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ON POLYGONS OF ORDER n IN PROJECTIVE n-SPACE, WITH AN APPLICATION TO STRICTLY CONVEX CURVES

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In this paper we shall be concerned with open and closed polygons of order n in the real projective space R^n. Such polygons were first treated by D. Derry [2].

In Section 1 geometric and analytical conditions that an open polygon π be of order n are given, and it is shown that there exist hyperplanes which have no points in common with π. In the next section we consider open polygons in the affine space and introduce the concept of monotone sequences of points. In Section 3 the closed polygons are treated, and in Section 4 it is proved that if a hyperplane H has at least n points in common with a strictly convex curve, then these points are the vertices of a polygon of order $n-1$, i.e. form a monotone sequence of points in H.

1. Open polygons of order n.

1.1. In the real projective space R^n, $n \geqq 2$, let m points $P_1, P_2, \ldots, P_m$ be given. If consecutive points are joined by segments, we obtain an *open* polygon with the *vertices* $P_1, P_2, \ldots, P_m$ and the *sides* $P_1P_2, P_2P_3, \ldots, P_{m-1}P_m$. The polygon may be closed by addition of one of the segments joining P_m and P_1.

If a hyperplane H has points in common with the side P_iP_{i+1}, there are the following possibilities: H can intersect the side in a vertex or in an interior point of the side (a sidepoint), or H can contain the segment P_iP_{i+1}. In the last case we shall say that H has the points P_i and P_{i+1} in common with the polygon, disregarding the interior points of the side.

An open or closed polygon π is called a polygon of order n provided it satisfies

1° the *dimension* condition: the polygon π is not contained in a hyperplane, and

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2° the *order* condition: no hyperplane has more than n points in common with π.

A polygon of order n is denoted π_n. From 1° it follows, that the polygon has at least $n+1$ vertices.

1.2. An open polygon $P_1P_2\ldots P_m$ can be represented by coordinates in the following way. The vertex P_i, $i=1,2,\ldots,m$, is determined by a vector $x_i=(x_i^0,x_i^1,\ldots,x_i^n)$ and the side P_iP_{i+1} by the parametric equation

$$x = \lambda_i x_i + \mu_i x_{i+1}\,, \tag{1}$$

where the homogeneous parameters λ_i and μ_i, for all the points on the side, either have the same sign or opposite signs. We normalize the vectors x_i and the parameters λ_i and μ_i in the following way. The homogeneous vector x_1 is chosen arbitrarily. Next the vector x_2 is chosen so that the parameters λ_1 and μ_1 for the points on the side P_1P_2 have the *same* sign. Then x_3 is chosen so that the parameters λ_2 and μ_2 for the points on the side P_2P_3 have the *same* sign. In this manner we continue until all the vectors are normalized. Finally we normalize the parameters λ_i and μ_i choosing all of them positive and with the sum $\lambda_i+\mu_i=1$.

Assume $m \geqq n+1$. To each set of $n+1$ vertices $P_{i_1},P_{i_2},\ldots P_{i_{n+1}}$ there is attached a determinant formed by corresponding coordinate vectors:

$$D(i_1,i_2,\ldots,i_{n+1}) = |x_{i_1}\,x_{i_2}\,\ldots\,x_{i_{n+1}}|\,. \tag{2}$$

It vanishes if and only if the $n+1$ points are linearly dependent i.e. are contained in a hyperplane.

1.3. An open polygon π of order n can be characterized by two different conditions, a purely geometric one and a coordinate condition:

THEOREM 1.1. *An open polygon π with at least $n+1$ vertices has order n if and only if no hyperplane through n vertices of the polygon has other points in common with it.*

THEOREM 1.2. *An open polygon π with at least $n+1$ vertices has order n if and only if the determinants D have the same sign for all increasing sequences of indices $i_1<i_2<\ldots<i_{n+1}$.*

To prove the theorems we consider the three statements:

A. An open polygon with at least $n+1$ vertices has order n.

B. No hyperplane through n vertices of the polygon has other points in common with the polygon.

C. All the determinants $D(i_1,i_2,\ldots,i_{n+1})$ have the same sign.

If we can show that

$$A \Rightarrow B \Rightarrow C \Rightarrow A ,$$

Theorems 1.1 and 1.2 will be proved.

It is obvious that $A \Rightarrow B$. To show that $B \Rightarrow C$ we verify that

$$\operatorname{sign} D(1,2,\ldots,n,n+1) = \operatorname{sign} D(i_1,i_2,\ldots,i_n,i_{n+1}) \tag{3}$$

for any sequence of increasing indices $i_1 < i_2 < \ldots < i_{n+1}$.

