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HARDY INEQUALITIES FOR THE HEISENBERG LAPLACIAN ON CONVEX BOUNDED POLYTOPES

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Abstract

We prove a Hardy-type inequality for the gradient of the Heisenberg Laplacian on open bounded convex polytopes on the first Heisenberg group. The integral weight of the Hardy inequality is given by the distance function to the boundary measured with respect to the Carnot-Carathéodory metric. The constant depends on the number of hyperplanes, given by the boundary of the convex polytope, which are not orthogonal to the hyperplane $x_3 = 0$.

1. Introduction

Consider the first Heisenberg group given by $\mathbb{R}^3$, equipped with the group law

$$(x_1, x_2, x_3) \boxplus (y_1, y_2, y_3) := \big(x_1 + y_1, x_2 + y_2, x_3 + y_3 - \tfrac{1}{2}(x_1 y_2 - x_2 y_1)\big),$$

and the sub-gradient $\nabla_{\mathbb{H}} := (X_1, X_2)$ given by

$$X_1 := \partial_{x_1} + \tfrac{1}{2} x_2 \partial_{x_3}, \quad X_2 := \partial_{x_2} - \tfrac{1}{2} x_1 \partial_{x_3},$$

for $x := (x_1, x_2, x_3) \in \mathbb{R}^3$. We recall that the vector fields $X_1, X_2, X_3 := [X_2, X_1] = \partial_{x_3}$ form a basis of the Lie algebra of left-invariant vector fields on $\mathbb{H}$ and that the sub-elliptic operator

$$\Delta_{\mathbb{H}} := -X_1^2 - X_2^2$$

is the Heisenberg Laplacian , also called the Kohn Laplacian. There is a considerable amount of literature concerning the Hardy-type inequality

$$\int_{\mathbb{H}} \frac{|u(x)|^2}{\|x\|_{\mathbb{H}}^4} (x_1^2 + x_2^2)\, \mathrm{d}x \le \int_{\mathbb{H}} \left|\nabla_{\mathbb{H}} u(x)\right|^2 \mathrm{d}x, \quad u \in C_0^\infty(\mathbb{H} \setminus \{0\}), \qquad (1)$$

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where

$$\|x\|_{\mathbb{H}}^4 := (x_1^2 + x_2^2)^2 + 16x_3^2.$$

For the proof of (1) we refer to [10], [1], [21], see also various improvements obtained in [2] and [23]. The anisotropic norm $\|x\|_{\mathbb{H}}$, which appears in (1), is referred to in the literature as the Korányi-Folland gauge or Kaplan gauge. For the sake of brevity we will use the latter notation and call it the Kaplan gauge.

In this paper we deal with sharp Hardy inequalities for the Heisenberg-Laplacian on bounded domains. In particular we consider the following problem: given a bounded domain $\Omega \subset \mathbb{R}^3$, we would like to find the best constant c for which the inequality

$$\int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\,\mathrm{d}x \le c^2 \int_\Omega \left|\nabla_{\mathbb{H}} u(x)\right|^2 \mathrm{d}x, \quad \forall u \in C_0^\infty(\Omega) \tag{2}$$

holds, where $\delta_C(x)$ is the Carnot-Carathéodory distance (C-C distance in the sequel) between x and the boundary of Ω, see Section 2 for its definition. For more details on the C-C distance we refer to [6], [7]. With respect to the well-studied inequality (1), less is known about the validity of (2), especially if one is interested in explicit constants. In [8] the authors proved that for every Ω with a $C^{1,1}$ regular boundary there exists $c > 0$ such that (2) is valid. Later it was shown by Yang, [24] that if Ω is a ball with respect to the C-C distance, then (2) holds with $c = 2$.

The fundamental problem of deriving inequalities of the form (2) lies in the fact, that we a priori don't know much about domains which are the most natural for a Hardy inequality on $\mathbb{H}$. In comparison to the Euclidean setting it is well-known that if Ω is *convex* then

$$\int_\Omega \frac{|u(x)|^2}{\operatorname{dist}(x, \partial\Omega)^2}\,\mathrm{d}x \le 4 \int_\Omega \left|\nabla u(x)\right|^2 \mathrm{d}x \tag{3}$$

holds for all $u \in C_0^\infty(\Omega)$, and the constant 4 is sharp independently of Ω, see e.g. [3], [22], [9], [16], [5], [4], [13].

In this paper we prove that for open bounded convex polytopes Ω we obtain a constant depending on the number of hyperplanes of $\partial\Omega$, which are not orthogonal to the hyperplane $x_3 = 0$. Under an additional geometrical assumption the constant in (2) for convex polytopes can be improved, see Theorem 6.2. It is even possible to show that for any $c > 2$ there exists an open bounded convex domain such that (2) is fulfilled, which is almost a sharp result since we prove that for any bounded domain Ω the following inequality holds:

$$\inf_{u\in C_0^\infty(\Omega)} \frac{\displaystyle\int_\Omega \left|\nabla_{\mathbb{H}}\, u(x)\right|^2 \mathrm{d}x}{\displaystyle\int_\Omega |u(x)|^2 \delta_C(x)^{-2}\,\mathrm{d}x} \le \frac{1}{4}.$$

This shows that at least some convex domains are more compatible with the Heisenberg group structure than we expect them to be.

In [15] Luan and Yang proved on the half-space $\Omega := \{x \in \mathbb{H} | x_3 > 0\}$ that for any $u \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \frac{x_1^2 + x_2^2}{4x_3^2} |u(x)|^2\,\mathrm{d}x \le 4 \int_\Omega \left|\nabla_{\mathbb{H}}\, u(x)\right|^2 \mathrm{d}x. \tag{4}$$

This result was recently generalized by Larson [14] to any convex domain. Under an additional convexity condition, where $H(x)$ denotes the tangent plane to x, we can replace the weight on the left-hand side by

$$\omega(x) := \inf_{y\in\partial\Omega\cap H(x)} d_C(x, y), \tag{5}$$

see Theorem 3.1. This result turns out to be (4) for the case of the half-space.

The paper is organized as follows. In the next section we introduce necessary notation. Main results are formulated in Section 3 and the proof of each Theorem is done in a separate section.

