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NON-KOSZUL QUADRATIC GORENSTEIN TORIC RINGS

KAZUNORI MATSUDA

Abstract

Koszulness of Gorenstein quadratic algebras of small socle degree is studied. In this paper, we construct non-Koszul Gorenstein quadratic toric ring such that its socle degree is more than 3 by using stable set polytopes.

Introduction

Let K be a field and $S = K[x_1, \dots, x_n]$ a polynomial ring over K. Let $R = S/I$ be a standard graded K-algebra with respect to the grading $\deg x_i = 1$ for all $1 \leq i \leq n$, where I is a homogeneous ideal of S. Let R_+ denote the homogeneous maximal ideal of R. For an R-module M, we denote $\beta_{ij}^R(M)$ by the (i, j)-th graded Betti number of M as an R-module.

The Koszul algebra was originally introduced by Priddy (note that he also considered non-commutative algebras).

Definition 0.1 ([32]). A standard graded K-algebra R is said to be *Koszul* if the residue field $K = R/R_+$ has a linear R-free resolution as an R-module, that is, all non-zero entries of matrices representing the differential maps in the graded minimal free resolution of K are homogeneous of degree one. In other words, $\beta_{ij}^R(K) = 0$ holds if $i \neq j$.

Example 0.2.
(1) Polynomial rings are Koszul (consider the Koszul complex).
(2) Let $R = K[X]/(X^2)$. Then R is Koszul since

$$\cdots \xrightarrow{X} R \xrightarrow{X} R \longrightarrow K \longrightarrow 0$$

is a linear R-resolution of K.

Since $\beta_{2j}^R(K) = 0$ for all $j > 2$, hence Koszul algebras are *quadratic*, where $R = S/I$ is said to be quadratic if I is generated by homogeneous

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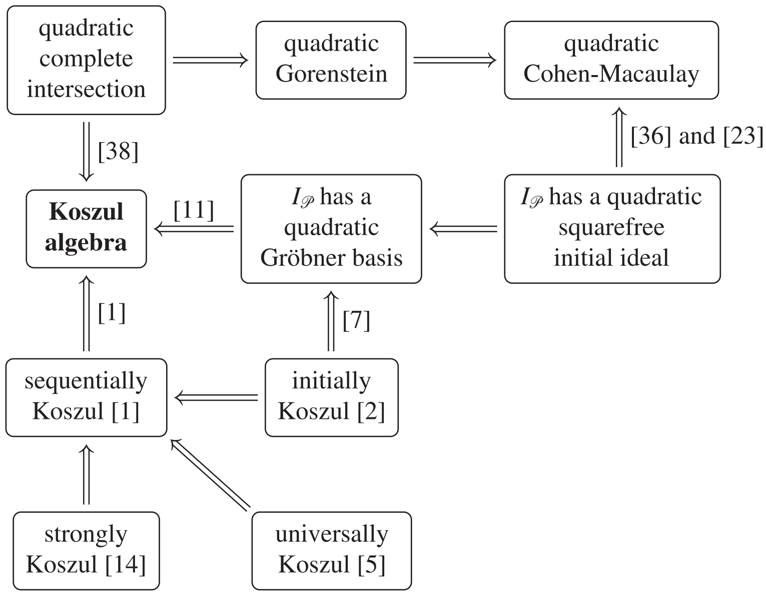


FIGURE 1

elements of degree 2. Every quadratic complete intersection is Koszul by Tate's theorem [38]. Moreover, $R = S/I$ is Koszul if I has a quadratic Gröbner bases by Fröberg's theorem [11] and the fact that $\beta_{ij}^{R}(K) \leq \beta_{ij}^{R'}(K)$ for all i, j and for all monomial order $<$ on S, where $R' = S/\mathrm{in}_<(I)$. The notion of Koszul algebra has played an important role in the research on graded K-algebras, and various Koszul-like algebras have been introduced, e.g., universally Koszul [5], strongly Koszul [14], initially Koszul [2], sequentially Koszul [1], etc.

Koszulness of toric rings of integral convex polytopes is studied. Let $\mathscr{P} \subset \mathbb{R}^n$ be an integral convex polytope, i.e., a convex polytope each of whose vertices belongs to $\mathbb{Z}^n$, and let $\mathscr{P} \cap \mathbb{Z}^n = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$. Assume that $\mathbb{Z}\mathbf{a}_1 + \cdots + \mathbb{Z}\mathbf{a}_m = \mathbb{Z}^n$. Let $K[X^{\pm 1}, t] := K[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, t]$ be the Laurent polynomial ring in $n + 1$ variables over K. Given an integer vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we put $X^{\mathbf{a}}t = x_1^{a_1} \cdots x_n^{a_n} t \in K[X^{\pm 1}, t]$. The *toric ring* of $\mathscr{P}$, denoted by $K[\mathscr{P}]$, is the subalgebra of $K[X^{\pm 1}, t]$ generated by $\{X^{\mathbf{a}_1}t, \dots, X^{\mathbf{a}_m}t\}$ over K. Note that $K[\mathscr{P}]$ can be regarded as a standard graded K-algebra by setting $\deg X^{\mathbf{a}_i}t = 1$. The *toric ideal* $I_{\mathscr{P}}$ is the kernel of a surjective ring homomorphism $\pi\colon K[Y] = K[y_1, \dots, y_m] \to K[\mathscr{P}]$ defined by $\pi(y_i) = X^{\mathbf{a}_i}t$ for $1 \leq i \leq m$. Then $K[\mathscr{P}] \cong K[Y]/I_{\mathscr{P}}$. It is known that $I_{\mathscr{P}}$ is generated by homogeneous binomials.

