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REMARKS ON COMPACT MAPPINGS

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In Sections 1 and 2 of this note simple proofs are given for two theorems on compact mappings. Theorem 1 states a sufficient condition for a mapping to be onto and may be considered as a generalization of the Fundamental Theorem of Algebra. Theorem 2 deals with the inverse images of points under compact mappings. Results related to Theorem 1 have been obtained previously, in particular by G. T. Whyburn [3, Theorem 5.2, p. 5] and [4, Theorem 1.1, p. 78] and recently by M. Reinbach [2, Lemma 1, p. 1399]. However, these results neither contain nor are contained in that of the present note. Reinbach's lemma deals with convergence spaces. In Section 3 this lemma and Theorem 1 below are linked together by means of the notion of k-space introduced by J. R. Kelley [1, pp. 230, 240]. The author is indebted to Mr. Anton Jensen for suggesting the use of that notion in this context and for valuable discussions.

Notation and definitions. In the following T_1 and T_2 denote topological spaces. All mappings are assumed to be continuous.

A mapping f is said to be open (closed) if it maps open (closed) sets onto open (closed) sets.

A mapping $f: T_1 \to T_2$ is said to be compact if $f^{-1}(C)$ is compact for every compact set $C \subseteq T_2$.

A mapping $f: T_1 \to T_2$ is said to be locally topological if every $x \in T_1$ possesses an open neighbourhood $O(x)$ such that the restriction of f to $O(x)$ is a homeomorphism.

1.

Lemma 1. *Let T_2 be a locally compact Hausdorff space. If $f: T_1 \to T_2$ is a compact mapping, then f is closed.*

Proof. Let $F \subseteq T_1$ be closed, and let $x \in \overline{f(F)}$. If $O(x)$ is an open neighbourhood of x with compact closure, $f^{-1}(\overline{O(x)})$ is closed and compact in T_1, and we have

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$$A = F \cap f^{-1}(f(F) \cap O(x)) \subseteqq f^{-1}(\overline{O(x)}) .$$

Hence $\bar{A} \subseteqq f^{-1}(\overline{O(x)})$, which shows that $\bar{A}$ is compact. Consequently $f(\bar{A})$ is compact and therefore closed. Since $f(F) \cap O(x) \subseteqq f(\bar{A})$ and $\bar{A} \subseteqq F$, we have

$$\overline{f(F) \cap O(x)} \subseteqq f(\bar{A}) \subseteqq f(F) .$$

Now, every neighbourhood of x intersects $O(x)$ in a neighbourhood which has points in common with $f(F)$. Hence

$$x \in \overline{f(F) \cap O(x)} \subseteqq f(F) ,$$

which proves the lemma.

THEOREM 1. *Let T_2 be a locally compact, connected Hausdorff space. If $f: T_1 \to T_2$ is a compact mapping such that $f(T_1)$ is open, then $f(T_1) = T_2$.*

PROOF. From Lemma 1 it follows that $f(T_1)$ is closed. Hence $f(T_1)$ is both open and closed in the connected space T_2.

2.

LEMMA 2. *Let T_1 be a Hausdorff space and T_2 a locally compact, connected Hausdorff space. If $f: T_1 \to T_2$ is a compact and locally topological mapping, then, for each $x \in T_2$, the set $f^{-1}(x)$ is finite, and the number of its points is constant on T_2.*

PROOF. $\{x\} \subseteqq T_2$ is compact, and thus $f^{-1}(x)$ is compact in T_1. For each $y \in f^{-1}(x)$ there exists an open neighbourhood $O(y)$ of y such that $f: O(y) \to f(O(y))$ is a homeomorphism. Since this open covering of $f^{-1}(x)$ by the neighbourhoods $O(y)$ has a finite subcovering, $f^{-1}(x)$ must be finite.

Let $n(x)$, $x \in T_2$, denote the number of points in $f^{-1}(x)$, and let $n_0 = \min n(x)$ be assumed at x_0. Then there exists an open set $U \subseteqq T_2$ such that every $y \in f^{-1}(x_0)$ has an open neighbourhood $O(y)$ which, by f, is mapped homeomorphically onto U. Let V denote the set of those $x \in T_2$ for which $f^{-1}(x)$ contains more than n_0 points. Putting

$$G = \bigcup_{y \in f^{-1}(x_0)} O(y) ,$$

we have $f(T_1 \backslash G) \supseteqq U \cap V$ since every point of V has at least one inverse point outside G. Now f is closed (Lemma 1) and, since $T_1 \backslash G$ is closed, it follows that

$$f(T_1 \backslash G) \supseteqq \overline{U \cap V} .$$

This implies $x_0 \notin \overline{U \cap V}$. We can therefore conclude that the set of points $x \in T_2$ for which $f^{-1}(x)$ consists of n_0 points is open.