Consider first the determinants

$$D(1,2,\ldots,n,n+1) \quad \text{and} \quad D(1,2,\ldots,n,i_{n+1}) , \tag{4}$$

where $i_{n+1} > n+1$. The hyperplane

$$|x_1\, x_2 \ldots x_n\, x| = 0 \tag{5}$$

has only the vertices $P_1,P_2,\ldots,P_n$ in common with the polygon π. Putting $x = \lambda x_{n+1} + \mu x_{n+2}$ we get

$$\lambda D(1,2,\ldots,n,n+1) + \mu D(1,2,\ldots,n,n+2) = 0 . \tag{6}$$

None of the two determinants in (6) is equal to zero, and they must have the same sign since the hyperplane (5) does not intersect the side $P_{n+1}P_{n+2}$. Analogously we find that the last determinant in (6) has the same sign as the determinant $D(1,2,\ldots,n,n+3)$, and continuing in this manner we see that the two determinants in (4) have the same sign.

Next we consider the determinants

$$D(1,2.\ldots,n-1,n,i_{n+1}) \quad \text{and} \quad D(1,2,\ldots,n-1,i_n,i_{n+1}) , \tag{7}$$

where $i_n > n$. The hyperplane

$$|x_1\, x_2 \ldots x_{n-1}\, x\, x_{i_{n+1}}| = 0 \tag{8}$$

has only the points $P_1,P_2,\ldots,P_{n-1},P_{i_{n+1}}$ in common with the polygon and, consequently, no points in common with the sub-polygon $P_nP_{n+1}\ldots P_{i_n}$. Exactly as above it is seen that the two determinants in (7) have the same sign, and continuing in this manner we finally find that the equation (3) is true.

Now it only remains to prove that $C \Rightarrow A$. Since the determinants D are different from zero, any $n+1$ vertices in π are linearly independent, i.e. the dimension condition 1° is satisfied.

To show that the order condition 2° holds, we choose on the polygon the ordered sequence of $n+1$ points $Q_1,Q_2,\ldots,Q_{n+1}$ determined by the $n+1$ vectors $y_1,y_2,\ldots,y_{n+1}$. The points Q_i are either points on different sides of π or vertices, which are not endpoints of those sides. Consequently, each y_k is either an x_i (corresponding to a vertex of π) or a linear combination, with positive coefficients, of two consecutive vectors x_j and x_{j+1} (corresponding to a sidepoint on π). The determinant

$$D' = |y_1 \, y_2 \, \dots \, y_{n+1}|$$

may be decomposed into a sum of determinants of type (2) with positive coefficients, and these determinants cannot all vanish because at least $n+1$ different vectors x_i must occur in the expression for D'. Hence D' is different from zero, and the $n+1$ points Q_i cannot be situated in a hyperplane. Thus the order condition is satisfied, and we have proved that $C \Rightarrow A$. This finishes the proof of Theorems 1.1 and 1.2.

1.4. We now turn to the last theorem on open polygons of order n:

THEOREM 1.3. *For every open polygon of order n there exist hyperplanes which have no point in common with it.*

Let the vertices of π_n be denoted $P_1, P_2, \dots, P_m$, $m \geqq n+1$. We consider the hyperplane H which contains the n vertices $P_1, P_2, \dots, P_n$ and consequently the sub-polygon $P_1 \dots P_n$. Since π_n has order n the hyperplane H has only the vertex P_n in common with the side $P_n P_{n+1}$ and no point in common with the sub-polygon $P_{n+1} \dots P_m$.

We now consider another hyperplane H' which passes through a point P'_1 on the complementary segment of the side P_1P_2, a point P'_2 on the complementary segment of the side P_2P_3, and so on, finally a point P'_n on the complementary segment of the side P_nP_{n+1}. The points P'_i are chosen in the "neighbourhood" of the points P_i such that the n points P'_i are linearly independent, and furthermore such that H' has no point in common with the sub-polygon $P_{n+1} \dots P_m$. However, the hyperplane H' has no point in common with the polygon $P_1 \dots P_{n+1}$ either. For, otherwise, H' would contain the whole polygon $P_1 \dots P_{n+1}$, which is impossible, since the $n+1$ vertices $P_1, P_2, \dots, P_{n+1}$ are linearly independent. Consequently, the hyperplane H' has the desired property.

2. Monotone sequences of points.

2.1. In the *affine* space of n dimensions we consider a polygon π with the vertices $P_1, P_2, \dots, P_m$, $m \geqq n+1$. If we add a "first coordinate" $x_i^0 = 1$ to the affine coordinates $(x_i^1, x_i^2, \dots, x_i^n)$ of P_i, we obtain coordinate vectors x_i which satisfy the conditions of normalization in Section 1.2 so that the sides P_iP_{i+1} are determined by equation (1), where λ_i and μ_i are positive with the sum 1.