Note that the implications in Figure 1 hold. In addition, the following is known.

(1) Conca-De Negri-Rossi posed a conjecture that the defining ideal of a strongly Koszul algebra has a quadratic Gröbner bases [6, Question 13(1)]. This conjecture is true for the toric ring of edge polytope [18], order polytope [14], stable set polytope [26] and cut polytope [34].

(2) A squarefree strongly Koszul toric ring is compressed [27, Theorem 2.1], where $K[\mathscr{P}] \cong K[Y]/I_{\mathscr{P}}$ is said to be *compressed* if $\sqrt{\text{in}_<(I_{\mathscr{P}})} = \text{in}_<(I_{\mathscr{P}})$ for any reverse lexicographic order $<$ on $K[Y]$. In particular, a squarefree strongly Koszul toric ring is quadratic Cohen-Macaulay.

(3) Many toric rings associated with integral convex polytopes whose toric ideals has a quadratic Gröbner bases are constructed (e.g., [3], [15], [17], [19], [20], [21]). In other words, many Koszul toric rings associated with integral convex polytopes are constructed.

(4) A quadratic algebra is not always Koszul (see [30, Example 2.1], [33, Example 3] and [37, Theorem 3.1]). Note that both of these examples are Cohen-Macaulay but are not Gorenstein.

On the other hand, Koszulness of Gorenstein quadratic algebras is studied. For a standard graded K-algebra $R = \oplus_{i \geq 0} R_i$ with $\dim R = d$, we denote by

$$H_R(t) = \sum_{i \geq 0} \dim_K R_i t^i = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d}$$

the *Hilbert series* of R, where $h_s \neq 0$, and we say that $h(R) := (h_0, h_1, \ldots, h_s)$ is the *h-vector* of R and the index s is the *socle degree* of R. It is known that $h_0 = 1$ and if R is Gorenstein then $h_i = h_{s-i}$ for all $0 \leq i \leq \lfloor s/2 \rfloor$ ([35, Theorem 4.4]). Conca-Rossi-Valla proved that if R is a quadratic Gorenstein with $h(R) = (1, n, 1)$ (in this case $n \geq 2$ since R is quadratic) then R is Koszul [7, Proposition 2.12].

The case for $s = 3$ is also studied. Let R be a quadratic Gorenstein with $h(R) = (1, n, n, 1)$ (in this case $n \geq 3$ since R is quadratic). If $n = 3$, then R is quadratic complete intersection, hence R is Koszul. Conca-Rossi-Valla proved that R is Koszul if $n = 4$ [7, Theorem 6.15] and Caviglia proved that R is Koszul if $n = 5$ in his unpublished master thesis. The case for $n \geq 6$ is still open.

In this note, we focus on (4). In Section 1, we remark about known result of toric rings and toric ideals of stable set polytopes, and construct non-Koszul quadratic Gorenstein toric rings by using stable set polytopes. In Section 2, we present some questions.

1. Stable set polytope and non-Koszul quadratic Gorenstein toric ring

The stable set polytope is an integral convex polytope associated with stable sets of a simple graph.

Let G be a finite simple graph on the vertex set $[n] = \{1, 2, \dots, n\}$ and let $E(G)$ denote the set of edges of G. Recall that a finite graph is *simple* if it possesses no loops or multiple edges. We denote by $\overline{G}$ the complement graph of G.

Given a subset $W \subset [n]$, we define the (0, 1)-vector $\rho(W) = \sum_{i \in W} \mathbf{e}_i \in \mathbb{R}^n$, where $\mathbf{e}_i$ is the i-th unit coordinate vector of $\mathbb{R}^n$. In particular, $\rho(\emptyset)$ is the origin of $\mathbb{R}^n$.

A subset $W \subset [n]$ is said to be *stable* if $\{i, j\} \notin E(G)$ for all $i, j \in W$ with $i \neq j$. Note that the empty set and each single-element subset of $[n]$ are stable. By definition, W is a stable set of G if and only if W is a clique of $\overline{G}$. Let $S(G)$ denote the set of all stable sets of G. The *stable set polytope* of a simple graph G, denoted by $\mathcal{Q}_G$, is the convex hull of $\{\rho(W) \mid W \in S(G)\}$. By definition, $\mathcal{Q}_G$ is a (0, 1)-polytope and $K[\mathcal{Q}_G] = K\big[t \cdot \prod_{i \in W} x_i \mid W \in S(G)\big] \subset K[x_1, \dots, x_n, t]$. Note that $\dim K[\mathcal{Q}_G] = n + 1$. Let $K[Y] = K[y_W \mid W \in S(G)]$ be the polynomial ring over K. Now we define a surjective ring homomorphism $\pi \colon K[Y] \to K[\mathcal{Q}_G]$ by $\pi(y_W) = t \cdot \prod_{i \in W} x_i$ and let $I_{\mathcal{Q}_G} = \ker \pi$.

