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EXISTENCE PROOFS FOR MIXED PROBLEMS FOR HYPERBOLIC DIFFERENTIAL EQUATIONS IN TWO INDEPENDENT VARIABLES BY MEANS OF THE CONTINUITY METHOD

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The aim of this paper is to give existence proofs for mixed boundary problems, on the one hand for a hyperbolic system of linear differential equations of the first order in two independent variables, and on the other hand for a linear hyperbolic differential equation of arbitrary order in two independent variables. Existence theorems for boundary problems of the types considered in this paper have been given previously by Campbell and Robinson [1] by means of Picard's iteration method. Our fundamental tool will be a priori inequalities for these types of boundary problems, which have been obtained in [13].

In the theory of hyperbolic differential equations a priori estimates, or estimates of the Friedrichs-Lewy [4] type, as they often are called here, have proved to be an efficient tool for existence proofs in boundary problems. Thus, for Cauchy's problem estimates of this type have been used in existence proofs by Schauder [11], Petrovskiĭ [10], Friedrichs [2], Leray [9], Lax [8], and Gårding [5], and for mixed problems for a hyperbolic second order equation Krzyżański and Schauder [6] used a priori estimates to prove the existence of solutions.

In the papers quoted above different methods have been added to supplement the a priori inequalities. In some of the papers, for instance, it turned out that if it is possible to solve the problem in the special case of analytic coefficients and boundary data, which in the case of Cauchy's problem is granted by the Cauchy-Kovalevsky theorem, it is possible to derive results also for the non-analytic case. The method in this paper is also to reduce the problem to a special case, and from this to derive the general result by means of the continuity method.

The continuity method consists in considering a family of operators A^λ $(0 \leqq \lambda \leqq 1)$ such that A^1 is the operator, for which we want to prove

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the existence of an inverse, and A^0 is known to have an inverse. The proof is then obtained by means of a continuous variation of the parameter λ. In the theory of partial differential equations this method for existence proofs has been frequently used for non-linear equations. For linear elliptic and parabolic differential equations it has for instance been used by Ladyženskaya [7] [14].

By using normed spaces, in which the norms in addition to the ordinary L^2-norms contain L^2-norms of the boundary values, it is possible to reduce the general boundary problem for the first order system to the Cauchy problem for first order equations with only one dependent variable each. Further it is possible to reduce the boundary problem for the n^{th} order equation to a special boundary problem for a special n^{th} order equation. Then this special equation is reduced to a simple first order system.

In the reduction of the n^{th} order equation to a first order system, we use some results from the differential calculus in L^2. In order to facilitate the reading, these results have been given in a separate part. One of these results is the identity between the classes of functions, which are n times differentiable in $L^2(V)$ in the strong and weak senses, respectively, where V has a sufficiently smooth boundary. Similar results have been given previously by Friedrichs [3] and Sobolev [12], but their results are valid only for functions vanishing at the boundary of V.

The plan of the paper is the following.

In Part I we treat the differential calculus in L^2. In Section I.1 the identity between weak and strong derivatives in a region with a sufficiently smooth boundary is proved. Since the proof is the same in R^m as in R^2, we have given the proof in R^m although we only need the result in the case $m=2$. In Section I.2 we discuss the simultaneous integrability of the equations

$$\left(\frac{\partial}{\partial t}\right)^{k-1}\left(\frac{\partial}{\partial x}\right)^{n-k} u = w_k, \qquad k = 1, \ldots, n\,.$$

In Part II we treat the mixed boundary problem for the hyperbolic first order system. In Section II.1 we introduce notations and formulate the boundary problem. In Section II.2 the problem is reduced to the form $\mathsf{L}u=\mathsf{F}$, where L is an operator from one Hilbert space to another. The main result is formulated in Theorem II.1, which states that the boundary problem for arbitrary boundary data has a strong solution. Here we also introduce a family of auxiliary operators $\mathsf{L}^\lambda (0 \leqq \lambda \leqq 1)$ with $\mathsf{L}^1=\mathsf{L}$. This family is investigated in Section II.3 and there we also obtain the solvability of the equation $\mathsf{L}^0 u=\mathsf{F}$. Throughout this section the estimates obtained in [13] are fundamental. In Section II.4 the proof of Theorem II.1 is completed. In Section II.5 we obtain estimates for higher order derivatives analogous to those in [13]. In Section II.6 these results are used to derive a result corresponding to Theorem II.1 for a special system with stronger regularity assumptions, which will be needed in Part III.

