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# ON NON-HERMITIAN TOEPLITZ MATRICES[1]

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## 1. Introduction.

Let $f(\theta)$ be a (not necessarily real) function of class $L(0,2\pi)$ with Fourier series

$$f(\theta) \sim \sum_{-\infty}^{\infty} c_k e^{ik\theta} .$$

We are interested in the solution of the system of equations

$$\sum_{k=0}^{n} c_{j-k} h_k = g_j, \qquad 0 \leqq j \leqq n , \tag{1.1}$$

and in relating the "size" of the solution $\{h_k\}_0^n$ to that of $\{g_j\}_0^n$ by means of a norm inequality. We shall extend some results of Baxter's [1] for this problem.

A periodic function $F(e^{i\theta})$ of class $L$ over $(0,2\pi)$ is said to be of power series type [3] if

$$\int_0^{2\pi} F(e^{i\theta})\, e^{ik\theta}\, d\theta = 0, \qquad k = 1,2,\ldots .$$

The following statement is a rephrased version of the main theorem of [1]. (The exponent $-1$ denotes the reciprocal.)

THEOREM A. *Suppose $f(\theta) = A(e^{i\theta})B(e^{i\theta})$, where $A(e^{i\theta})$, $B(e^{-i\theta})$, $A^{-1}(e^{i\theta})$, and $B^{-1}(e^{-i\theta})$ are of power series type, and possess absolutely convergent Fourier series. Then there exist constants $K_{A,B}$, and $N_{A,B}$, depending only on the functions A and B, with the following property:*

*If n is a positive integer and $\{h_k\}_0^n$, $\{g_k\}_0^n$ satisfy* (1.1), *then*

$$\sum_0^n |h_k| \leqq K_{A,B} \sum_0^n |g_k|, \qquad n \geqq N_{A,B} . \tag{1.2}$$

Let $g(\theta) = \sum_0^n g_k e^{ik\theta}$, $h(\theta) = \sum_0^n h_k e^{ik\theta}$. If $F \in L^r$, we put, as usual,

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$$M_r[F] = \left\{\frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^r \, d\theta\right\}^{1/r} .$$

We will see that, by using $M_r$ norms ($r > 1$) to replace (1.2), one can weaken the hypothesis of Theorem $A$ to obtain the following.

THEOREM 1. *Suppose* $f(\theta) = A(e^{i\theta})B(e^{i\theta})$ *where* $A(e^{i\theta})$, $B(e^{-i\theta})$, $A^{-1}(e^{i\theta})$, *and* $B^{-1}(e^{-i\theta})$ *are bounded measurable functions of power series type, and at least one of the two functions* $A^{-1}(e^{i\theta})$, $B^{-1}(e^{-i\theta})$ *is continuous. Then for any* $r > 1$ *there exist constants* $K_{A,B,r}$, $N_{A,B,r}$ *such that whenever* (1.1) *is satisfied,*

(1.3) $$M_r[h] \leqq K_{A,B,r} M_r[g], \qquad n \geqq N_{A,B,r} .$$

While the conclusion (1.3) is, because of the different norm, not the same as (1.2) this will be seen to be of no importance so far as the applications which are of interest to us (Sections 4 and 5) are concerned. The basic structure of the proof of (1.3) is the same as in [1]. In addition to the introduction of the norms $M_r$, and the use of classical facts regarding conjugate Fourier series, the principal technical innovation in the present contribution is the use made of Cesàro means.

## 2. Proof of Theorem 1.

It is convenient to follow Baxter in the use of the operators $F^+(\theta)$ and $F^-(\theta)$. However it is necessary to define these for a broader class of functions $F(\theta)$ than merely the functions with absolutely convergent Fourier series considered in [1]. Suppose $F(\theta) \in L^r$, $r > 1$, and

$$F(\theta) \sim \sum_{-\infty}^{\infty} a_k e^{ik\theta} .$$

Then the conjugate function ([3, p. 253]) $\tilde{F}(\theta)$ is defined a.e., and is also of class $L^r$, and

$$\tilde{F}(\theta) \sim -i \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} (\operatorname{sign} k)\, a_k e^{ik\theta} .$$

We define

$$F^+(\theta) = \tfrac{1}{2}[F(\theta) + a_0 + i\tilde{F}(\theta)] \sim \sum_{0}^{\infty} a_k e^{ik\theta} ,$$

$$F^-(\theta) = \tfrac{1}{2}[F(\theta) - a_0 - i\tilde{F}(\theta)] \sim \sum_{-\infty}^{-1} a_k e^{ik\theta} .$$

By Riesz' theorem and Minkowski's inequality there exist constants $C_r$, not depending on $F$, such that

(2.1) $$M_r[F^+] \leqq C_r M_r[F], \quad M_r[F^-] \leqq C_r M_r[F] \qquad (r > 1) .$$

(In the simplest case, $r=2$, we can clearly take $C_2=1$.) An alternative but equivalent definition of the functions $F^+(\theta)$ and $F^-(\theta)$ a.e. is obtained by means of the Abel sums of the respective formal Fourier series.