To state known results of the toric ring $K[\mathcal{Q}_G]$ and the toric ideal $I_{\mathcal{Q}_G}$ of the stable set polytope $\mathcal{Q}_G$ of a simple graph G, we introduce some classes of graphs. About terminologies for the graph theory, see [8].

A *cycle* graph with length n, denoted by C_n, is a connected graph which satisfies $E(C_n) = \big\{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{1, n\}\big\}$. An *odd cycle* is a cycle such that its length is odd.

A graph G is said to be *perfect* if the chromatic number of every induced subgraph of G is equal to the size of the largest clique of that subgraph. A graph G is perfect if and only if both G and $\overline{G}$ are $(C_{2n+3}, n \geq 1)$-free [4].

The *comparability* graph $G(P)$ of a partially ordered set $P = ([n], <_P)$ is the graph such that $V(G(P)) = [n]$ and $\{i, j\} \in E(G(P))$ if and only if $i <_P j$ or $j <_P i$. A graph G is said to be *comparability* if G is the comparability graph of some partially ordered set. Forbidden induced subgraphs of comparability graphs are known (see [25, p. 13]).

A graph G is said to be *bipartite* if there exist V_1, V_2 with $V_1 \cup V_2 = V(G)$ and $V_1 \cap V_2 = \emptyset$ such that if $\{i, j\} \in E(G)$ then either $i \in V_1$ and $j \in V_2$ or $i \in V_2$ and $j \in V_1$. It is known that a graph G is bipartite if and only if G is $(C_{2n+1}, n \geq 1)$-free.

A graph G is said to be *almost bipartite* (see [10, p. 87]) if there exists a vertex v such that the induced subgraph $G_{[n]\setminus v}$ is bipartite.

REMARK 1.1. The following facts are known.

(1) Let G be a perfect graph. Then $K[\mathscr{Q}_G]$ is Gorenstein if and only if all maximal cliques of G have the same cardinality [31, Theorem 2.1(b)].

(2) Let $G(P)$ be the comparability graph of a partially ordered set P. Then $K[\mathscr{Q}_{G(P)}]$ is Koszul since $\mathscr{Q}_{G(P)}$ is equal to the chain polytope of P and the toric ideal of a chain polytope has a squarefree quadratic initial ideal (see [16, Corollary 3.1]).

(3) If G is almost bipartite, then $K[\mathscr{Q}_G]$ is Koszul since its toric ideal $I_{\mathscr{Q}_G}$ has a squarefree quadratic initial ideal (see [10, Theorem 8.1]).

(4) Let G be a graph such that $\overline{G}$ is bipartite. Then $K[\mathscr{Q}_G]$ is quadratic if and only of it is Koszul [28, Corollary 3.4].

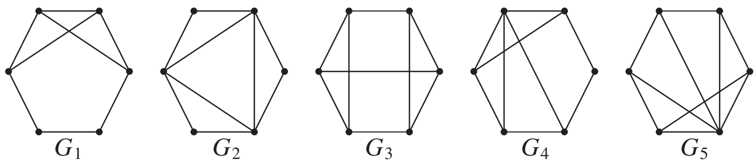
Hence, if $K[\mathscr{Q}_G]$ is quadratic but not Koszul, then G is neither a comparability graph nor almost bipartite, and $\overline{G}$ is not bipartite. From this fact and the classifications of these graphs, we have:

PROPOSITION 1.2. *Let G be a graph on $[n]$. If $K[\mathscr{Q}_G]$ is non-Koszul quadratic Gorenstein, then $n \geq 7$, that is,* $\dim K[\mathscr{Q}_G] \geq 8$.

PROOF. First, we assume that $n \leq 5$. Then G is a comparability graph if G is not C_5. Since C_5 is almost bipartite, we have that $K[\mathscr{Q}_G]$ is Koszul if $n \leq 5$ from Remark 1.1(2) and (3).

Next, we assume that $n = 6$. If G is not connected, then G is a comparability graph if G is not $C_5 \cup K_1$. Since $C_5 \cup K_1$ is almost bipartite, we have that $K[\mathscr{Q}_{G(P)}]$ is Koszul.