In Part III we treat the mixed boundary problem for a hyperbolic equation of arbitrary order. Section III.1 is analogous to Section II.1 and in Section III.2 the problem is given an abstract form and is written as an equation $\mathsf{M}u=\mathsf{G}$. The main result is formulated in Theorem III.1, which states that the boundary problem for arbitrary boundary data has a strong solution. The proof uses a family $\mathsf{M}^{\lambda}(0\leqq\lambda\leqq 1)$ of axiliary operators such that $\mathsf{M}^1=\mathsf{M}$ and is analogous to the proof of Theorem II.1. The existence of solutions of the equation $\mathsf{M}^0u=\mathsf{G}$, which is assumed in Section III.2, is proved in Section III.3. For the sake of completeness, we state in Section III.4 the estimates for the higher order derivatives, corresponding to those in Section II.5.

The notations in this paper coincide with the notations in the paper [13]. Therefore Sections II.1 and III.1 have been condensed, and the reader is referred to [13] for details.

I wish to express my gratitude to Professor Lars Hörmander for several inspiring discussions on the subject of this paper.

I. Some results from the differential calculus in $L^2(V)$.

I.1. Weak and strong derivatives. Let V be a bounded region in the m-dimensional euclidean space R^m. We denote by $L^2(V)$ the set of real-valued functions $u(x)=u(x_1, \ldots, x_m)$, which are square integrable in Lebesgue's sense in V and define

$$\|u\| = \left\{\int_V u^2\,dx\right\}^{\frac{1}{2}},$$

where dx is the element of volume in R^m.

Let $C^n(V)$ be the set of functions, which are n times continuously differentiable in V and correspondingly for the closure $\overline{V}$ of V. We denote by p the m-tuple $(p_1, \ldots, p_m)$ of non-negative integers and $|p|=\Sigma_{i=1}^m p_i$ and write

$$D^p u = \left(\frac{\partial}{\partial x_1}\right)^{p_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{p_m} u\,.$$

We shall recall two definitions of the derivatives of functions in $L^2(V)$.

We say that $u\in L^2(V)$ has a weak derivative $D^p u=u_p\in L^2(V)$ of order

$|p|$ if for every $\psi(x) \in C^{|p|}(V)$, which vanishes outside a compact subset of V, we have

$$\int_V u_p(x)\,\psi(x)\,dx = (-1)^{|p|} \int_V u(x)\,D^p\psi(x)\,dx\,. \tag{I.1.1}$$

For the weak derivatives we have the following

Lemma I.1. *Assume that u has a weak derivative $D^{p_1}u \in L^2(V)$ and that $D^{p_1}u$ has a weak derivative $D^{p_2}(D^{p_1}u) \in L^2(V)$. Then u has a weak derivative $D^{p_1+p_2}u \in L^2(V)$, and $D^{p_1+p_2}u = D^{p_2}(D^{p_1}u)$.*

Proof. Let $\psi(x) \in C^{|p_1+p_2|}(V)$ be an arbitrary function which vanishes outside a compact subset of V. Then by repeated use of the definition of weak derivatives we have

$$\begin{aligned}\int_V u(x)\,D^{p_1+p_2}\psi(x)\,dx &= \int_V u(x)\,D^{p_1}D^{p_2}\psi(x)\,dx \\ &= (-1)^{|p_1|}\int_V D^{p_1}u(x)\,D^{p_2}\psi(x)\,dx \\ &= (-1)^{|p_1+p_2|}\int_V D^{p_2}\big(D^{p_1}u(x)\big)\,\psi(x)\,dx\,,\end{aligned}$$

which proves Lemma I.1.

