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NOTE ON RAMANUJAN'S FUNCTION $\tau(n)$

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The function $\tau(n)$ is defined by

$$\sum_{1}^{\infty} \tau(n)x^n = x\prod_{1}^{\infty}(1-x^n)^{24} .$$

It is well known that for certain moduli the residue of $\tau(n)$ can be expressed by the function $\sigma_k(n)$, the sum of the k-th powers of the divisors of n. The strongest results obtained in this direction are the following:

$$\begin{array}{lrll}
(1) & \tau(8n+1) \equiv & \sigma_{11}(8n+1) & (\operatorname{mod} 2^{11}) ,\\
(2) & \tau(8n+3) \equiv & 1217\sigma_{11}(8n+3) & (\operatorname{mod} 2^{13}) ,\\
(3) & \tau(8n+5) \equiv & 1537\sigma_{11}(8n+5) & (\operatorname{mod} 2^{12}) ,\\
(4) & \tau(8n+7) \equiv & 705\sigma_{11}(8n+7) & (\operatorname{mod} 2^{14}) ,\\
(5) & \tau(3n+1) \equiv & \sigma_{11}(3n+1) & (\operatorname{mod} 3^{5}) ,\\
(6) & \tau(3n+2) \equiv & 53\sigma_{11}(3n+2) & (\operatorname{mod} 3^{6}) ,\\
(7) & \tau(n) \equiv & 5n^2\sigma_7(n)-4n\sigma_9(n) & (\operatorname{mod} 5^{3}) \text{ if } (n,5) = 1 ,\\
(8) & \tau(n) \equiv & n\sigma_3(n) & (\operatorname{mod} 7) ,\\
(9) & \tau(n) \equiv & 0 & (\operatorname{mod} 23) \text{ if } (n/23) = -1 ,\\
(10) & \tau(n) \equiv & \sigma_{11}(n) & (\operatorname{mod} 691) ,
\end{array}$$

where (n/p) is Legendre's symbol. The congruences (1)–(6) were proved in [3], (7) is due to Bambah and Chowla [1], (8) to Wilton [5], (9) to Hardy [2], and (10) to Ramanujan [4].

The object of this note is to prove the congruence

$$(11) \qquad \tau(n) \equiv n\sigma_9(n) \pmod{7^2} \quad \text{if } (n/7) = -1 .$$

We put

$$P = 1-24\sum\sigma(n)x^n, \qquad Q = 1+240\sum\sigma_3(n)x^n, \qquad R = 1-504\sum\sigma_5(n)x^n ,$$

where $\sigma(n)=\sigma_1(n)$, and the sums are taken from 1 to ∞. It is known (cf. [4]) that

$$\begin{aligned}
1 + 480\sum \sigma_7(n)x^n &= Q^2 ,\\
1008\sum n\sigma_5(n)x^n &= Q^2-PR ,\\
1584\sum n\sigma_9(n)x^n &= 3Q^3+2R^2-5PQR ,\\
1728\sum \tau(n)x^n &= Q^3-R^2 .
\end{aligned}$$

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Combining these equations, and noticing that $R \equiv 1 \pmod{7}$, we easily verify the congruence

$$(12) \qquad 7 + \sum \{n\sigma_9(n) + 13\tau(n)\}x^n \equiv 7Q\left(1 + 4\sum \{\sigma_7(n) - n\sigma_5(n)\}x^n\right) \pmod{7^2}.$$

Now, if $(n, 7) = 1$ we have

$$n\sigma_5(n) = n^6\sigma_{-5}(n) \equiv \sigma(n) \pmod{7}.$$

Since $\sigma_7(n) \equiv \sigma(n) \pmod{7}$ for all n, we get

$$\sigma_7(n) - n\sigma_5(n) \equiv \begin{cases} 0 & \pmod{7} \quad \text{if } (n,7) = 1 \\ \sigma(n) & \pmod{7} \quad \text{if } (n,7) = 7. \end{cases}$$

Thus, returning to (12) we obtain

$$(13) \qquad 7 + \sum \{n\sigma_9(n) + 13\tau(n)\}x^n \equiv 7\left(1 + 2\sum \sigma_3(n)x^n\right)\left(1 + 4\sum \sigma(7n)x^{7n}\right) \pmod{7^2}.$$

Further, if $(n, 7) = 1$, we have

$$\sigma_3(n) = n^3\sigma_{-3}(n) \equiv n^3\sigma_3(n) \pmod{7},$$

and hence

$$(14) \qquad \sigma_3(n) \equiv 0 \pmod{7} \quad \text{if } (n/7) = -1.$$

From (13) and (14) we conclude that

$$n\sigma_9(n) + 13\tau(n) \equiv 0 \pmod{7^2} \quad \text{if } (n/7) = -1,$$

which implies (11), cf. (8) and (14).

ADDED IN PROOF: In connection with formula (8) it may be noticed that D. H. Lehmer has shown that

$$(n^3 - 1)\,\tau(n) \equiv 30n\sigma_{15}(n) + 16n\sigma_9(n) - (12n^4 - 15n)\,\sigma_3(n) \pmod{7^2}.$$

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