Assume that G is connected. From the classifications of comparability and almost bipartite graphs, G is one of the following (see [26, p. 10]):



Then we can see that

- $K[\mathscr{Q}_{G_1}]$ is not Gorenstein since $h(K[\mathscr{Q}_{G_1}]) = (1, 7, 10, 3)$,
- $K[\mathscr{Q}_{G_2}]$ is Koszul; indeed, we can check that the Gröbner bases of $I_{\mathscr{Q}_{G_2}}$ with respect to the reverse lexicographic order induced by the ordering

$$y_{\{3,6\}} > y_{\emptyset} > y_{\{1\}} > \cdots > y_{\{6\}} > y_{\{1,4\}} > y_{\{2,4\}} > y_{\{2,5\}} > y_{\{2,6\}} > y_{\{4,6\}} > y_{\{2,4,6\}}$$

is quadratic,

- $\overline{G_3}$ is C_6, hence it is bipartite,
- $K[\mathscr{Q}_{G_4}]$ is not Gorenstein since $h(K[\mathscr{Q}_{G_4}]) = (1, 6, 8, 2)$,
- $K[\mathscr{Q}_{G_5}]$ is Koszul since $I_{\mathscr{Q}_{G_5}} = I_{\mathscr{Q}_{C_5}}$ and $I_{\mathscr{Q}_{C_5}}$ has a quadratic Gröbner bases.

Therefore we have the desired conclusion.

For each integer $k \geq 3$, the complement of an odd cycle C_{2k+1}, denoted by $\overline{C_{2k+1}}$, is neither a comparability graph nor almost bipartite. Moreover, we note that $\overline{C_{2k+1}}$ is not perfect and $S(\overline{C_{2k+1}}) = \big\{\emptyset, \{1\}, \{2\}, \dots, \{2k+1\}, \{1,2\}, \{2,3\}, \dots, \{2k, 2k+1\}, \{1, 2k+1\}\big\}$.

Let $K[Y] = K[y_\emptyset, y_{\{1\}}, y_{\{2\}}, \dots, y_{\{2k+1\}}, y_{\{1,2\}}, y_{\{2,3\}}, \dots, y_{\{2k,2k+1\}}, y_{\{1,2k+1\}}]$. Now we study the toric ring

$$K\big[\mathscr{Q}_{\overline{C_{2k+1}}}\big] \cong \frac{K[Y]}{I_{\mathscr{Q}_{\overline{C_{2k+1}}}}}.$$

PROPOSITION 1.3. *We have the following:*

(1) $K\big[\mathscr{Q}_{\overline{C_{2k+1}}}\big]$ *is quadratic Cohen-Macaulay for all* $k \geq 3$*;*

(2) $K\big[\mathscr{Q}_{\overline{C_{2k+1}}}\big]$ *is not Gorenstein for all* $k \geq 4$*;*

(3) $K\big[\mathscr{Q}_{\overline{C_7}}\big]$ *is Gorenstein;*

(4) $I_{\mathscr{Q}_{\overline{C_{2k+1}}}}$ *possesses no quadratic Gröbner bases for all* $k \geq 3$.

PROOF. (1) Note that $\alpha(\overline{C_{2k+1}}) = 2$ and C_{2k+1} satisfies the odd cycle condition (see [12, p. 167]). Hence, by applying $G = \overline{C_{2k+1}}$ to [28, Theorem 2.1], we have that $\mathscr{Q}_{\overline{C_{2k+1}}}$ is a normal polytope. Thus $K\big[\mathscr{Q}_{\overline{C_{2k+1}}}\big]$ is normal Cohen-Macaulay from [36] and [23].

Next, we will determine generators of the toric ideal $I_{\mathscr{Q}_{\overline{C_{2k+1}}}}$. By applying $G = \overline{C_{2k+1}}$ to [28, Theorem 3.2], we have that $I_{\mathscr{Q}_{\overline{C_{2k+1}}}} = I_{\mathscr{P}_{C_{2k+1}}} + J$, where $I_{\mathscr{P}_{C_{2k+1}}}$ is the toric ideal of the edge ring of C_{2k+1} and J is generated by the following $4k+2$ quadratic binomials:

$$\begin{aligned}
&y_{\{i\}}y_{\{i+1\}} - y_\emptyset y_{\{i,i+1\}} \quad (1 \leq i \leq 2k),\\
&y_{\{1\}}y_{\{2k+1\}} - y_\emptyset y_{\{1,2k+1\}},\\
&y_{\{i\}}y_{\{i+1,i+2\}} - y_{\{i+2\}}y_{\{i,i+1\}} \quad (1 \leq i \leq 2k-1),\\
&y_{\{2k\}}y_{\{1,2k+1\}} - y_{\{1\}}y_{\{2k,2k+1\}}, \quad y_{\{2k+1\}}y_{\{1,2\}} - y_{\{2\}}y_{\{1,2k+1\}}.
\end{aligned}$$

Since C_{2k+1} is an odd cycle, $I_{\mathscr{P}_{C_{2k+1}}} = (0)$ from [28, Proposition 3.1]. Hence $K\big[\mathscr{Q}_{\overline{C_{2k+1}}}\big]$ is quadratic. Therefore $K\big[\mathscr{Q}_{\overline{C_{2k+1}}}\big]$ is quadratic Cohen-Macaulay.

