \text{for all } x \in G . \tag{1.5}$$

Thus $\mathscr{V}$ is the filter of neighbourhoods of e for a topological group with a common left and right uniform structure $\mathscr{U}$ having the properties (a), (b), (c).

Finally, let $\mathscr{U}'$ be some uniform structure with the properties (a), (b), (c), and let $\mathscr{V}'$ be the filter of neighbourhoods of e for the associated topological group. For any $V \in \mathscr{V}'$ there exists a sequence $\{V_n\}_{n=1,2,\ldots}$ from $\mathscr{V}'$ having the properties (c), (d). Hence $\mathscr{V}' \subset \mathscr{V}$, and so $\mathscr{U}'$ is a coarser structure than $\mathscr{U}$.

The symbols $\mathscr{U}_B$ and $\mathscr{T}_B$ will be used in the sequel to denote the uniform structure of Theorem 1 and its associated topology.

COROLLARY. *Every topological group admits a maximal, compact representation.*

2. Equivalence of almost periodicity and $\mathscr{U}_B$-uniform continuity.

We recall that a complex valued function f on a topological group G is termed *left* (*right*) *almost periodic* if it is continous and the set of *left translates* f_s:$f_s(x)=f(s^{-1}x)$, $s \in G$ (respectively, *right translates* f^s:$f^s(x)=f(xs)$, $s \in G$) is totally bounded with respect to the pseudo-metric $\|f-g\|_\infty = \sup_{x \in G} |f(x)-g(x)|$ on the set of continous functions on G. Evidently f is left almost periodic if and only if it is continuous and the set

$$E(f,\varepsilon) = \{s\colon \ \|f_s - f\|_\infty \leqq \varepsilon\}$$

is left relatively dense for every $\varepsilon > 0$.

THEOREM 2. *A complex valued function f on a topological group $(G,\mathscr{T})$ is left (right) almost periodic if and only if it is uniformly continuous with respect to the structure $\mathscr{U}_B$.*

PROOF. 1) We assume f to be left almost periodic and define a pseudo-metric on G by the formula

$$d_f(x,y) = \|f_x - f_y\|_\infty . \tag{2.1}$$

The pseudo-metric d_f is evidently left invariant, that is $d_f(sx,sy) = d_f(x,y)$ for every $s \in G$, and the closed ε-neighbourhoods of d_f have the form $E(f,\varepsilon)$. By the definition of left almost periodicity, every $E(f,\varepsilon)$ is left relatively dense, and so d_f determines a totally bounded uniform structure $\mathscr{U}_f$. In virtue of Proposition 2, $\mathscr{U}_f$ admits an associated structure $\tilde{\mathscr{U}}_f$ which is seen to be defined by the pseudo metric $\tilde{d}_f(x,y) = \sup_z d_f(xz,yz)$ having closed ε-neighbourhoods of the form

$$H(f,\varepsilon) = \{s\colon \ \sup_{x,y} |f(xy) - f(xsy)| \leqq \varepsilon\} . \tag{2.2}$$

By Proposition 2, $\tilde{\mathscr{U}}_f$ is totally bounded, and so there exists a finite sequence $a_1,\ldots,a_n$ of elements of G such that the ε-neighbourhood of an arbitrary $a \in G$ contains some a_i.

In virtue of (2.2) this means that for every $a \in G$ there exists an i, $i = 1,\ldots,n$, such that

$$|f(xay) - f(xa_iy)| \leqq \varepsilon \qquad \text{for all } x,y \in G . \tag{2.3}$$

Using this formula we shall prove that the set

$$W = \{t\colon \ |f(a_ita_j) - f(a_ia_j)| < \varepsilon;\ i,j = 1,\ldots,n\}$$

which is an open neighbourhood of e in the original topology, is contained in $H(f,5\varepsilon)$.

For a given x there exists an i, $i = 1,\ldots,n$, such that

$$|f(xty) - f(a_ity)| \leqq \varepsilon \qquad \text{for all } t,y \in G . \tag{2.4}$$

Similarly, for a given y there exists a_j, $j = 1,\ldots,n$, such that

$$|f(a_ity) - f(a_ita_j)| \leqq \varepsilon \qquad \text{for all } t \in G . \tag{2.5}$$

By application of (2.4) and (2.5) we obtain the following inequality for any $t \in W$:

$$\begin{aligned} |f(xty) - f(xy)| \leqq\ & |f(xty) - f(a_ity)| + \\ & + |f(a_ity) - f(a_ita_j)| + |f(a_ita_j) - f(a_ia_j)| + \\ & + |f(a_ia_j) - f(a_iy)| + |f(a_iy) - f(xy)| \leqq 5\varepsilon . \end{aligned}$$

Since x,y were arbitrary elements of G, this means $t \in H(f,5\varepsilon)$. Hence we have proved $W \subset H(f, 5\varepsilon)$, and so $H(f, 5\varepsilon)$ is a $\mathcal{T}$-neighbourhood of e. Then the uniform structure $\tilde{\mathcal{U}}_f$ has all the properties (a), (b), (c) of Theorem 1, and so $\tilde{\mathcal{U}}_f$ is coarser than $\mathcal{U}_B$.