Suppose the hypotheses of Theorem 1 are satisfied, with $A^{-1}(e^{i\theta})$ continuous. We have $f(\theta)h(\theta)-g(\theta)\in L^\infty$, and, by (1.1),

$$\int_0^{2\pi}[f(\theta)h(\theta)-g(\theta)]\,e^{-ik\theta}\,d\theta = 0, \qquad 0\leqq k\leqq n\,.$$

Hence

$$G_1(\theta) = e^{i(n+1)\theta}[e^{-i(n+1)\theta}(fh-g)]^+ \sim \sum_{n+1}^{\infty} g_k e^{ik\theta} \tag{2.2}$$

and

$$G_2(\theta) = [fh-g]^- \sim \sum_{-\infty}^{-1} g_k e^{ik\theta} \tag{2.3}$$

are of class $L^r$, and

$$f(\theta)h(\theta) = G_1(\theta)+g(\theta)+G_2(\theta)\,. \tag{2.4}$$

Thus

$$[Be^{-i(n+1)\theta}h]^+ = [A^{-1}e^{-i(n+1)\theta}G_1+A^{-1}e^{-i(n+1)\theta}g+A^{-1}e^{-i(n+1)\theta}G_2]^+\,.$$

Since $B(e^{-i\theta})$ is of power series type, the left side is zero. The term $A^{-1}e^{-i(n+1)\theta}G_1$ is also of power series type. Hence

$$A^{-1}e^{-i(n+1)\theta}G_1 = -[A^{-1}e^{-i(n+1)\theta}g]^+ - [A^{-1}e^{-i(n+1)\theta}G_2]^+\,. \tag{2.5}$$

If $\sigma_{n+1}(\theta)$ is any linear combination of $\{e^{ik\theta}\}$, $0\leqq k\leqq n+1$, and

$$\delta_{n+1}(\theta) = A^{-1}(e^{i\theta})-\sigma_{n+1}(\theta)\,,$$

then, in view of (2.3),

$$[\sigma_{n+1}e^{-i(n+1)\theta}G_2]^+ = 0\,.$$

Therefore,

$$[A^{-1}e^{-i(n+1)\theta}G_2]^+ = [\delta_{n+1}e^{-i(n+1)\theta}G_2]^+\,. \tag{2.6}$$

In particular, let $\sigma_n$ be the $n^{\text{th}}$ Cesàro$-1$ mean of the Fourier series of $A^{-1}(e^{i\theta})$. By Fejér's theorem,

$$\lim_{n\to\infty}\sup_\theta|\delta_{n+1}(\theta)| = 0\,. \tag{2.7}$$

Substituting (2.6) into (2.5), and solving for $G_1$, yields, after application of (2.1),

$$M_r[G_1] \leqq C_r\sup_\theta|A(e^{i\theta})|\,\{\sup_\theta|A^{-1}(e^{i\theta})|\;M_r[g]+\sup_\theta|\delta_{n+1}(\theta)|\;M_r[G_2]\}\,. \tag{2.8}$$

On the other hand, starting again with (2.4), and multiplying both sides by $B^{-1}$ we obtain

$$(Ah)^- = 0 = (B^{-1}G_1)^-+(B^{-1}g)^-+(B^{-1}G_2)^-\,.$$

Since $(B^{-1}G_2)^- = B^{-1}G_2$,

$$G_2 = -B(B^{-1}G_1)^- - B(B^{-1}g)^- .$$

Therefore

$$M_r[G_2] \leqq C_r \sup_\theta |B(e^{i\theta})| \sup_\theta |B^{-1}(e^{i\theta})| \{M_r[G_1] + M_r[g]\} . \tag{2.9}$$

Combining (2.8) and (2.9), and taking into account (2.7), we find that, when $n \geqq N_{A,B,r}$,

$$M_r[G_1] \leqq K_1 M_r[g], \qquad M_r[G_2] \leqq K_2 M_r[g] ,$$

where $K_1$, $K_2$ depend only on $A$, $B$, and $r$. Hence, by (2.4),

$$M_r[h] \leqq (1+K_1+K_2) \sup |A^{-1}B^{-1}| M_r[g], \qquad n \geqq N_{A,B,r} ,$$

completing the proof of (1.3). In the case when $A^{-1}(e^{i\theta})$ is only bounded, but $B^{-1}(e^{i\theta})$ is continuous, the proof is similar.