2. Preliminaries and notation

The *tangent plane* to $x := (x_1, x_2, x_3) \in \mathbb{H}$ is given by

$$\begin{aligned} H(x) &:= \left\{ y \in \mathbb{H} \;\middle|\; \left\langle \left(-\frac{x_2}{2}, \frac{x_1}{2}, 1 \right), y - x \right\rangle = 0 \right\}, \\ &= \left\{ y \in \mathbb{H} \;\middle|\; x_1 y_2 - x_2 y_1 = 2(x_3 - y_3) \right\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^3$.

Let us briefly recall the definition of the C-C distance $d_C(x, y)$. We call a Lipschitz curve $\gamma\colon [a, b] \to \mathbb{H}$ parametrized by $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ *horizontal* if

$$\gamma'(t) \in \operatorname{span}\left\{ \left(1, 0, \frac{\gamma_2(t)}{2}\right), \left(0, 1, -\frac{\gamma_1(t)}{2}\right) \right\}.$$

The C-C distance between x and y is then defined as

$$d_C(x, y) := \inf_{\gamma} \int_a^b \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}\, \mathrm{d}t, \tag{6}$$

where the infimum is taken over all horizontal curves γ connecting x and y.

We define the C-C and Kaplan distance functions for an open bounded domain Ω by

$$\delta_C(x) := \inf_{y \in \partial\Omega} d_C(x, y), \quad \delta_K(x) := \inf_{y \in \partial\Omega} \|(-y) \boxplus x\|_{\mathbb{H}}.$$

If $x \in \Omega^c$, we set $\delta_K(x) := 0$ and $\delta_C(x) := 0$. With these prerequisites we can state the main results.

3. Main results

THEOREM 3.1. *Let $\Omega \subset \mathbb{H}$ be open bounded, and let the connected components of $H(x) \cap \Omega$ be convex for all $x \in \Omega$. Then*

$$\int_{\Omega} \frac{|u(x)|^2}{\omega(x)^2}\, \mathrm{d}x \leq 4 \int_{\Omega} \left|\nabla_{\mathbb{H}} u(x)\right|^2 \mathrm{d}x \tag{7}$$

holds for all $u \in C_0^\infty(\Omega)$, where $\omega(\cdot)$ is defined in (5) and we have

$$\omega(x) = \inf_{y \in \partial\Omega \cap H(x)} \|(-y) \boxplus x\|_{\mathbb{H}} = \inf_{y \in \partial\Omega \cap H(x)} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

We call the weight $\omega(\cdot)$ the reduced C-C distance. The proof of (7) is done in the following way. We prove the Hardy inequality for each separate X_j, where the distance function is given by the C-C metric generated by X_j for $j \in \{1, 2\}$. Then we apply the hyperplane separation theorem in the same way as E. B. Davies did for the proof of (3) for convex domains, see [9].

THEOREM 3.2. *Let $\Omega \subset \mathbb{H}$ be an open bounded convex polytope, and let $m \in \mathbb{N}$ be the number of hyperplanes of $\partial\Omega$ which are not orthogonal to the hyperplane $x_3 = 0$. Then*

$$\frac{1}{5}\left(\frac{3^{3/2}\sqrt{2}}{c_m} + 1\right)^{-4/3} \int_{\Omega} \frac{|u(x)|^2}{\delta_C(x)^2}\, \mathrm{d}x \leq \int_{\Omega} \left|\nabla_{\mathbb{H}} u(x)\right|^2 \mathrm{d}x \tag{8}$$

holds for all $u \in C_0^\infty(\Omega)$, where c_m is defined as the unique positive solution of the following equality

$$\sqrt{c_m^2 + 16}\left(1 + \frac{c_m}{3^{3/2}\sqrt{2}}\right)^{2/3} {c_m}^{4/3} = \frac{1}{2^{7/3} 3\pi m}.$$

For the uniqueness we use the intermediate value theorem and the monotonicity of the functions on the left-hand side.

In addition, we prove that for c_m the following holds

$$\frac{1}{c_m} \leq m^{8/9}\pi^{8/9} 3 \cdot 2^{19/9}\sqrt{2^{-4/3}\pi^{-2/3}+16}\left(1+\frac{1}{3^{3/2}2^{7/6}\pi^{1/3}}\right)^{2/3}, \tag{9}$$

which yields a result with an explicit constant in (8).

The strategy of the proof of Theorem 3.2 consists of two steps. We use Theorem 3.1 for a bounded convex polytope. Then we take into account the following Hardy inequality

$$\int_\Omega \frac{|u(x)|^2}{d_C(x,0)^2}\,\mathrm{d}x \leq \int_\Omega \left|\nabla_{\mathbb{H}} u(x)\right|^2 \mathrm{d}x, \tag{10}$$

for all $u \in C_0^\infty(\Omega)$, which was proved in [11] and [24]. The sum of the weight functions is then comparable to the distance function to the hyperplanes of the given polytope, respectively the Kaplan gauge, which is equivalent to the distance function respectively the C-C metric.

We can improve the constant in Theorem 3.2 under an additional geometrical assumption, which is discussed in Section 6. The main consequence of that result is the following:

THEOREM 3.3. *For any $\varepsilon > 0$ there exists a bounded convex domain Ω such that*

$$\int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\,\mathrm{d}x \leq (2+\varepsilon)^2 \int_\Omega \left|\nabla_{\mathbb{H}} u(x)\right|^2 \mathrm{d}x$$

for all $u \in C_0^\infty(\Omega)$.

The last result has an almost optimal constant since we prove that for any bounded domain Ω the inequality

$$\inf_{u\in C_0^\infty(\Omega)} \frac{\displaystyle\int_\Omega \left|\nabla_{\mathbb{H}} u(x)\right|^2 \mathrm{d}x}{\displaystyle\int_\Omega |u(x)|^2 \delta_C(x)^{-2}\,\mathrm{d}x} \leq \frac{1}{4}$$

holds, see Theorem 6.5.

4. Restricted C-C distance and its connection to the Euclidean distance

4.1. The natural restriction of $\partial\Omega$

In this section we show that the reduced distance $\omega(\cdot)$, defined by (5), can be expressed in terms of a simple explicit formula. In particular, we show that ω coincides with the distance to the boundary in the Kaplan gauge as well as the projection onto the (x_1, x_2)-hyperplane of the Euclidean metric.