With these coordinates of x_i in the expression (2), the sign of the determinant $D(i_1, i_2, \ldots, i_{n+1})$ indicates the *orientation* of the affine space determined by the $n+1$ points $P_{i_1}, P_{i_2}, \ldots, P_{i_{n+1}}$, taken in this order.

In his paper [4], J. Hjelmslev has called a sequence of points $P_1, P_2, \ldots, P_m$ in the *affine space* a *monotone sequence* if any $n+1$ of the points, taken in the natural order, determine the *same orientation* of the space, i.e. the determinants D have the same sign for any choice of increasing indices. As a consequence of Theorem 1.2 we get

THEOREM 2.1. *A sequence of points $P_1, P_2, \ldots, P_m$ in the affine R^n is monotone if and only if the polygon with the sides $P_1P_2, P_2P_3, \ldots, P_{m-1}P_m$ has order n.*

2.2. From Theorem 1.3 it follows that, by a suitable choice of a hyperplane at infinity any open polygon of order n, can be placed in an affine space, and in this space the vertices form a monotone sequence. Hence we introduce in the *projective* space the notion of monotone sequence of points in the following way:

A sequence of points $P_1, P_2, \ldots, P_m$, $m \geqq n+1$, in a projective R^n, is called monotone, if the points in this order are the vertices of some open polygon of order n.

As a consequence of Theorem 1.2 we find that a necessary and sufficient condition that a sequence of points $P_1, P_2, \ldots, P_m$ be monotone is that it is possible to normalize their coordinate vectors $x_1, x_2, \ldots, x_m$ such that the determinants $D(i_1, i_2, \ldots, i_{n+1})$ have the same sign for all sets of increasing indices. The sides $P_1P_2, P_2P_3, \ldots, P_{m-1}P_m$ must then be chosen in accordance with (1), the parameters λ_i and μ_i being positive with the sum 1.

If $m = n+1$, the only condition for monotonicity is that $D(1, 2, \ldots, n+1) \neq 0$, i.e., the $n+1$ points are linearly independent. No matter how we choose the sides of the polygon and how we arrange the points, the polygon will have order n.

3. Closed polygons of order n.

3.1. Corresponding to Theorem 1.1 for open polygons, we shall prove for closed polygons

THEOREM 3.1. *A closed polygon π with at least $n+2$ vertices ($n+2$ sides) has order n if and only if no hyperplane through n vertices of the polygon has other points in common with it.*

It is obvious that the above condition that a closed polygon be of order n is necessary. The condition is also sufficient which may be seen as follows.

The dimension condition 1° is clearly satisfied. To show that the order condition holds, we assume the existence of a hyperplane H having at least $n+1$ points in common with π. If the number of points of intersection is exactly $n+1$, there is at least one side in π which has at most one endpoint, but no other points in common with H. This side may be thought of as removed. If there are more than $n+1$ points of intersection we remove an arbitrary side. In both cases an open polygon remains having at least $n+1$ points in common with the hyperplane H, and for which the condition in Theorem 3.1 is satisfied. In virtue of Theorem 1.1 the open polygon has order n; this implies that the hyperplane H cannot exist. Thus the closed polygon has order n.

It may be noted that the theorem is not true if π has exactly $n+1$ vertices and sides, say the 3-side of odd order in the plane or the 4-side of even order in the usual space.

3.2. If an arbitrary side in a closed polygon of order n is removed we get an open polygon of order n. Conversely, an open polygon of order n with the vertices $P_1, P_2, \ldots, P_m$ may be closed by the addition of one of the segments $P_m P_1$ without increasing its order.

Let H denote a hyperplane having no points in common with the open polygon π_n. The hyperplane intersects one of the segments $P_m P_1$ and has thus one point in common with one of the closed polygons $P_1 P_2, \ldots, P_m, P_1$ and no point in common with the other. Now, the minimum and maximum numbers of points which a hyperplane has in common with a *closed* polygon have the same parity, and since the maximum number, the *order* of the closed polygon, must be either n or $n+1$, there are two possibilities:

If n is an *even* number and we close π_n by the segment which has *no* point in common with H we get a closed polygon of even order, i.e. of order n, while the order increases to $n+1$ if we use the complementary segment. If n is an *odd* number the polygon will be closed without increase of order by the addition of the segment which intersects H, while the order increases to $n+1$ if the other segment is used.

It is then obvious that for n even, and only in this case, a closed polygon of order n may be considered as situated in an affine space R^n.

3.3. In § 2 we have called a sequence of points $P_1, P_2, \ldots, P_m$ a monotone sequence if the points are the vertices of some open polygon of order n. If the polygon is closed by the addition of one of the segments

P_mP_1, without increase of order, and we remove another side $P_{i-1}P_i$ of the polygon, we obtain a new open polygon of order n. It then follows that if $P_1, P_2, \ldots, P_m$ is a monotone sequence of points in the projective R^n this property will be unchanged if we choose any of the points as the first point of the sequence, preserving the cyclic order of the points.