(2) For an integral convex polytope $\mathscr{P} \subset \mathbb{R}^n$, we define

$$\mathrm{cone}(\mathscr{P}) := \{(\alpha, t) \mid \alpha \in t\mathscr{P} \cap \mathbb{Z}^n, t \in \mathbb{Z}_{\geq 0}\} \subset \mathbb{R}^{n+1}$$

as the cone of $\mathscr{P}$. By (1), we can regard $\mathrm{cone}(\mathscr{Q}_{\overline{C_{2k+1}}})$ as a positive toroidal monoid and $K[\mathscr{Q}_{\overline{C_{2k+1}}}]$ is the semigroup ring defined by $\mathrm{cone}(\mathscr{Q}_{\overline{C_{2k+1}}})$. Hence, from [35, Theorem 6.7] (see also [24, Corollary 5.11]), it is enough to show that $\mathrm{cone}(\mathscr{Q}_{\overline{C_{2k+1}}})$ has two minimal interior lattice points to prove that $K[\mathscr{Q}_{\overline{C_{2k+1}}}]$ is not Gorenstein.

Assume that $k \geq 4$. First, $(1, 1, \dots, 1, k+1) \in \mathbb{R}^{2k+2}$ is a minimal interior lattice point of $\mathrm{cone}(\mathscr{Q}_{\overline{C_{2k+1}}})$ for all $k \geq 4$. Moreover,

$$(\underset{1}{2}, 1, 1, \underset{4}{2}, 1, 1, \dots, \underset{2k-1}{2}, 1, 1, k+3) \in \mathbb{R}^{2k+2} \quad (k \equiv 1 \bmod 3),$$
$$(\underset{1}{2}, 1, 1, \underset{4}{2}, 1, 1, \dots, \underset{2k-3}{2}, 1, 1, \underset{2k}{2}, 1, k+3) \in \mathbb{R}^{2k+2} \quad (k \equiv 2 \bmod 3),$$
$$(\underset{1}{2}, 1, 1, \underset{4}{2}, 1, 1, \dots, \underset{2k-5}{2}, 1, 1, \underset{2k-2}{3}, 1, 1, 1, k+3) \in \mathbb{R}^{2k+2} \quad (k \equiv 0 \bmod 3)$$

are also minimal interior lattice points of $\mathrm{cone}(\mathscr{Q}_{\overline{C_{2k+1}}})$. Therefore we have that $K[\mathscr{Q}_{\overline{C_{2k+1}}}]$ is not Gorenstein for all $k \geq 4$.

(3) Assume $k = 3$. From the proof of (1), we have that the toric ideal $I_{\mathscr{Q}_{\overline{C_7}}}$ of the toric ring $K[\mathscr{Q}_{\overline{C_7}}]$ is generated by the following 14 binomials:

$$\begin{gathered}
y_{\{1\}}y_{\{2\}} - y_\emptyset y_{\{1,2\}}, \quad y_{\{2\}}y_{\{3\}} - y_\emptyset y_{\{2,3\}}, \quad y_{\{3\}}y_{\{4\}} - y_\emptyset y_{\{3,4\}}, \\
y_{\{4\}}y_{\{5\}} - y_\emptyset y_{\{4,5\}}, \quad y_{\{5\}}y_{\{6\}} - y_\emptyset y_{\{5,6\}}, \quad y_{\{6\}}y_{\{7\}} - y_\emptyset y_{\{6,7\}}, \\
y_{\{1\}}y_{\{7\}} - y_\emptyset y_{\{1,7\}}, \quad y_{\{1\}}y_{\{2,3\}} - y_{\{3\}}y_{\{1,2\}}, \quad y_{\{2\}}y_{\{3,4\}} - y_{\{4\}}y_{\{2,3\}}, \\
y_{\{3\}}y_{\{4,5\}} - y_{\{5\}}y_{\{3,4\}}, \quad y_{\{4\}}y_{\{5,6\}} - y_{\{6\}}y_{\{4,5\}}, \quad y_{\{5\}}y_{\{6,7\}} - y_{\{7\}}y_{\{5,6\}}, \\
y_{\{6\}}y_{\{1,7\}} - y_{\{1\}}y_{\{6,7\}}, \quad y_{\{7\}}y_{\{1,2\}} - y_{\{2\}}y_{\{1,7\}}.
\end{gathered}$$

Let $S := K[Y]$ and $K[\mathscr{Q}_{\overline{C_7}}] \cong S/I_{\mathscr{Q}_{\overline{C_7}}}$. By using Macaulay2 [13], we can see that

$$\begin{aligned}
0 &\to S(-11) \to S(-9)^{14} \to S(-7)^{36} \oplus S(-8)^{21} \to S(-6)^{126} \\
&\quad \to S(-5)^{126} \to S(-3)^{21} \oplus S(-4)^{36} \to S(-2)^{14} \to S \to S/I_{\mathscr{Q}_{\overline{C_7}}} \to 0
\end{aligned}$$

is a minimal free S-resolution of $S/I_{\mathscr{Q}_{\overline{C_7}}}$. Hence we have that $K[\mathscr{Q}_{\overline{C_7}}] \cong S/I_{\mathscr{Q}_{\overline{C_7}}}$ is Gorenstein.