THEOREM 4.1. *Let* $\Omega \subset \mathbb{H}$ *be open and bounded. Then*

$$\omega(x) = \inf_{y\in\partial\Omega\cap H(x)} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = \inf_{y\in\partial\Omega\cap H(x)} \|(-y) \boxplus x\|_{\mathbb{H}}$$

holds for all $x \in \Omega$.

For the proof we need the following:

LEMMA 4.2. *For all* $x, y \in \mathbb{H}$, *we have*

$$\frac{1}{\pi^2} d_C(x,y)^4 \le \|(-y)\boxplus x\|_{\mathbb{H}}^4 \le d_C(x,y)^4. \tag{11}$$

Moreover, both inequalities are sharp.

PROOF. Using the left-invariance of $d_C(x, y)$, respectively the group law on $\mathbb{H}$, we transform (11) into

$$\frac{1}{\pi^2} d_C(y^{-1}\boxplus x, 0)^4 \le \|(-y)\boxplus x\|_{\mathbb{H}}^4 \le d_C(y^{-1}\boxplus x, 0)^4.$$

We know that $y^{-1} = -y$. Therefore it is sufficient to prove

$$\frac{1}{\pi^2} d_C(z,0)^4 \le \|z\boxplus 0\|_{\mathbb{H}}^4 \le d_C(z,0)^4 \quad \forall\, z \in \mathbb{H}.$$

The arc joining geodesics starting from the origin were computed in [18] and [17]. The parametrization of these arcs is given by

$$z = \gamma_{k,\theta}(t) := \begin{cases} z_1(t,k,\theta) = \dfrac{\cos(\theta)-\cos(kt+\theta)}{k}, \\ z_2(t,k,\theta) = \dfrac{\sin(kt+\theta)-\sin(\theta)}{k}, \\ z_3(t,k,\theta) = \dfrac{kt-\sin(kt)}{2k^2}, \end{cases} \tag{12}$$

where $t \in [0, 2\pi/|k|]$, $\theta \in [0, 2\pi)$ and $k \in \mathbb{R} \setminus \{0\}$. This means that for the given point $z := \gamma_{k,\theta}(t) \in \mathbb{H}$, one has $d(\gamma_{k,\theta}(t), 0) = t$. We extend this formula to the case $k = 0$ by taking the limit for $k \to 0$. This gives

$$z = \gamma_{0,\theta}(t) := \begin{cases} z_1(t, 0, \theta) = t \sin(\theta), \\ z_2(t, 0, \theta) = t \cos(\theta), \\ z_3(t, 0, \theta) = 0. \end{cases}$$

For the computation of $d_C(z, 0)$ we use (12). It is then sufficient to calculate the supremum and the infimum of

$$\frac{\left\|\gamma_{k,\theta}(t) \boxplus 0\right\|_{\mathbb{H}}^4}{d_C(z, 0)^4} = \frac{4\big(1 - \cos(kt)\big)^2 + 4\big(kt - \sin(kt)\big)^2}{(tk)^4}.$$

This leads to estimating the function

$$g(\tau) := \frac{4}{\tau^4}\big((1 - \cos(\tau))^2 + (\tau - \sin(\tau))^2\big),$$

with $0 \le \tau \le 2\pi$, because $t \in [0, 2\pi/|k|]$. To proceed, we show that the function $g(\tau)$ is non-increasing on $[0, 2\pi]$. By differentiating the function $g(\tau)$ several times we find that the latter is non-increasing on $[0, 2\pi]$ which implies that the same is true for g. Hence

$$\frac{1}{\pi^2} = g(2\pi) \le g(\tau) \le \lim_{\tau \to 0+} g(\tau) = 1.$$

The sharpness of that inequality is an immediate consequence.

Proof of Theorem 4.1. Let $x \in \Omega$ and let $y \in \partial\Omega \cap H(x)$. Consider the curve $\gamma\colon [0, 1] \to \mathbb{H}$ given by the parametrization $\gamma(t) = (1 - t)x + ty$, $t \in [0, 1]$. Obviously γ connects x and y. Moreover, since $y \in H(x)$, it is easily verified that γ is horizontal. Indeed, we have

$$\gamma'(t) = (y_1 - x_1)\left(1, 0, \frac{\gamma_2(t)}{2}\right) + (y_2 - x_2)\left(0, 1, -\frac{\gamma_1(t)}{2}\right).$$

By definition of the C-C distance, see equation (6), it thus follows that

$$d_C(x, y) \le \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Using $y \in \partial\Omega \cap H(x)$ we see that

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \|(-y) \boxplus x\|_{\mathbb{H}} .$$

Then we apply Lemma 4.2 to obtain the following chain of inequalities

$$d_C(x,y) \le \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = \|(-y)\boxplus x\|_{\mathbb{H}} \le d_C(x,y).$$

Taking the infimum over $y \in \partial\Omega \cap H(x)$ yields the result.

4.2. The Hardy inequality involving ω

We need the following auxiliary result.

LEMMA 4.3. *Let Ω be an open bounded domain in $\mathbb{H}$. Then*

$$\int_\Omega \left(\frac{|u(x)|^2}{d_1(x)^2} + \frac{|u(x)|^2}{d_2(x)^2} \right) \mathrm{d}x \le 4 \int_\Omega \left| \nabla_{\mathbb{H}} u(x) \right|^2 \mathrm{d}x$$

holds for all $u \in C_0^\infty(\Omega)$, where the distances $d_1(x)$ and $d_2(x)$ are given by

$$d_1(x) := \inf_{s\in\mathbb{R}} \{|s| > 0 \mid x + s(1,0,x_2/2) \notin \Omega\},$$

$$d_2(x) := \inf_{s\in\mathbb{R}} \{|s| > 0 \mid x + s(0,1,-x_1/2) \notin \Omega\}.$$

PROOF. Let $u \in C_0^\infty(\Omega)$. First we show that

$$\int_\Omega \frac{|u(x)|^2}{d_1(x)^2} \,\mathrm{d}x \le 4 \int_\Omega |X_1 u(x)|^2 \,\mathrm{d}x. \tag{13}$$