4. Strictly convex curves.

4.1. The notion of strict convexity has been introduced by M. Barner [1]. In the projective R^n a closed, n times differentiable curve is called strictly convex if, for any $n-1$ points of the curve, there exists at least one hyperplane having these, but no other points in common with the curve. The definition is easily extended to open curves.

In a recent paper [3] the author has stated some properties of *plane* strictly convex curves among which we recall the following. Let a line l have the points $P_1, P_2, \ldots, P_m$, $m \geqq 2$, in common with a curve c of bounded order, the points being taken in the order determined by the parametrization of the curve. By the *chord* belonging to the arc P_iP_{i+1} we shall mean that one of the two segments P_iP_{i+1} whose union with the arc P_iP_{i+1} is a closed curve of *even* order. It is shown that if c is strictly convex, the chords $P_1P_2, P_2P_3, \ldots, P_{m-1}P_m$ (and, if c is closed, in addition P_mP_1) make up an *open* or *closed polygon of order* 1, i.e. a segment P_1P_m or the whole line l, according as the curve c is open or closed. In the following this theorem will be generalized to curves in spaces of arbitrary dimensions.

4.2. Let a hyperplane H have the points $P_1, P_2, \ldots, P_m$, $m \geqq n$, in common with the strictly convex curve c, the points being taken in the order determined by the parametrization of the curve. The curve c may be open or closed, and we assume that c is of bounded order, i.e. any hyperplane has at most finitely many points in common with the curve. As for plane curves, the *chord* P_iP_{i+1} is defined as that segment P_iP_{i+1} which together with the arc P_iP_{i+1} forms a closed curve of *even* order. A hyperplane which intersects the chord has at least one point in common with the corresponding arc, and the number of intersections will be odd if the hyperplane does not contain any tangent to the curve.

The chords $P_1P_2, P_2P_3, \ldots, P_{m-1}P_m$, and, if c is closed, in addition P_mP_1, form a polygon π in the hyperplane H. It will be proved that π has order $n-1$.

We consider simultaneously the two cases where c (and π) is open and where c is closed and π has at least $n+1$ vertices, whereas we afterwards treat the special case where c is closed and π has exactly n vertices (and n sides). In the first cases we may apply the analogous Theorems 1.1 and 3.1 with $n-1$ instead of n.

We consider in H a linear subspace R^{n-2}, containing $n-1$ vertices of the polygon π. This subspace cannot contain a new vertex. Otherwise no hyperplane through the given $n-1$ vertices, having only these points in common with the curve, could exist, which contradicts the assumption that c is strictly convex. If R^{n-2} had an interior point S of a chord in common with π, every hyperplane through the $n-1$ vertices and S would have at least one more point in common with the curve (situated on the arc corresponding to the chord), contrary to the definition of strictly convex curves. From Theorems 1.1 and 3.1 it follows that the polygon π (in H) has order $n-1$ since no R^{n-2} in H through $n-1$ vertices has other points in common with the polygon.

There only remains the case where c is closed and π has n vertices. The order of π must be either $n-1$ or n, and from the definition of chords it follows that the order of π and the order of c have the same parity. But since c is strictly convex, the order of c has the same parity as the number $n-1$, and consequently π has the order $n-1$. Thus we have proved

THEOREM 4.1. *If a hyperplane H has the points $P_1, P_2, \ldots, P_m$, $m \geqq n$, in common with a strictly convex curve of bounded order in R^n, the chords $P_1P_2, P_2P_3, \ldots, P_{m-1}P_m$ (and if c is closed, in addition P_mP_1) form a polygon of order $n-1$.*

4.3. Using the concept of monotone sequence in R^n (cf. § 2) we find from Theorem 4.1 that the common points of a hyperplane H and a strictly convex curve c of bounded order always form a *monotone sequence* in H, and that the sides of the corresponding polygon of order $n-1$ are chords belonging to the curve.

We may then, as in the plane, say that the curve c is *linearly monotone* along the hyperplane H, and if we call a curve, as a whole, linearly monotone if it is linearly monotone along any hyperplane having at least n points in common with the curve, we can give Theorem 4.1 the same formulation as Theorem 4.1 in the paper [3] quoted:

THEOREM 4.2. *Any strictly convex curve of bounded order in R^n is linearly monotone.*

For plane curves the converse problem, whether a linearly monotone curve is strictly convex, has been solved [3]. For curves in spaces of 3 or more dimensions this problem remains unsolved.

ADDED IN PROOF: It has been drawn to the authors attention that Theorem 1.2 for polygons in an affine space has been proved by Schoenberg and Whitney [5, p. 142].

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