Since $\tilde{d}_f$ is both right and left invariant, we can conclude that

$$\tilde{d}_f(x,y) \leqq \varepsilon \Rightarrow \tilde{d}_f(x^{-1},y^{-1}) \leqq \varepsilon \Rightarrow d_f(x^{-1},y^{-1}) \leqq \varepsilon \Rightarrow |f(x)-f(y)| \leqq \varepsilon .$$

Hence f is uniformly continuous with respect to $\tilde{\mathcal{U}}_f$ and hence also with respect to the finer structure $\mathcal{U}_B$.

Similarly we prove that a right almost periodic function is $\mathcal{U}_B$-uniformly continuous.

2) We assume f to be $\mathcal{U}_B$-uniformly continuous. Then f is also continuous in the initial topology which is finer than $\mathcal{T}_B$. For a given $\varepsilon > 0$ we determine a symmetric $\mathcal{T}_B$-neighbourhood V of e such that

$$xy^{-1} \in V \Rightarrow |f(x)-f(y)| < \varepsilon .$$

Then $(yz)(xz)^{-1} = yx^{-1} \in V$ for all $z \in G$, and so

$$|f(xz)-f(yz)| < \varepsilon .$$

Hence

$$\|f_{x^{-1}} - f_{y^{-1}}\|_\infty \leqq \varepsilon .$$

Writing $y=e$, we obtain $V \subset E(f, \varepsilon)$, and the left relative density of V accomplishes the proof.

Remark. As shown above $H(f,\varepsilon)$ is a symmetric invariant and relatively dense ε-neighbourhood of e. Now simple calculations prove that the sequence $\{V_n\}$, $V_n = H(f, \varepsilon \cdot 2^{-n})$, $n = 1, 2, \ldots$, possesses the properties (d), (e) of Theorem 1 with $V = H(f,\varepsilon)$. Thus $H(f,\varepsilon)$ is also a $\mathcal{T}_B$-neighbourhood of e.

Finally, we recall that any totally bounded uniform structure is determined by the pseudo-metrics

$$g(s,t) = |f(s)-f(t)|, \qquad f \text{ uniformly continuous} . \tag{2.6}$$

Thus, by the relation

$$H(f,\varepsilon) \subset \{s: \; |f(s)-f(e)| \leqq \varepsilon\}, \qquad \varepsilon > 0,$$

the sets $H(f,\varepsilon)$, f left almost periodic, $\varepsilon > 0$, form a fundamental system of neighbourhoods of e for $\mathcal{T}_B$.

From Theorem 2 we may immediately deduce a series of standard properties of almost periodic functions.

First: *The collections of left and of right almost periodic functions are* *identical, and hence the indications "left", "right" may be omitted.* Second: *Every almost periodic function is bounded and uniformly continuous in the initial left and right uniform structures.* Third: *The collection of almost periodic functions is closed with respect to (pointwise) addition and multiplication, multiplication by complex numbers, complex conjugation, and passage to uniform limits.* Fourth: *Every continuous function on a compact group is almost periodic.* Fifth: *A function f on a topological group is almost periodic if and only if there exists a continuous function $\hat{f}$ on the group $\hat{G}$ of the maximal compact representation ϱ, such that*

$$f(x) = \hat{f}(\varrho(x)), \qquad \text{for all } x \in G .$$

Sixth: *There exists a unique, positive, linear functional M (mean value) on the linear space of almost periodic functions, which takes the value* 1 *for the unit function and is left (and right) invariant in the sense that*

$$M(f_s) = M(f) \qquad \text{for all } s \in G .$$

Seventh: *The maximal compact representation of G is $1-1$ into $\hat{G}$ if and only if the set of almost periodic functions separates the points of G, whereas $\hat{G}$ reduces to one point if and only if the set of almost periodic functions consists of the constants only.*

The first three statements are obvious. The fourth statement follows from the explicite characterization of $\mathscr{T}_B$ in Theorem 1, which gives $\mathscr{T}_B = \mathscr{T}$ in the compact case. Now, the fifth statement follows from the extendability of uniformly continuous functions, and then the sixth statement is obtained by application of the existence and uniqueness of the Haar integral on G. Finally the seventh statement follows from the characterization of totally bounded uniform structures in terms of the pseudo-metrics (2.7).