To this end we define the following coordinate transformation

$$F(t,\varphi,\theta) := \begin{cases} x_1(t,\varphi,\theta) = t + \varphi, \\ x_2(t,\varphi,\theta) = \theta, \\ x_3(t,\varphi,\theta) = t\theta/2, \end{cases} \tag{14}$$

where $(t,\varphi,\theta) \in A := \{(t,\varphi,\theta) \in \mathbb{R}^3 \mid \theta \ne 0\}$. It can be easily checked that $F\colon A \mapsto \mathrm{Ran}(A)$ is a diffeomorphism and that the determinant of F is equal to $\theta/2$. For a given $x \in \Omega^c$ we set $u(x) = 0$. If $x = F(t,\varphi,\theta)$ for fixed $\theta \in \mathbb{R} \setminus \{0\}$ and $\varphi \in \mathbb{R}$, we see that there exists a constant $c \in \mathbb{R}$ such that $F(c,\varphi,\theta) = \hat{x} \in \partial\Omega$ satisfies $d_1(x) = d_C(x,\hat{x})$. By $\{a_j\}_{j\in\mathbb{N}}$ we denote the increasing sequence such that $F(a_j,\varphi,\theta) \in \partial\Omega$. Thus for a fixed $x \in \Omega$ we immediately see that there exists a $k \in \mathbb{N}$ such that

$$\begin{aligned} d_1(F(t,\varphi,\theta)) &= d_C(F(t,\varphi,\theta), F(a_k,\varphi,\theta)) \\ &= d_C(F(t,\varphi,\theta), F(t,\varphi,\theta) + (a_k - t)(1,0,\theta/2)) \\ &= |a_k - t|. \end{aligned}$$

Using the last observation, we apply the transformation F to find that to prove (13) it suffices to show that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{j=1}^{\infty}\int_{a_j}^{a_{j+1}} \frac{|u(t,\varphi,\theta)|^2}{\delta_j(t)^2}\,\mathrm{d}t\,\frac{|\theta|}{2}\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ \le 4\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{j=1}^{\infty}\int_{a_j}^{a_{j+1}} |\partial_t u(t,\varphi,\theta)|^2\,\mathrm{d}t\,\frac{|\theta|}{2}\,\mathrm{d}\theta\,\mathrm{d}\varphi, \quad (15)$$

where $\delta_j(t) := \inf(a_{j+1}-t, t-a_j)$. Hence the one-dimensional Hardy inequality in the t-direction then implies that (15) holds which in turn yields (13). It remains to prove

$$\int_{\Omega}\frac{|u(x)|^2}{d_2(x)^2}\,\mathrm{d}x \le 4\int_{\Omega}|X_2u(x)|^2\,\mathrm{d}x. \quad (16)$$

This is done in the same way as (13) replacing the transformation of (14) by

$$\tilde{F}(t,\varphi,\theta) := \begin{cases} x_1(t,\varphi,\theta) = \theta, \\ x_2(t,\varphi,\theta) = t+\varphi, \\ x_3(t,\varphi,\theta) = -t\theta/2, \end{cases}$$

for $(t,\varphi,\theta)\in A$. Summing up (13) and (16) then completes the proof.

Proof of Theorem 3.1. Let $a := (a_1,a_2,a_3)\in\partial\Omega\cap H(x)$ be such that

$$\omega(x) = \sqrt{(x_1-a_1)^2+(x_2-a_2)^2}.$$

The existence of such a is guaranteed by the compactness of $\overline{\Omega}$ and the continuity of the distance. We know that all connected components of $H(x)\cap\Omega$ are convex. Therefore we assume without loss of generality that $H(x)\cap\Omega$ consists of a single connected component which is convex. Next we apply the hyperplane separation theorem, which implies that the hyperplane

$$T := \left\{ y\in\mathbb{H} \;\middle|\; \left\langle \begin{pmatrix} x_1-a_1 \\ x_2-a_2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1-a_1 \\ y_2-a_2 \\ y_3 \end{pmatrix} \right\rangle = 0 \right\} \quad (17)$$

separates $H(x)\cap\Omega$ from the point $a\in\partial\Omega$. We consider the definition of $d_1(x)$, see Lemma 4.3, and compute the intersection point of the line $c(s) = x+s(1,0,x_2/2)^t$ for $s\in\mathbb{R}$ with the hyperplane (17). This yields

$$s = -\frac{(x_1-a_1)^2+(x_2-a_2)^2}{x_1-a_1}.$$

At this point we apply the hyperplane separation theorem again to infer that

$$d_1(x) \leq \frac{(x_1 - a_1)^2 + (x_2 - a_2)^2}{|x_1 - a_1|}.$$

Now we do the same computation for $d_2(x)$ and obtain

$$d_2(x) \leq \frac{(x_1 - a_1)^2 + (x_2 - a_2)^2}{|x_2 - a_2|}.$$

Altogether we get

$$\begin{aligned}
&\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} \\
&\quad \geq \frac{(x_2 - a_2)^2}{((x_1 - a_1)^2 + (x_2 - a_2)^2)^2} + \frac{(x_1 - a_1)^2}{((x_1 - a_1)^2 + (x_2 - a_2)^2)^2} \\
&\quad = \frac{1}{\omega(x)^2}.
\end{aligned}$$

We recall that the point $a \in \partial\Omega \cap H(x)$ was chosen such that the equality $\omega(x) = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}$ holds, which proves inequality (7).

REMARK 4.4. For $p \geq 2$ it is possible to get an L^p version of Theorem 3.1 as well. In Lemma 4.3 we use the L^p version of the one-dimensional Hardy inequality, which holds for $p > 1$. Then we mimic the last proof and apply for $p \geq 2$ Jensen's inequality

$$(a^2 + b^2)^{p/2} = 2^{p/2}(a^2/2 + b^2/2)^{p/2} \leq 2^{p/2-1}(a^p + b^p),$$

for $a, b > 0$.