## 3. Hypotheses on *f* alone.

By means of a lemma, various forms of which are more or less well known, but whose proof shall be included for the sake of completeness, the hypotheses on $A$ and $B$ in Theorem 1 can be replaced by direct hypotheses on $f$.

LEMMA 3.1. *Suppose $v(\theta) \in L$ has a Fourier series of power series type, and $e^{v(\theta)} \in L$. Then the Fourier series of $e^{v(\theta)}$ is also of power series type.*

PROOF. There exists ([3, pp. 288–289]) a function $F(z)$, regular for $|z| < 1$, such that, for almost all $\theta$,

$$v(\theta) = \lim_{r \to 1} F(re^{i\theta}) ,$$

and with the property that $F$ is the Poisson integral of $v$,

$$F(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} v(t)\, dt .$$

Let $G(z) = e^{F(z)}$, $|z| < 1$. Then

$$\lim_{r \to 1} G(re^{i\theta}) = e^{v(\theta)} \quad \text{a.e.}$$

On the other hand,

$$|G(re^{i\theta})| = \exp\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\operatorname{Re}v(t)\,dt\right]$$
$$\leqq \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}e^{\operatorname{Re}v(t)}\,dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}|e^{v(t)}|\,dt\,.$$

Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi}|G(re^{i\theta})|\,d\theta \leqq \frac{1}{2\pi}\int_0^{2\pi}|e^{v(t)}|\,dt, \qquad 0 \leqq r < 1\,,$$

that is, $G(z)$ belongs to the Hardy class $H$. It follows that the radial limit values of $G$ are of power series type.

As an immediate corollary we can state the following.

THEOREM 2. *If the two functions* $(\log f)^+$, $(\log f)^-$ *exist and are bounded, and if at least one of them is continuous, then*

$$M_r[h] \leqq K_{f,r}M_r[g], \qquad n \geqq N_{f,r}, \quad r > 1\,.$$

PROOF. We take $A=\exp[(\log f)^+]$, $B=\exp[(\log f)^-]$, and apply Theorem 1.

## 4. Non-vanishing of the determinants.

We consider the $(n+1)\times(n+1)$ determinants

$$D_n[f] = \det(c_{i-j})$$

corresponding to the matrix of system (1.1). Since $M_r[g]=0$ implies $M_r[h]=0$, which in turn implies that all $h_k=0$, $0\leqq k\leqq n$, we have

THEOREM 3. *If the hypotheses of Theorem* 1 *or Theorem* 2 *are satisfied then* $D_n[f]\neq 0$ *for all sufficiently large* $n$.

*Further remarks.*

In the classical case [2] when $(c_{i-j})$ is Hermitian, the condition

$$f \geqq 0, \qquad \log f \in L \tag{4.1}$$

is sufficient to insure that $D_n[f]\neq 0$. In the general case when $(c_{i-j})$ is not necessarily Hermitian the condition $\log f\in L$ is not sufficient to guarantee that $D_n[f]\neq 0$ for even a single $n$ as the example $f(\theta)=e^{i\theta}$ shows.

The function

$$f(\theta) = (1-e^{i\theta})(1-e^{-i\theta}) = 2(1-\cos\theta) \tag{4.2}$$

satisfies (4.1), and therefore $D_n[f] \neq 0$, $n=1,2,\ldots$. (This can also be seen more directly.) But, unfortunately, we would not have been able to conclude this from Theorem 3, since neither $(1-e^{i\theta})^{-1}$ nor $(1-e^{-i\theta})^{-1}$ are even integrable. In fact, it is impossible to generalize Theorem 3, in the sense to be described below, so as to include (4.2).

Let us call the class of functions $\mathscr{F}$ *admissible* if $\mathscr{F}$ has the property that the conditions

$$\begin{cases} f(\theta) = \alpha(e^{i\theta})\beta(e^{-i\theta}) \\ \alpha(e^{i\theta}) \in \mathscr{F},\ \alpha^{-1}(e^{i\theta}) \in \mathscr{F},\ \beta(e^{i\theta}) \in \mathscr{F},\ \beta^{-1}(e^{i\theta}) \in \mathscr{F} \end{cases}$$

imply

$$D_n[f] \neq 0, \quad n \geqq N_{\alpha,\beta}\,.$$

Theorem A shows that the class of functions with absolutely convergent Fourier series of power series type constitutes an admissible class. In view of the present results we know that the larger class of merely continuous functions with Fourier series of power series type is admissible. The example

$$f(\theta) = (1-e^{i\theta})(1-e^{-i\theta})^{-1} = -e^{i\theta}$$

shows, however, that there exists no admissible class containing both the functions $(1-e^{i\theta})$ and $(1-e^{i\theta})^{-1}$.