We say that $u \in L^2(V)$ has a strong derivative $D^p u = u_p \in L^2(V)$ of order $|p|$ if there exists a sequence of functions $u^\mu \in C^{|p|}(\overline{V})$ $(\mu = 1, 2, \ldots)$ such that

$$\|u^\mu - u\| \to 0 \quad \text{and} \quad \|D^p u^\mu - u_p\| \to 0 \quad \text{when} \quad \mu \to \infty\,. \tag{I.1.2}$$

We shall be concerned with two corresponding extensions of $C^n(\overline{V})$, namely the sets $L_w^{2,n}(V)$ and $L_s^{2,n}(V)$ of functions in $L^2(V)$ for which all derivatives $D^p u$, $|p| \leqq n$, exist in the weak and strong sense, respectively, and belong to $L^2(V)$. It is obvious that

$$L_s^{2,n}(V) \subseteq L_w^{2,n}(V)\,.$$

In fact, let u be an arbitrary function in $L_s^{2,n}(V)$ and let p be an arbitrary m-tuple with $|p| \leqq n$. Then $D^p u = u_p$ exists in the strong sense, and there is a sequence of functions $u^\mu \in C^{|p|}(\overline{V})$ $(\mu = 1, 2, \ldots)$ satisfying (I.1.2). After an integration by parts, we get for every $\psi(x) \in C^{|p|}(V)$, which vanishes outside a compact subset of V

$$\int_V D^p u^\mu(x)\,\psi(x)\,dx = (-1)^{|p|}\int_V u^\mu(x)\,D^p\psi(x)\,dx\,. \tag{I.1.3}$$

Making $\mu \to \infty$ in (I.1.3) we obtain (I.1.1) from (I.1.2).

Our purpose is to prove that, if the boundary S of V is sufficiently regular, the sets $L_w^{2,n}(V)$ and $L_s^{2,n}(V)$ coincide. We therefore make the following assumption concerning the boundary S of V (cf. fig. 1):

(R) *There exists a continuously turning direction $\vec{v}(P)$ defined on S, which points to the interior of V, and an angle θ, such that at an arbitrary point P of S, the double cone with vertex P, the axis along $\vec{v}(P)$ and top angle θ, in a neighbourhood of P has only P in common with S.*

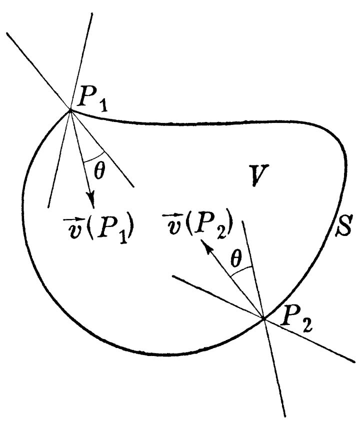

Fig. 1.

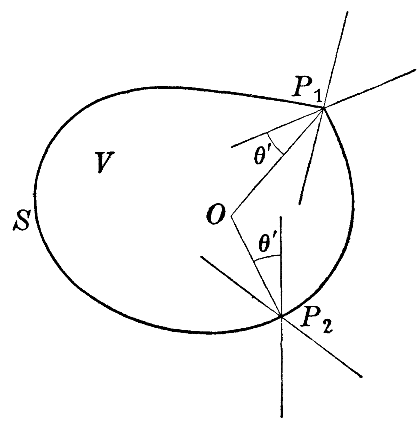

Fig. 2.

We can now state our result.

THEOREM I.1. *If V has the property (R), the sets $L_w^{2,n}(V)$ and $L_s^{2,n}(V)$ coincide, that is the set of functions which have weak derivatives in $L^2(V)$ of all orders $\leqq n$ equals the set of functions which have strong derivatives in $L^2(V)$ of all orders $\leqq n$. Further, for every $u \in L_w^{2,n}(V) = L_s^{2,n}(V)$ there exists a sequence $u^\mu \in C^n(\overline{V})$ ($\mu = 1, 2, \ldots$) such that for all $|p| \leqq n$*

$$D^p u^\mu \to D^p u \qquad \textit{when} \qquad \mu \to \infty .$$

In the proof we first assume that V has the following property (cf. fig. 2):

(R') *V is star-shaped with respect to a point O in V, and such that every point P of the boundary S is the vertex of a cone with OP as its axis and with a constant top angle θ'. This cone has in a neighbourhood of P only P in common with S.*

The region V is said to be star-shaped with respect to the point O in V if for Q in V all points on the segment OQ belong to V. In what follows, O is supposed to be the origin in R^m.

If V has the property (R'), there exists a constant ϱ such that if ε is small enough, then for every $x \in \bar{V}$ the sphere with centre $x(1-\varepsilon)$ and radius $\varrho\varepsilon$ is entirely in V.