5. Proof of the Hardy inequalities for open bounded convex polytopes

In this section we give the proof of Theorem 3.2. First we have to give some lower estimates for the Kaplan distance function to hyperplanes which are not orthogonal to the $x_3 = 0$ hyperplane. Therefore we need the following:

LEMMA 5.1. *Let $p > 0$ and $q \in \mathbb{R} \setminus \{0\}$. Consider*

$$z^3 + pz = q,$$

for $z \in \mathbb{R}$. Then there exists a unique real solution and it satisfies

$$|z| \geq \frac{|q^{1/3}|}{3}\left(1 + \frac{p\sqrt{p}}{|q|3\sqrt{3}}\right)^{-2/3}.$$

PROOF. First we consider the case $q > 0$. Then Cardano's formula gives the unique real solution

$$\begin{aligned} z &= \big(q/2 + \sqrt{q^2/4 + p^3/27}\big)^{1/3} + \big(q/2 - \sqrt{q^2/4 + p^3/27}\big)^{1/3} \\ &= \frac{1}{3}\int_{-q/2+\sqrt{q^2/4+p^3/27}}^{q/2+\sqrt{q^2/4+p^3/27}} s^{-2/3}\,\mathrm{d}s \geq \frac{q}{3}\big(q/2 + \sqrt{q^2/4 + p^3/27}\big)^{-2/3} \\ &\geq \frac{q}{3}\big(q + \sqrt{p^3/27}\big)^{-2/3} \end{aligned}$$

The case $q < 0$ is treated in the same way.

PROPOSITION 5.2. *Let $x \in \mathbb{H}$ and $a > 0$. We consider*

$$\Pi := \{y \in \mathbb{H} \mid n_1 y_1 + n_2 y_2 + n_3 y_3 = c\},$$

where $n_1, n_2, n_3, c \in \mathbb{R}$ and $n_3 \neq 0$. When $(-2n_2/n_3 + x_1)^2 + (2n_1/n_3 + x_2)^2 \leq a|-c/n_3 + x_3 + x_1 n_1/n_3 + x_2 n_2/n_3|$ we have

$$\Big(\inf_{y\in\Pi} \|(-y) \boxplus x\|_{\mathbb{H}}\Big)^2 \geq \frac{4|-c/n_3 + x_3 + x_1 n_1/n_3 + x_2 n_2/n_3|}{3^3}\left(1 + \frac{a}{3^{3/2}\sqrt{2}}\right)^{-2},$$

and for $(-2n_2/n_3 + x_1)^2 + (2n_1/n_3 + x_2)^2 \geq a|-c/n_3 + x_3 + x_1 n_1/n_3 + x_2 n_2/n_3|$ we have

$$\Big(\inf_{y\in\Pi} \|(-y) \boxplus x\|_{\mathbb{H}}\Big)^2 \geq \frac{4|-c/n_3 + x_3 + x_1 n_1/n_3 + x_2 n_2/n_3|^2}{(-2n_2/n_3 + x_1)^2 + (2n_1/n_3 + x_2)^2}\left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3}.$$

PROOF. First of all we consider the case $n_1 = n_2 = c = 0$ and $n_3 = 1$. Let $y \in \mathbb{H}$ such that $y_3 = 0$ and fix $x := (x_1, x_2, x_3) \in \mathbb{H}$ with $x_3 \neq 0$. We set $z_1 := y_1 - x_1$ and $z_2 := y_2 - x_2$ and consider

$$\begin{aligned} &\|(-y) \boxplus x\|_{\mathbb{H}}^4 \\ &\quad = \big((y_1 - x_1)^2 + (y_2 - x_2)^2\big)^2 + 16\big(y_3 - x_3 - \tfrac{1}{2}y_1 x_2 + \tfrac{1}{2}y_2 x_1\big)^2 \qquad (18) \\ &\quad = (z_1^2 + z_2^2)^2 + 16\big(-x_3 - \tfrac{1}{2}z_1 x_2 + \tfrac{1}{2}z_2 x_1\big)^2. \end{aligned}$$

Then we compute the minimum of the right-hand side as a function of x. We assume that $x_1 \neq 0$, since $x_1 = 0$ is a null set and δ_K is continuous, see (24),

(26) and Lemma 4.2. The y_1 and y_2 derivatives then yield respectively

$$(z_1^2 + z_2^2)4z_1 - x_2 16\big(-x_3 - \tfrac{1}{2}z_1x_2 + \tfrac{1}{2}z_2x_1\big) = 0,$$
$$(z_1^2 + z_2^2)4z_2 + x_1 16\big(-x_3 - \tfrac{1}{2}z_1x_2 + \tfrac{1}{2}z_2x_1\big) = 0.$$

Since $x_1 \neq 0$, we easily deduce that $z_1^2 + z_2^2 \neq 0$ and obtain

$$z_1 = \frac{-z_2x_2}{x_1}.$$

Inserting this in (18) yields

$$\|(-y) \boxplus x\|_{\mathbb{H}}^4 = z_2^4 \frac{(x_2^2 + x_1^2)^2}{x_1^4} + 16\left(-x_3 + \tfrac{1}{2}z_2 \frac{x_2^2 + x_1^2}{x_1}\right)^2.$$

We compute the critical points with respect to y_2 and obtain

$$\|(-y) \boxplus x\|_{\mathbb{H}}^4 = z_2^4 \frac{(x_2^2 + x_1^2)^2}{x_1^4} + z_2^6 \frac{(x_2^2 + x_1^2)^2}{x_1^6},$$

where z_2 is the unique real solution of

$$z_2^3 + 2z_2x_1^2 = \frac{4x_3x_1^3}{x_2^2 + x_1^2}, \quad p := 2x_1^2, \quad q := \frac{4x_3x_1^3}{x_2^2 + x_1^2}.$$

Using the estimate in the previous Lemma, we get

$$|z_2| \geq \frac{4^{1/3}|x_3|^{1/3}|x_1|}{3(x_1^2 + x_2^2)^{1/3}} \left(1 + \frac{x_1^2 + x_2^2}{|x_3| 3^{3/2}\sqrt{2}}\right)^{-2/3} \tag{19}$$

For the case $x_1^2 + x_2^2 \leq a|x_3|$, we use

$$\|(-y) \boxplus x\|_{\mathbb{H}}^4 \geq z_2^6 \frac{(x_2^2 + x_1^2)^2}{x_1^6},$$

and (19) to get

$$\Big(\inf_{y \in \mathbb{H}, y_3 = 0} \|(-y) \boxplus x\|_{\mathbb{H}}\Big)^2 \geq \frac{4|x_3|}{3^3}\left(1 + \frac{a}{3^{3/2}\sqrt{2}}\right)^{-2}.$$

For the case $x_1^2 + x_2^2 \geq a|x_3|$, we use (19) again for

$$\|(-y) \boxplus x\|_{\mathbb{H}}^4 \geq z_2^4 \frac{(x_2^2 + x_1^2)^2}{x_1^4},$$

which yields

$$\Big(\inf_{y\in\mathbb{H},\, y_3=0} \|(-y)\boxplus x\|_{\mathbb{H}}\Big)^2 \geq \frac{4|x_3|^2}{(x_2^2+x_1^2)}\left(\frac{3^{3/2}\sqrt{2}}{a}+1\right)^{-4/3},$$