## 5. The ratios $D_{n-1}[f]/D_n[f]$.

We shall prove the following

THEOREM 4. *If the hypothesis of Theorem 2 is satisfied then*

$$\lim_{n\to\infty} \frac{D_{n-1}[f]}{D_n[f]} = \exp\left[-\frac{1}{2\pi}\int_0^{2\pi} \log f\, d\theta\right]. \tag{5.1}$$

The basic method of proof, following [1], is to consider the solution of (1.1) for the case $g_j = \delta_{j0}$, namely,

$$\sum_{k=0}^{n} c_{j-k} h_k^{(n)} = \delta_{j0}, \quad 0 \leqq j \leqq n, \qquad h^{(n)}(\theta) = \sum_{k=0}^{n} h_k^{(n)} e^{ik\theta}, \tag{5.2}$$

where the superscript is used as a reminder of the fact that the solution vector $\{h_k^{(n)}\}$ depends on $n$. We have

$$h_0^{(n)} = D_{n-1}[f]/D_n[f]\,.$$

LEMMA 5.1. *Suppose the hypothesis of Theorem 2 is satisfied. Let* $h^{(n)}(\theta)$ *be defined by* (5.2), *and suppose that*

$$H(\theta) \sim \sum_{0}^{\infty} H_k e^{ik\theta}$$

*is a function of class* $L^2$ *for which*

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)H(\theta)e^{-ik\theta}\,d\theta = \delta_{k0}, \qquad 0 \leqq k < \infty . \tag{5.3}$$

*Then there exist constants* $K_f$, $N_f$, *depending only on* $f$, *such that*

$$\sum_{k=0}^{n} [H_k - h_k^{(n)}]^2 \leqq K_f \sum_{k=n+1}^{\infty} H_k^2, \qquad n \geqq N_f . \tag{5.4}$$

PROOF. Since

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)h^{(n)}(\theta)e^{-ik\theta}\,d\theta = \delta_{k0}, \qquad 0 \leqq k \leqq n ,$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)[H^{(n)}(\theta) - h^{(n)}(\theta)]e^{-ik\theta}\,d\theta = u_k = \frac{1}{2\pi}\int_0^{2\pi} [H^{(n)}(\theta) - H(\theta)]f(\theta)e^{-ik\theta}\,d\theta , \qquad 0 \leqq k \leqq n ,$$

with

$$H^{(n)}(\theta) = \sum_{0}^{n} H_k e^{ik\theta} .$$

By Theorem 2,

$$\begin{aligned} \left\{\sum_{k=0}^{n} [H_k - h_k^{(n)}]^2\right\}^{\frac{1}{2}} &\leqq K_{f,2}\left\{\sum_{k=0}^{n} u_k^2\right\}^{\frac{1}{2}} \\ &\leqq M_2\{[H^{(n)} - H]f\}K_{f,2} \\ &\leqq K_{f,2}\sup|f|\, M_2[H^{(n)} - H], \qquad n \geqq N_{f,2} . \end{aligned}$$

PROOF OF (5.1). The function

$$H(\theta) = \left[\frac{1}{2\pi}\int_0^{2\pi} B(e^{it})dt\right]^{-1} A^{-1}(e^{i\theta})$$

is of class $L^2$ and satisfies (5.3). By (5.4),

$$\lim_{n\to\infty} h_0^{(n)} = H_0 = \left[\frac{1}{2\pi}\int_0^{2\pi} A^{-1}(e^{i\theta})\,d\theta\right]\left[\frac{1}{2\pi}\int_0^{2\pi} B(e^{i\theta})\,d\theta\right]^{-1} . \tag{5.5}$$

Let $p(z)$ be the function, regular for $|z| < 1$, with non-tangential boundary values $[\log f(\theta)]^+$. The function $e^{-p(z)}$ has non-tangential boundary values $A^{-1}(e^{i\theta})$ almost everywhere, and, by Lemma 3.1, $e^{-p(z)}$ is the Poisson integral of $A^{-1}(e^{i\theta})$. Hence

$$(5.6)\qquad \frac{1}{2\pi}\int_0^{2\pi} A^{-1}(e^{i\theta})\,d\theta = e^{-p(0)} = \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} [\log f(\theta)]^+\,d\theta\right\}.$$

Similarly,

$$(5.7)\qquad \frac{1}{2\pi}\int_0^{2\pi} B(e^{i\theta})\,d\theta = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} [\log f(\theta)]^-\,d\theta\right\} = 1\,.$$

Substituting (5.6) and (5.7) into (5.5) gives (5.1).

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