(4) Assume that there exists a monomial order $<$ on $K[Y]$ such that the Gröbner bases of $I_{\mathscr{Q}_{\overline{C_{2k+1}}}}$ with respect to $<$ is quadratic. We may assume that $y_{\{1\}}y_{\{2,3\}} < y_{\{3\}}y_{\{1,2\}}$. Then $y_{\{3\}}y_{\{4,5\}} < y_{\{5\}}y_{\{3,4\}}$ since $y_{\{5\}}y_{\{1,2\}}y_{\{3,4\}} -$

$y_{\{1\}}y_{\{2,3\}}y_{\{4,5\}} \in I_{\mathscr{Q}_{\overline{C_{2k+1}}}}$ and its initial monomial is $y_{\{5\}}y_{\{1,2\}}y_{\{3,4\}}$. Since $y_{\{7\}}y_{\{3,4\}}y_{\{5,6\}} - y_{\{3\}}y_{\{4,5\}}y_{\{6,7\}} \in I_{\mathscr{Q}_{\overline{C_{2k+1}}}}$ and its initial monomial is $y_{\{7\}}y_{\{3,4\}}y_{\{5,6\}}$, we have $y_{\{5\}}y_{\{6,7\}} < y_{\{7\}}y_{\{5,6\}}$. By repeating this argument, we have

$$\begin{aligned}
y_{\{1\}}y_{\{2,3\}} &< y_{\{3\}}y_{\{1,2\}},\\
y_{\{3\}}y_{\{4,5\}} &< y_{\{5\}}y_{\{3,4\}},\\
&\vdots\\
y_{\{2k-1\}}y_{\{2k,2k+1\}} &< y_{\{2k+1\}}y_{\{2k-1,2k\}},\\
y_{\{2k+1\}}y_{\{1,2\}} &< y_{\{2\}}y_{\{1,2k+1\}},\\
y_{\{2\}}y_{\{3,4\}} &< y_{\{4\}}y_{\{2,3\}},\\
y_{\{4\}}y_{\{5,6\}} &< y_{\{6\}}y_{\{4,5\}},\\
&\vdots\\
y_{\{2k-2\}}y_{\{2k-1,2k\}} &< y_{\{2k\}}y_{\{2k-2,2k-1\}},\\
y_{\{2k\}}y_{\{1,2k+1\}} &< y_{\{1\}}y_{\{2k,2k+1\}}.
\end{aligned}$$

These inequalities lead to a contradiction. Hence we have the desired conclusion.

We can check that $K\left[\mathscr{Q}_{\overline{C_7}}\right]$ is not Koszul by using Macaulay2. For convenience, we introduce how to check that $K\left[\mathscr{Q}_{\overline{C_7}}\right]$ is not Koszul (see [37, p. 289]).

Let $S = K[Y]$, $I := I_{\mathscr{Q}_{\overline{C_7}}}$ and $R := K\left[\mathscr{Q}_{\overline{C_7}}\right] \cong S/I$. We compute the infinite resolution of K over R up to homological degree 3 by using command `LengthLimit`. We must input S and I in advance.

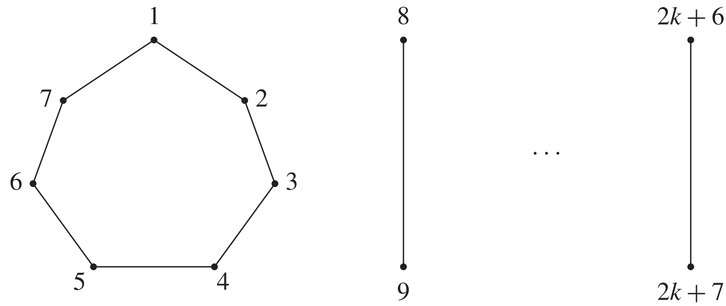
```
i3 : R = S/I

o3 = R

o3 : QuotientRing

i4 = betti res(coker vars R, LengthLimit => 3)
                0  1   2   3
o4 = total : 1 15 119 687
           0: 1 15 119 686
           1: .  .   .   1

o4 : BettiTally
```

Hence we have $\beta^R_{34}(K) = 1$. Thus R is not Koszul. Therefore we have

Corollary 1.4. *The toric ring $K\left[\mathscr{Q}_{\overline{C_7}}\right]$ is non-Koszul quadratic Gorenstein.*

We can construct an infinite family of non-Koszul quadratic Gorenstein toric rings by using stable set polytopes.