To obtain the result for a general hyperplane, we consider

$$\begin{aligned}\inf_{y\in\Pi} \|(-y)\boxplus x\|_{\mathbb{H}} &= \inf_{y\in\Pi} \|(-(v\boxplus y))\boxplus(v\boxplus x)\|_{\mathbb{H}} \\ &= \inf_{(-v)\boxplus q\in\Pi} \|(-q)\boxplus(v\boxplus x)\|_{\mathbb{H}},\end{aligned}$$

where $q := (q_1, q_2, q_3) \in \mathbb{H}$, and $v \in \mathbb{H}$ is set

$$v := \frac{1}{n_3}(-2n_2, 2n_1, -c).$$

Then $(-v)\boxplus q \in \Pi$ is equivalent to $q_3 = 0$, which yields the result.

Proof of Theorem 3.2. Let us assume that Ω is an open bounded convex polytope. Let $m \in \mathbb{N}$ be the number of hyperplanes of $\partial\Omega$, which are not orthogonal to the hyperplane $y_3 = 0$. We denote these hyperplanes by Π_j for $1 \leq j \leq m$. Thus there exist $n_{1,j}, n_{2,j}, n_{3,j}, c_j \in \mathbb{R}$ such that

$$\Pi_j := \{y \in \mathbb{H} \mid n_{1,j}y_1 + n_{2,j}y_2 + n_{3,j}y_3 = c_j\},$$

where $n_{3,j} \neq 0$ for $1 \leq j \leq m$. Write $n_j \in \mathbb{R}^3$ for the unit normal of Π_j. We use Lemma 4.3 and inequality (10) to obtain

$$\begin{aligned}\int_\Omega \left(\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} + \frac{1}{m}\sum_{j=1}^{m}\frac{1}{d_C(x,a_j)^2}\right)|u(x)|^2\,\mathrm{d}x \\ \leq 5\int_\Omega \left|\nabla_{\mathbb{H}}u(x)\right|^2\mathrm{d}x, \qquad (20)\end{aligned}$$

for $u \in C_0^\infty(\Omega)$, where

$$a_j := \frac{1}{n_{3,j}}(2n_{2,j}, -2n_{1,j}, c_j).$$

The aim is to give a pointwise estimate for the weights on the left-hand side from below. We take $b \in \partial\Omega$ such that $\delta_K(x) = \|(-b)\boxplus x\|_{\mathbb{H}}$, which exists since $\partial\Omega$ is compact and δ_K is continuous.

The first case is $b \in \Pi_j$ for a fixed j. Since Ω is convex we compute the intersection points $d_1(x)$ and $d_2(x)$ with Π_j. The hyperplane separation theorem yields then

$$\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} \geq \frac{1}{4} \frac{(-2n_{2,j} + x_1 n_{3,j})^2 + (2n_{1,j} + x_2 n_{3,j})^2}{|-c_j + \langle x, n_j \rangle|^2}.$$

Let $a > 0$. We use Proposition 5.2 for the case $(-2n_{2,j}/n_{3,j} + x_1)^2 + (2n_{1,j}/n_{3,j} + x_2)^2 \geq a|-c/n_3 + x_3 + x_1 n_1/n_3 + n_2 x_2/n_3|$ and get

$$\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} \geq \left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3} \left(\inf_{y \in \Pi_j} \|(-y) \boxplus x\|_{\mathbb{H}}\right)^{-2}.$$

For the case $(-2n_{2,j}/n_{3,j} + x_1)^2 + (2n_{1,j}/n_{3,j} + x_2)^2 \leq a|-c/n_3 + x_3 + x_1 n_1/n_3 + n_2 x_2/n_3|$, we use Lemma 4.2 to get

$$\begin{aligned} \frac{1}{m} \sum_{k=1}^{m} \frac{1}{d_C(x, a_k)^2} &\geq \frac{1}{\pi m} \frac{1}{\|-a_j \boxplus x\|_{\mathbb{H}}^2} \\ &\geq \frac{1}{\pi m \sqrt{a^2 + 16}} |-c/n_3 + x_3 + x_1 n_1/n_3 + n_2 x_2/n_3|^{-1} \end{aligned}$$

and then again Proposition 5.2 yields

$$\begin{aligned} &\frac{1}{m} \sum_{k=1}^{m} \frac{1}{d_C(x, a_k)^2} \\ &\qquad \geq \frac{1}{4 \cdot 3^3 \pi m \sqrt{a^2 + 16}} \left(1 + \frac{a}{3^{3/2}\sqrt{2}}\right)^{-2} \left(\inf_{y \in \Pi_j} \|(-y) \boxplus x\|_{\mathbb{H}}\right)^{-2}. \end{aligned}$$

We choose $a > 0$ such that

$$\frac{1}{4 \cdot 3^3 \pi m \sqrt{a^2 + 16}} \left(1 + \frac{a}{3^{3/2}\sqrt{2}}\right)^{-2} = \left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3},$$

which obviously exists. The positive constant, which satisfies that equation is denoted by c_m. If we summarise our estimates, the weight function in (20) is then bounded from below by

$$\begin{aligned} &\left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3} \left(\inf_{y \in \Pi_j} \|(-y) \boxplus x\|_{\mathbb{H}}\right)^{-2} \\ &\qquad \geq \left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3} \|(-b) \boxplus x\|_{\mathbb{H}}^{-2}, \end{aligned}$$

where we used $b \in \Pi_j$. We recall that b was chosen so that $\delta_K(x) = \|(-b) \boxplus x\|_{\mathbb{H}}$.