Let $\varphi(x)$ be a positive, infinitely differentiable function with its support in the unit sphere and with

$$\int_{R^m} \varphi(x)\,dx = 1 .$$

For $u \in L^2(V)$ we then define a modified mollifier operator J_ε,

$$J_\varepsilon u(x) = \frac{1}{(\varrho\varepsilon)^m} \int_V u(y)\, \varphi\left(\frac{x(1-\varepsilon)-y}{\varrho\varepsilon}\right) dy , \qquad x \in \bar{V} .$$

Here the integrand is non-vanishing only for such points y that $|y-x(1-\varepsilon)| < \varrho\varepsilon$, that is, in a sphere lying entirely in V. $J_\varepsilon u(x)$ is in $C^n(\bar{V})$ for any n and the differentiation of $J_\varepsilon u(x)$ may be performed under the integral sign. Further, one can prove that

$$J_\varepsilon u \to u \quad \text{in} \quad L^2(V) \qquad \text{when} \quad \varepsilon \to 0 .$$

The proof of this fact is analogous to the proof in [3] and [12] of the corresponding result for the ordinary mollifier operator J_ε,

$$J_\varepsilon u(x) = \frac{1}{\varepsilon^m} \int_V u(y)\, \varphi\left(\frac{x-y}{\varepsilon}\right) dy , \qquad x \in \bar{V} . \tag{I.1.4}$$

We now suppose that $u \in L_w^{2,n}(V)$. The weak derivatives $D^p u = u_p$ then exist and belong to $L^2(V)$ for all $|p| \leqq n$. We shall prove that u_p is the strong derivative $D^p u$ for $|p| \leqq n$, too. More precisely, we shall prove that for all $|p| \leqq n$

$$D^p J_\varepsilon u \to u_p \quad \text{in} \quad L^2(V) \qquad \text{when} \quad \varepsilon \to 0 .$$

We obtain

$$\begin{aligned} D^p J_\varepsilon u(x) &= \frac{1}{(\varrho\varepsilon)^m} \int_V u(y) D_x{}^p\, \varphi\left(\frac{x(1-\varepsilon)-y}{\varrho\varepsilon}\right) dy \\ &= \frac{(1-\varepsilon)^{|p|}}{(\varrho\varepsilon)^m} \int_V u(y)\,(-1)^{|p|} D_y{}^p\, \varphi\left(\frac{x(1-\varepsilon)-y}{\varrho\varepsilon}\right) dy . \end{aligned}$$

According to the definition (I.1.1) of a weak derivative, and since

$$\varphi\left(\frac{x(1-\varepsilon)-y}{\varrho\varepsilon}\right)$$

vanishes in the neighbourhood of S if $x \in \bar{V}$, we obtain

$$D^p J_\varepsilon u(x) = \frac{(1-\varepsilon)^{|p|}}{(\varrho\varepsilon)^m} \int_V u_p(y)\, \varphi\left(\frac{x(1-\varepsilon)-y}{\varrho\varepsilon}\right) dy = (1-\varepsilon)^{|p|} J_\varepsilon u_p(x)\,.$$

But $J_\varepsilon u_p \to u_p$ in $L^2(V)$ when $\varepsilon \to 0$, and hence $D^p J_\varepsilon u = (1-\varepsilon)^{|p|} J_\varepsilon u_p \to u_p$ in $L^2(V)$ when $\varepsilon \to 0$. Theorem I.1 is therefore proved for this special shape of the region.

We now turn to the general case. We first prove that there is a covering of $\overline{V}$ by means of a finite number of open sets O_i $(i=1, \ldots, N)$ such that $O_i \cap \overline{V}$ has the property (R').

An arbitrary point P of S can be considered as the vertex of a double cone, with its axis along $\vec{v}(P)$ and with top angle $\frac{1}{2}\theta$. If this cone is submitted to a sufficiently small translation in the direction of $\vec{v}(P)$, the open component of V defined by the translated cone, having P on its boundary, gives after a suitable similarity transformation with respect to the translated vertex an open set whose intersection with $\overline{V}$ has the property (R') if O is a point on the axis sufficiently close to the translated vertex (cf. fig. 3). If we complete this covering of S by means of a covering of V consisting of spherical open neighbourhoods of the points of V, we get an open covering of $\overline{V}$. Now $\overline{V}$ is a compact set in R^m and we can thus find a finite subcovering $\bigcup_{i=1}^N O_i$ such that $O_i \cap \overline{V}$ $(i=1, \ldots, N)$ have the property (R').