PROPOSITION 1.5. *Let $k \geq 1$ be an integer. Let G be a graph on $[2k+7]$ such that $\overline{G} = C_7 \cup K_2 \cup \cdots \cup K_2$ and the labeling of vertices is as follows:*



Then we have

(1) *$K[\mathcal{Q}_G]$ is quadratic Gorenstein such that*

$$H_{K[\mathcal{Q}_G]}(t) = \frac{(1+7t+14t^2+7t^3+t^4)(1+t)^k}{(1-t)^{2k+8}}.$$

(2) *$K[\mathcal{Q}_G]$ is not Koszul.*

PROOF. (1) By [28, Theorem 3.2], we have that the toric ideal $I_{\mathcal{Q}_G}$ is generated by the following binomials:

$$y_{\{i\}}y_{\{i+1\}} - y_{\emptyset}y_{\{i,i+1\}} \quad (1 \leq i \leq 6),$$
$$y_{\{1\}}y_{\{7\}} - y_{\emptyset}y_{\{1,7\}}, \quad y_{\{i\}}y_{\{i+1,i+2\}} - y_{\{i+2\}}y_{\{i,i+1\}} \quad (1 \leq i \leq 5),$$
$$y_{\{6\}}y_{\{1,7\}} - y_{\{1\}}y_{\{6,7\}}, \quad y_{\{7\}}y_{\{1,2\}} - y_{\{2\}}y_{\{1,7\}},$$
$$y_{\{2i\}}y_{\{2i+1\}} - y_{\emptyset}y_{\{2i,2i+1\}} \quad (4 \leq i \leq k+3).$$

Let $K[Y] = K[y_W \mid W \in S(G)]$. Then $K[\mathcal{Q}_G] \cong K[Y]/I_{\mathcal{Q}_G}$. Note that $\mathbf{y} = y_{\emptyset}, y_{\{1\}} - y_{\{2,3\}}, y_{\{2\}} - y_{\{3,4\}}, \ldots, y_{\{5\}} - y_{\{6,7\}}, y_{\{6\}} - y_{\{1,7\}}, y_{\{7\}} - y_{\{1,2\}}, y_{\{8\}} - y_{\{9\}}, \ldots, y_{\{2k+6\}} - y_{\{2k+7\}}, y_{\{8,9\}}, \ldots, y_{\{2k+6,2k+7\}}$ is a regular sequence of $K[Y]/I_{\mathcal{Q}_G}$. Hence we have

$$\frac{K[Y]}{I_{\mathcal{Q}_G} + (\mathbf{y})} \cong \frac{K[y_{\{1\}}, y_{\{2\}}, \ldots, y_{\{7\}}]}{I_7} \otimes_K \frac{K[y_{\{2i\}} \mid 4 \leq i \leq k+3]}{(y_{\{2i\}}^2 \mid 4 \leq i \leq k+3)}.$$

Thus the Hilbert series of $K[Y]/I_{\mathcal{Q}_G} + (\mathbf{y})$ is $(1+7t+14t^2+7t^3+t^4)(1+t)^k$. Therefore we have the desired conclusion.

(2) $K\left[\mathscr{Q}_{\overline{C_7}}\right]$ is a combinatorial pure subring (see [29]) of $K[\mathscr{Q}_G]$. Since $K\left[\mathscr{Q}_{\overline{C_7}}\right]$ is not Koszul, hence $K[\mathscr{Q}_G]$ is not Koszul by [29, Proposition 1.3].

PROPOSITION 1.6. *Let G be a graph. Let $h(K[\mathscr{Q}_G]) = (h_0, h_1, \ldots, h_s)$ be the h-vector of $K[\mathscr{Q}_G]$. If $K[\mathscr{Q}_G]$ is non-Koszul quadratic Gorenstein, then $h_1 \geq 7$.*

PROOF. First, note that $h_1 = \operatorname{codim} R = \operatorname{embdim} R - \dim R$ and $\operatorname{embdim} K[\mathscr{Q}_G] = \#S(G) = 1 + n + \#\{W \in S(G) \mid \#W \geq 2\}$. Hence $h_1 = \#\{W \in S(G) \mid \#W \geq 2\}$.

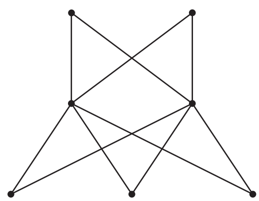
Assume that $h_1 \leq 6$. Let $\alpha(G) := \max\{\#W \mid W \in S(G)\}$. Then we have $\alpha(G) \leq 3$ since $\#\{W \in S(G) \mid \#W \geq 2\} > 6$ if there exists $W \in S(G)$ with $\#W \geq 4$. Moreover, $\alpha(G) \neq 1$ since if $\alpha(G) = 1$, then $K[\mathscr{Q}_G]$ is isomorphic to a polynomial ring, hence $K[\mathscr{Q}_G]$ is Koszul. Thus $\alpha(G) = 2, 3$.

Now let us consider the complement $\overline{G}$ of G. Then $\omega(\overline{G}) = \alpha(G)$ holds, where $\omega(\overline{G}) = \max\{\#C \mid C \text{ is a clique of } \overline{G}\}$. Thus $\omega(\overline{G}) = 2, 3$. In addition, we may assume that $\overline{G}$ has no isolated vertex. Indeed, if $\overline{G}$ has an isolated vertex v, then $K[\mathscr{Q}_G] \cong K[\mathscr{Q}_{G\setminus v}] \otimes_K K[y_v]$ and $I_{\mathscr{Q}_G} = I_{\mathscr{Q}_{G\setminus v}}$. From these fact and [9, Proposition 3.1], we have that $K[\mathscr{Q}_G]$ is non-Koszul quadratic Gorenstein if and only of $K[\mathscr{Q}_{G\setminus v}]$ is non-Koszul quadratic Gorenstein.