The second case is $b := (b_1, b_2, b_3) \in \partial\Omega$ when the hyperplane, which contains b, is orthogonal to the hyperplane $x_3 = 0$. We denote that hyperplane by Π. Because of the orthogonality condition, the hyperplane is parametrized by

$$\Pi_j := \{y \in \mathbb{H} \mid (b_1 - x_1)(y_1 - b_1) + (b_2 - x_2)(y_2 - b_2) = 0\}.$$

We use the hyperplane separation theorem again and compute the intersection points of $d_1(x), d_2(x)$ with Π_j obtaining

$$\begin{aligned}\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} &\geq \frac{(b_1 - x_1)^2}{\big((b_1 - x_1)^2 + (b_2 - x_2)^2\big)^2} \\ &\quad + \frac{(b_2 - x_2)^2}{\big((b_1 - x_1)^2 + (b_2 - x_2)^2\big)^2} \\ &= \frac{1}{(b_1 - x_1)^2 + (b_2 - x_2)^2} \geq \frac{1}{\|(-b) \boxplus x\|_{\mathbb{H}}^2}.\end{aligned}$$

At that point we use that b was chosen, such that $\delta_K(x) = \|(-b) \boxplus x\|_{\mathbb{H}}$ is fulfilled. Summarizing our estimates we arrive at

$$\left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3} \int_\Omega \frac{|u(x)|^2}{\delta_K(x)^2}\,\mathrm{d}x \leq 5 \int_\Omega \left|\nabla_{\mathbb{H}} u(x)\right|^2 \mathrm{d}x,$$

where Lemma 4.2 finally yields the result.

Proof of inequality (9). Let us assume that $c_m > 0$ satisfies

$$\sqrt{c_m^2 + 16}\left(1 + \frac{c_m}{3^{3/2}\sqrt{2}}\right)^{2/3} {c_m}^{4/3} = \frac{1}{2^{7/3} 3\pi m}.$$

It can be easily seen that

$$c_m \leq (4m\pi)^{-1/3} \leq (4\pi)^{-1/3}.$$

Thus we get the following estimate

$$\frac{1}{2^{7/3} 3\pi m} \leq \sqrt{(4\pi)^{-2/3} + 16}\left(1 + \frac{(4\pi)^{-1/3}}{3^{3/2}\sqrt{2}}\right)^{2/3} (4m\pi)^{-1/9} c_m,$$

which yields

$${c_m}^{-1} \leq m^{8/9}\pi^{8/9} 3 \cdot 2^{19/9}\sqrt{2^{-4/3}\pi^{-2/3} + 16}\left(1 + \frac{1}{3^{3/2} 2^{7/6}\pi^{1/3}}\right)^{2/3}.$$

6. Convex polytopes with improved constants

We prove that for some open bounded convex polytopes the constant in Theorem 3.2 can be improved. We discuss that behavior in detail for convex cylinders. At the end we show for the smallest constant $c > 0$ satisfying (2) that $2 \le c$, which is a similar result to the Euclidean case.

6.1. The improved version

ASSUMPTION 6.1. Let Ω be an open bounded convex polytope. Let $m \in \mathbb{N}$ denote the number of hyperplanes of $\partial\Omega$, which are not orthogonal to the hyperplane $x_3 = 0$. We denote these hyperplanes by Π_j for $1 \le j \le m$. Thus there exist $n_{1,j}, n_{2,j}, n_{3,j}, c_j \in \mathbb{R}$ such that

$$\Pi_j := \{y \in \mathbb{H} \mid n_{1,j}y_1 + n_{2,j}y_2 + n_{3,j}y_3 = c_j\},$$

where $n_{3,j} \neq 0$ for $1 \le j \le m$. We assume that there exists a constant $a > 0$ such that for all $x \in \Omega$ and all $j \in \{1, \dots, m\}$ holds

$$(-2n_{2,j}/n_{3,j} + x_1)^2 + (2n_{1,j}/n_{3,j} + x_2)^2 \\ \ge a|-c/n_3 + x_3 + x_1n_1/n_3 + n_2x_2/n_3|. \quad (21)$$

THEOREM 6.2. *Under Assumption 6.1, one has*

$$\left(\frac{3^{3/2}\sqrt{2}}{a} + 1\right)^{-4/3} \int_\Omega \frac{|u(x)|^2}{\delta_C(x)^2}\,\mathrm{d}x \le 4\int_\Omega \left|\nabla_{\mathbb{H}} u(x)\right|^2 \mathrm{d}x$$

for all $u \in C_0^\infty(\Omega)$.

PROOF. We use Lemma 4.3 to obtain

$$\int_\Omega \left(\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2}\right)|u(x)|^2\,\mathrm{d}x \le 4\int_\Omega \left|\nabla_{\mathbb{H}} u(x)\right|^2 \mathrm{d}x,$$

for $u \in C_0^\infty(\Omega)$, and proceed in the same way as in the proof of Theorem 3.2. We treat only the case $b \in \Pi_j$ with $\delta_K(x) = \|(-b) \boxplus x\|_{\mathbb{H}}$ since the other one is the same verbatim. By n_j we denote the unit normal to Π_j. Again we use the hyperplane separation theorem and get

$$\frac{1}{d_1(x)^2} + \frac{1}{d_2(x)^2} \ge \frac{1}{4}\frac{(-2n_{2,j} + x_1n_{3,j})^2 + (2n_{1,j} + x_2n_{3,j})^2}{|-c_j + \langle x, n_j\rangle|^2}.$$

Under Assumption 6.1, we use Proposition 5.2, yielding

$$\frac{1}{4}\frac{(-2n_{2,j}+x_1n_{3,j})^2+(2n_{1,j}+x_2n_{3,j})^2}{|-c_j+\langle x,n_j\rangle|^2} \geq \left(\frac{3^{3/2}}{a2^{-1/2}}+1\right)^{-4/3}\left(\inf_{y\in\Pi_j}\|(-y)\boxplus x\|_{\mathbb{H}}\right)^{-2}.$$

Since $b\in\Pi_j$, we use Lemma 4.2 and get the result.

REMARK 6.3. The last result can be extended to any convex bounded Ω as long as there exists a constant $a>0$ such that for any hyperplane, which separates Ω from points lying on its boundary, inequality (21) holds.