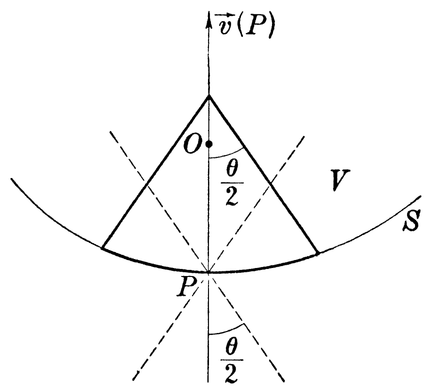

Fig. 3.

Now suppose that $u \in L_w^{2,n}(V)$. Then $u \in L_w^{2,n}(O_i \cap V)$ for $i=1, \ldots, N$ and there is a sequence $u_i^\mu \in C^n(\overline{O_i \cap V})$ such that

$$D^p u_i^\mu \to D^p u \quad \text{in} \quad L^2(O_i \cap V) \qquad \text{when} \quad \mu \to \infty$$

and $|p| \leqq n$. Let $h_i(x)$ $(i=1, \ldots, N)$ be functions in $C^n(R^m)$ vanishing outside O_i and such that $\Sigma_{i=1}^N h_i(x) = 1$ for all x in $\overline{V}$. We construct the sequence

$$u^\mu(x) = \sum_{i=1}^N h_i(x) u_i^\mu(x), \qquad \mu = 1, 2, \ldots\,.$$

It is easily seen that $u^\mu \to u$ in $L^2(V)$, and observing that $\Sigma_{i=1}^N D^p h_i(x) = 0$ $(0 < |p| \leqq n)$, we also get $D^p u^\mu \to D^p u$ in $L^2(V)$. This proves Theorem I.1 in the general case.

In Part III we shall restrict ourselves to regions V having the property

(R). According to Theorem I.1 there is then no reason to distinguish between weak and strong derivatives and we shall simply write generalized derivatives. The set $L_w^{2,n}(V)=L_s^{2,n}(V)$ will be denoted by $L^{2,n}(V)$.

We end this section by stating the following well-known

Lemma I.2. *A function $u \in L^{2,n}(V)$ with $D^p u=0$ for $|p|=n$ is a polynomial of degree $\leqq n-1$.*

Remark. Sobolev states in [12], p. 494, that for $u \in L_w^{2,n}(V)$ the ordinary mollifier J_ε, defined in (I.1.4), has the property $D^p J_\varepsilon u \to D^p u$ in $L^2(V)$ for all $|p| \leqq n$ and $\varepsilon \to 0$. This is, however, not true in general for a bounded region V if u does not vanish in the neighbourhood of S, which is easily seen by examples.

I.2. The simultaneous integrability of the equations $D^p u=w_p$, $|p|=n-1$. In this section we assume that V is a simply connected region in the xt-plane having the property (R) in Section I.1. We have the following

Theorem I.2. *Assume that $w_1, \ldots, w_n$ are n functions in $L^{2,1}(V)$ such that*

$$\frac{\partial}{\partial t} w_{i-1} = \frac{\partial}{\partial x} w_i \qquad \text{for} \quad i = 2, \ldots, n .$$

Then there exists a function $u \in L^{2,n}(V)$ such that

$$\left(\frac{\partial}{\partial t}\right)^{i-1} \left(\frac{\partial}{\partial x}\right)^{n-i} u = w_i, \qquad i = 1, \ldots, n .$$

The function u is uniquely determined up to an arbitrary additive polynomial of degree $\leqq n-2$.

The theorem is an easy consequence of Lemma I.1, Lemma I.2, and the following

Lemma I.3. *Assume that w_x and w_t are two functions in $L^{2,1}(V)$ such that*

$$\frac{\partial}{\partial t} w_x = \frac{\partial}{\partial x} w_t .$$

Then there exists a function $u \in L^{2,2}(V)$ such that

$$\frac{\partial u}{\partial x} = w_x \qquad \text{and} \qquad \frac{\partial u}{\partial t} = w_t .$$

The function u is uniquely determined up to an arbitrary additive constant.