Firstly, we assume that $\omega(\overline{G}) = 3$. Then $\overline{G}$ has a triangle. If $\overline{G}$ has two distinct triangles, then $\#\{W \in S(G) \mid \#W \geq 2\} = \#\{C : \text{clique of } \overline{G},\ \#C \geq 2\} \geq 7$, a contradiction. Hence $\overline{G}$ has just one triangle. From the above arguments, if $K[\mathscr{Q}_G]$ is non-Koszul quadratic Gorenstein and $\omega(\overline{G}) = 3$, then we have the following:

- $\#V(\overline{G}) = \#V(G) \geq 7$ (by Proposition 1.2);
- $\#\{C : \text{clique of } \overline{G},\ \#C \geq 2\} = \#\{W \in S(G) \mid \#W \geq 2\} \leq 6$;
- $\overline{G}$ has just one triangle.

Therefore, we have that $\overline{G} = K_3 \cup K_2 \cup K_2$. However, G is the comparability graph of a partially ordered set such that its Hasse diagram is as follows:



hence $K[\mathscr{Q}_G]$ is Koszul by Remark 1.1(2), a contradiction.

Next, we assume that $\omega(\overline{G}) = 2$. From Remark 1.1(4), $\overline{G}$ is not bipartite. Hence $\overline{G}$ has a C_5 as an induced subgraph. From the above arguments, if

$K[\mathscr{Q}_G]$ is non-Koszul quadratic Gorenstein and $\omega(\overline{G}) = 2$, then we have the following:

- $\#V(\overline{G}) = \#V(G) \geq 7$ (by Proposition 1.2);
- $\#\{C : \text{clique of } \overline{G},\ \#C \geq 2\} = \#\{W \in S(G) \mid \#W \geq 2\} \leq 6$;
- $\overline{G}$ has a C_5 as an induced subgraph.

Therefore, we have that $\overline{G} = C_5 \cup K_2$. Then

$$K[\mathscr{Q}_G] \cong \frac{K[y_\emptyset, y_{\{1\}}, y_{\{2\}}, \ldots, y_{\{7\}}, y_{\{1,2\}}, y_{\{2,3\}}, y_{\{3,4\}}, y_{\{4,5\}}, y_{\{1,5\}}, y_{\{6,7\}}]}{I_{\mathscr{Q}_G}}.$$

Now we can see that the Gröbner bases of the toric ideal $I_{\mathscr{Q}_G}$ with respect to the reverse lexicographic order induced by the ordering

$$y_{\{1,5\}} > y_\emptyset > y_{\{1\}} > y_{\{2\}} > \cdots > y_{\{7\}} > y_{\{1,2\}} > y_{\{2,3\}} > y_{\{3,4\}} > y_{\{4,5\}} > y_{\{6,7\}}$$

is quadratic. Hence $K[\mathscr{Q}_G]$ is Koszul, but this is a contradiction.

Therefore, we have that $h_1 \geq 7$, the desired conclusion.

2. Questions

As the end of this paper, we present some questions.

First, we recall that the toric ring $K\left[\mathscr{Q}_{\overline{C_7}}\right]$ is non-koszul quadratic Gorenstein and its h-vector is $(1, 7, 14, 7, 1)$. Moreover, by Proposition 1.6, $h_1 \geq 7$ if $K[\mathscr{Q}_G]$ is non-Koszul quadratic Gorenstein. Hence the following question is interesting.

QUESTION 2.1. Does there exist a non-Koszul quadratic Gorenstein algebra R such that $h(R) = (1, n_1, n_2, n_1, 1)$ and $n_1 \leq 6$?

Note that, in this case $n_1 \geq 4$ since R is quadratic.

Let G be a graph on $[n]$ and with $E(G)$ its edge set. The *edge ring* of G, denoted by $K[G]$, is defined by

$$K[G] := K[x_i x_j \mid \{i, j\} \in E(G)] \subset K[x_1, \ldots, x_n].$$

The second question is

QUESTION 2.2. Does there exist a graph G such that the edge ring $K[G]$ is non-Koszul quadratic Gorenstein?

In [30, Theorem 1.2], a criterion for the edge ring $K[G]$ of G to be quadratic is given. Moreover, in [22], a class of graphs with the property that the toric ideal I_G of the edge ring $K[G]$ of G is quadratic but I_G possesses no quadratic

Gröbner bases is studied. A graph G is said to be $(*)$-*minimal* if G satisfies the above property and every induced subgraph $H \subsetneq G$ does not satisfy the property. By the computation by using Macaulay2, we have that if G is $(*)$-minimal and the edge ring $K[G]$ is non-Koszul quadratic Gorenstein, then $n \geq 9$.

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KITAMI INSTITUTE OF TECHNOLOGY
KITAMI
HOKKAIDO 090-8507
JAPAN
E-mail: kaz-matsuda@mail.kitami-it.ac.jp