6.2. Convex cylinders

We indicate briefly that there are indeed domains satisfying Assumption 6.1. Therefore we consider domains of the form $\Omega=\omega\times(\alpha,\beta)$, where $\omega\subset\mathbb{R}^2$ is a bounded convex domain and $\alpha<\beta$. This domain is not a polytope but the hyperplanes which separate the points lying in $b\in\partial\omega\times(\alpha,\beta)$ are orthogonal to the hyperplane $x_3=0$. Thus the proof of Theorem 6.2 goes through and we get:

COROLLARY 6.4. *Let $\Omega=\omega\times(\alpha,\beta)$ such that $\alpha<\beta$ and $\omega\subset\mathbb{R}^2$ is a bounded convex domain. For fixed $a>0$, we assume that for all $x\in\Omega$ we have*

$$x_1^2+x_2^2\geq a|-\alpha+x_3| \quad \textit{and} \quad x_1^2+x_2^2\geq a|-\beta+x_3|.$$

Then the following is valid for all $u\in C_0^\infty(\Omega)$

$$\left(\frac{3^{3/2}\sqrt{2}}{a}+1\right)^{-4/3}\int_\Omega\frac{|u(x)|^2}{\delta_C(x)^2}\,\mathrm{d}x\leq 4\int_\Omega\left|\nabla_{\mathbb{H}}u(x)\right|^2\mathrm{d}x. \tag{22}$$

PROOF OF THEOREM 3.3. Let $a>0$ be fixed. We consider the following domain $\Omega_a:=B_1(p_a)\times(0,1)$, where $B_1(p_a)$ is the two-dimensional Euclidean ball with radius one centered at $p_a:=(\sqrt{a}+1,0)$. The conditions of the last Corollary can be checked easily, where $\alpha=0$ and $\beta=1$. Thus the Hardy inequality (22) holds, where the constant depends on $a>0$.

6.3. On the sharp constant

THEOREM 6.5. *Let $\Omega \subset \mathbb{H}$ be a bounded domain. Then one has*

$$\inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega \left|\nabla_{\mathbb{H}} u(x)\right|^2 \mathrm{d}x}{\int_\Omega |u(x)|^2 \delta_C(x)^{-2} \,\mathrm{d}x} \le \frac{1}{4}.$$

PROOF. It suffices to construct a sequence $u_n \in C_0^\infty(\Omega)$ such that

$$\lim_{n \to \infty} \frac{\int_\Omega \left|\nabla_{\mathbb{H}} u_n(x)\right|^2 \mathrm{d}x}{\int_\Omega |u_n(x)|^2 \delta_C(x)^{-2} \,\mathrm{d}x} = \frac{1}{4}.$$

To this end we consider the sequence

$$\tilde{u}_n(x) = \delta_C(x)^{1/2+1/n}, \quad n \in \mathbb{N},$$

and recall that $\delta_C(x)$ satisfies the Eikonal equation

$$\left|\nabla_{\mathbb{H}} \delta_C(x)\right|^2 = 1, \quad \text{for a.e. } x \in \Omega, \tag{23}$$

see [20, Thm 3.1]. Moreover, from [12] we know that

$$M \left\|x - y\right\|_e \le d_C(x, y) \le M^{-1} \left\|x - y\right\|_e^{1/2} \tag{24}$$

holds for some $M > 0$ and all $x, y \in \overline{\Omega}$. Hence the integral $\int_\Omega \delta_C(x)^{2/n-1} \,\mathrm{d}x < \infty$, and using (23) we easily find that

$$\frac{\int_\Omega \left|\nabla_{\mathbb{H}} \tilde{u}_n(x)\right|^2 \mathrm{d}x}{\int_\Omega |\tilde{u}_n(x)|^2 \delta_C(x)^{-2} \,\mathrm{d}x} = \left(\frac{1}{2} + \frac{1}{n}\right)^2, \quad \forall\, n \in \mathbb{N}.$$

Next we will show that δ_C is weakly differentiable with respect to X_1 and X_2 on Ω. Without loss of generality we consider only the case X_1. Let $u \in C_0^\infty(\Omega)$ be given. We must show

$$\int_\Omega X_1 u(x)\, \delta_C(x) \,\mathrm{d}x = -\int_\Omega u(x) X_1\, \delta_C(x) \,\mathrm{d}x. \tag{25}$$

Since we can extend these functions to the whole space we can integrate over $\mathbb{R}^3$. An application of the dominated convergence theorem then yields

$$\int_{\mathbb{R}^3} X_1 u(x)\delta_C(x)\,\mathrm{d}x = \lim_{h\to 0}\left(\int_{\mathbb{R}^3} \frac{u(x+h\tilde{x})}{h}\delta_C(x)\,\mathrm{d}x - \int_{\mathbb{R}^3} \frac{u(x)}{h}\delta_C(x)\,\mathrm{d}x\right),$$

where $\tilde{x} := (1, 0, x_2/2)$. We make the change of variables $x + h\tilde{x} \mapsto x$ to obtain

$$\int_{\mathbb{R}^3} X_1 u(x)\delta_C(x)\,\mathrm{d}x = \lim_{h\to 0}\left(\int_{\mathbb{R}^3} u(x)\frac{\delta_C(x-h\tilde{x})}{h}\,\mathrm{d}y - \int_{\mathbb{R}^3} \frac{u(x)}{h}\delta_C(x)\,\mathrm{d}y\right)$$
$$= -\lim_{h\to 0}\left(\int_{\mathbb{R}^3} u(x)\frac{\delta_C(x-h\tilde{x}) - \delta_C(x)}{-h}\,\mathrm{d}y\right).$$

Since any two points lying in $\mathbb{H}$ can be connected by a (not necessarily unique) geodesic, see [19], we can easily deduce

$$|\delta_C(x) - \delta_C(y)| \leq d_C(x, y), \quad \text{for all } x \in \mathbb{H}. \tag{26}$$

Taking that inequality and the application of the left-invariance of the C-C distance, we get $d_C(x - h\tilde{x}, x) = d_C(-he_1, 0) = |h|$, where $e_1 := (1, 0, 0) \in \mathbb{H}$. Hence we may apply the dominated convergence theorem again arriving at

$$\int_{\mathbb{R}^3} X_1 u(x)\,\delta_C(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3} u(x)\lim_{h\to 0}\frac{\delta_C(x-h\tilde{x}) - \delta_C(x)}{-h}\,\mathrm{d}y.$$

This limit exists almost everywhere on $\mathbb{H}$, see [20], since $\delta_C(x)$ fulfils (26). This proves (25), and therefore it follows that δ_C is weakly differentiable on Ω with respect to X_1. The case of X_2 is treated in the same way.

At this point it can be shown by a standard argument that δ_C can be approximated by $C_0^\infty(\Omega)$ functions and that the same is true for the sequence $\tilde{u}_n$.

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