It should be mentioned that somewhat deeper results concerning the interrelationship between finite dimensional unitary representations and almost periodic functions, such as the approximation theorem, L^2-expansion etc., can now be obtained by transfer of known properties of compact groups (Peter–Weyl Theorem). For details cf. e.g. [5, p. 165].

A set $\mathfrak{A}$ of complex valued functions on a topological group $(G, \mathscr{T})$ is termed a *left homogeneous set of almost periodic functions*, or briefly a *left homogeneous set*, if it is equicontinuous and the set $E(\mathfrak{A}, \varepsilon) = \{s\colon \|f_s - f\|_\infty \leqq \varepsilon, \text{ for all } f \in \mathfrak{A}\}$ is left relatively dense for every $\varepsilon > 0$. The concept of a *right homogeneous set* is defined correspondingly. Homogeneous sets of almost periodic functions on the complex plane were first studied by S. Bochner (who termed them "ausgezeichnete Mengen fastperiodischer Funktionen") [2, p. 143].

THEOREM 3. *A set* $\mathfrak{A}$ *of complex valued functions on a topological group* $(G, \mathcal{T})$ *is left (right) homogeneous if and only if it is uniformly equicontinuous with respect to* $\mathscr{U}_B$.

PROOF. 1) We assume $\mathfrak{A}$ to be a left homogeneous set and define a left invariant pseudo-metric on G by the formula:

$$d_{\mathfrak{A}}(x,y) = \sup_{f \in \mathfrak{A}} d_f(x,y) . \tag{2.7}$$

The pseudo-metric $d_{\mathfrak{A}}$ is evidently left invariant, and the ε-neighbourhoods of e have the form $E(\mathfrak{A}, \varepsilon)$. By the definition of a left homogeneous set, $d_{\mathfrak{A}}$ determines a totally bounded uniform structure $\mathscr{U}_{\mathfrak{A}}$. In virtue of Proposition 2, $\mathscr{U}_{\mathfrak{A}}$ admits a totally bounded uniform structure $\tilde{\mathscr{U}}_{\mathfrak{A}}$ compatible with the group structure and defined by the pseudo-metric $\tilde{d}_{\mathfrak{A}}(x,y) = \sup_z d_{\mathfrak{A}}(xz, yz)$. The closed ε-neighbourhoods of e are of the form

$$H(\mathfrak{A}, \varepsilon) = \{s:\ \sup_{x,y} |f(xy) - f(xsy)| \leqq \varepsilon \text{ for all } f \in \mathfrak{A}\} . \tag{2.8}$$

Proceeding as in the proof of Theorem 2, we determine $a_1, \ldots, a_n \in G$, such that every $a \in G$ admits an a_i, $1 \leqq i \leqq n$, for which

$$|f(yax) - f(ya_ix)| \leqq \varepsilon \qquad \text{for all } f \in \mathfrak{A} \text{ and all } x, y \in G . \tag{2.9}$$

By the equicontinuity of $\mathfrak{A}$, the set

$$W = \{t:\ |f(a_i t^{-1} a_j) - f(a_i a_j)| < \varepsilon;\ f \in \mathfrak{A},\ i, j = 1, \ldots, n\}$$

is a neighbourhood of e in the intitial topology, and again we get $W \subset H(\mathfrak{A}, 5\varepsilon)$. Hence $\tilde{\mathscr{U}}_{\mathfrak{A}}$ has the properties (a), (b), (c) of Theorem 1, and so it is coarser than $\mathscr{U}_B$.

By an argument similar to that of the proof of Theorem 2 (with $\tilde{d}_{\mathfrak{A}}$ in the place of $\tilde{d}_f$), we prove that $\mathfrak{A}$ is uniformly equicontinuous with respect to $\tilde{\mathscr{U}}_{\mathfrak{A}}$, and then also with respect to the finer structure $\mathscr{U}_B$.

Similarly we prove that a right homogeneous set is $\mathscr{U}_B$-uniformly equicontinuous.

2) We assume $\mathfrak{A}$ to be $\mathscr{U}_B$-uniformly equicontinuous, and proceed as in the second part of the proof of Theorem 2 to prove $\mathfrak{A}$ left homogeneous.

It follows in particular that a set $\mathfrak{A}$ of functions on G is left homogeneous if and only if it is right homogeneous, and hence the indications "left", "right" may be omitted.

COROLLARY. *A uniformly bounded homogeneous set of almost periodic functions on a topological group is conditionally compact.*

PROOF. Passage to the maximal compact representation and use of the Arzelà–Ascoli theorem (cf. e.g. [4, p. 266]).

From this corollary we obtain a theorem proved by Bochner [2, p. 143] for almost periodic functions on the real line, stating that *every uniformly bounded, homogeneous sequence of almost periodic functions admits a uniformly convergent subsequence.*

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UNIVERSITY OF OSLO, NORWAY