Clearly, V is open. For, if $z \in V$, there exist an open set $U \subseteq T_2$ containing z, and p $(>n_0)$ open, disjoint sets in T_1 each of which is mapped homeomorphically onto U. (Here we use that T_1 is a Hausdorff space.) Since T_2 is connected, V must be empty.

THEOREM 2. *Let $f:T_1 \to T_2$ be a compact mapping of a Hausdorff space T_1 into a locally compact Hausdorff space T_2. Let N denote the set of points of T_1 at which f is not locally topological. If $T_2 \backslash f(N)$ is connected and $f^{-1}f(N)$ is a proper subset of T_1, then $f(T_1)=T_2$, the set $f^{-1}(x)$ is finite for every $x \in T_2 \backslash f(N)$, and the number of its points is constant on $T_2 \backslash f(N)$.*

PROOF. The mapping

$$f\colon\ (T_1 \backslash f^{-1}f(N)) \to (T_2 \backslash f(N))$$

satisfies the conditions of Lemma 2. Indeed, $f(N)$ is closed because N is closed in T_1 (Lemma 1). Further, since T_2 is a locally compact Hausdorff space, T_2 is regular and thus $T_2 \backslash f(N)$ locally compact. It follows that the mapping in question is compact.

3.

A collection $\mathscr{K}$ of subsets of a topological space T with the property that $K \cap A \in \mathscr{K}$ whenever $K \in \mathscr{K}$ and $A \in T$ is closed will be called a k-collection in T.

Clearly, if $f:T_1 \to T_2$ is a mapping, and $\mathscr{K}_1$ is a k-collection in T_1 then the sets $f(K_1)$, $K_1 \in \mathscr{K}_1$, form a k-collection in T_2.

A mapping $f:T_1 \to T_2$ is said to be a k-mapping with respect to the k-collections $\mathscr{K}_1$ in T_1 and $\mathscr{K}_2$ in T_2 if $f(K_1) \in \mathscr{K}_2$ for $K_1 \in \mathscr{K}_1$ and $f^{-1}(K_2) \in \mathscr{K}_1$ for $K_2 \in \mathscr{K}_2$.

A topological space T is called a k-space with respect to the k-collection $\mathscr{K}$ if a set $A \subseteq T$ is closed if and only if $A \cap K \in \mathscr{K}$ for all $K \in \mathscr{K}$.

THEOREM 3. *If $f:T_1 \to T_2$ is a k-mapping with respect to the k-collections $\mathscr{K}_1$ in T_1 and $\mathscr{K}_2$ in T_2, and T_2 is a k-space with respect to $\mathscr{K}_2$, then f is closed.*

PROOF. Let $A \subseteq T_1$ be closed, and assume $K \in \mathscr{K}_2$. Then

$$A \cap f^{-1}(K) \in \mathscr{K}_1, \qquad f(A \cap f^{-1}(K)) = f(A) \cap K \in \mathscr{K}_2 .$$

Hence $f(A)$ is closed since T_2 is a k-space with respect to $\mathscr{K}_2$.

Obviously, the collection of the compact subsets of a topological space T is a k-collection.

LEMMA 3. *If T is a locally compact Hausdorff space, then T is a k-space with respect to the collection of compact sets.*

PROOF. Assume that $A \subseteq T$ has the property that $A \cap C$ is compact for each compact set $C \subseteq T$. If $x \in \bar{A}$, and $C(x)$ is a compact neighbourhood of x, then $C(x) \cap A$ is compact and therefore closed, which implies that $x \in A$.

Clearly, Lemma 1 is a simple consequence of Theorem 3 and Lemma 3.

A space T in which convergence $x_n \to x$ of a sequence (x_n) to a point x is defined, such that $x_n \to x$, $x_n \to y$ imply $x = y$ and $x_n \to x$ implies that every subsequence converges to x, is called a convergence space.

The subsets of a convergence space T which are closed in the sense that they contain the limit points of all convergent sequences contained in them determine a topology in T.

A subset B of a convergence space T is said to be sequentially compact, if every sequence in B contains a subsequence which converges to a point in B.

The collection of sequentially compact subsets of a space T is a k-collection.

LEMMA 4. *Let T be a convergence space, then T is a k-space with respect to the collection of sequentially compact sets.*

PROOF. Assume that $A \subseteq T$ has the property that $A \cap C$ is sequentially compact for each sequentially compact set C. If a sequence (x_n) contained in A converges to x, then $\{x_n\} \cup \{x\}$ is sequentially compact, and thus $A \cap (\{x_n\} \cup \{x\})$ is also sequentially compact. Hence, a subsequence of (x_n) converges to a point in A which must coincide with x.

Now, it is easy to see, that Theorem 3 and Lemma 4 imply Lemma 1 in [2, p. 1399].

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