In the case of three dimensions the corresponding result follows from Theorem II in Weyl's famous paper [15]. Also in the case of two dimensions the result follows from the paper quoted after obvious modifications. Since V satisfies the regularity assumption (R), we see that u, itself, and not only its derivatives, belongs to $L^2(V)$.

II. The hyperbolic first order system.

II.1. Notations and preliminaries. Let $C_m^i(\bar{V})$ $(i=0,1,\ldots;m=1,2,\ldots)$ be the set of real-valued vectors $u=(u_1(x,t),\ldots,u_m(x,t))$, which are i times continuously differentiable in the closure $\bar{V}$ of a simply connected region V in the xt-plane. For $m=1$, we shall usually omit the lower index and write $C^i(\bar{V})$.

For $u\in C_n^1(\bar{V})$ the equations

$$L_iu = (D_t-\alpha_iD_x)u_i+\sum_{k=1}^{n} a_{ik}u_k, \qquad i = 1,\ldots,n$$

$(D_t=\partial/\partial t,\ D_x=\partial/\partial x)$, where $\alpha_i\in C^1(\bar{V})$ and $a_{ik}\in C^0(\bar{V})$ define a linear hyperbolic operator

$$Lu = (L_1u,\ldots,L_nu)$$

from $C_n^1(\bar{V})$ to $C_n^0(\bar{V})$. For the sake of simplicity, we assume that $\alpha_1\leqq\alpha_2\leqq\ldots\leqq\alpha_n$.

The boundary S of V is supposed to be piece-wise smooth, and S_i $(i=0,\ldots,N)$ stands for the set of points of S at which i of the numbers $\nu_\tau-\alpha_k\nu_\xi$ $(k=1,\ldots,n)$ are <0. Here $\nu=(\nu_\xi,\nu_\tau)$ is the exterior normal of S. The parts of S_i on which ν_ξ is negative and positive are denoted by S_i^- and S_i^+, respectively (cf. fig. 4). The endpoints of S_i^-, S_i^+ and S_i are included in these sets so that they are all closed sets. We write

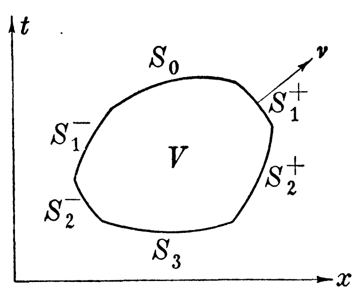

Fig. 4 $(n=3)$.

$$S^- = S_1^-\cup S_2^-\cup\ldots\cup S_{n-1}^-$$

and

$$S^+ = S_1^+\cup S_2^+\cup\ldots\cup S_{n-1}^+$$

and make the following assumptions about the boundary S:

(a) S^- has a positive distance to S^+;
(b) $\inf_S|\nu_\tau-\alpha_i\nu_\xi|>0,\quad i=1,\ldots,n$;
(c) the interior angle between the two tangents to S at a point belonging to two sets S_i is $<\pi$.

Let $C_m{}^i(S_k)$ $(i=0, \ldots; m=1, \ldots; k=0, \ldots, n)$ be the set of real-valued vectors $u=(u_1(s), \ldots, u_m(s))$, which are i times continuously differentiable with respect to the arc length s on S_k. For $m=1$ we shall usually omit the lower index and write $C^i(S_k)$. For $u \in C_n{}^0(\bar{V})$ the equations

$$l_{ik}(u) = l_{ik}(u_1, \ldots, u_n) = \sum_{j=1}^{n} l_{ik}{}^j u_j, \qquad k = 1, \ldots, i,$$

where $l_{ik}{}^j \in C^1(S_i)$ define a boundary operator

$$l_i u = (l_{i1}(u), \ldots, l_{ii}(u))$$

from $C_n{}^0(\bar{V})$ to $C_i{}^0(S_i)$. About the linear forms $l_{ik}(u)$ we assume that

$$\text{(II.1.1)} \qquad \begin{vmatrix} l_{i1}{}^1 & l_{i1}{}^2 & \ldots\ldots & l_{i1}{}^i \\ \ldots & \ldots & \ldots & \ldots \\ l_{ii}{}^1 & \ldots & \ldots & l_{ii}{}^i \end{vmatrix} \neq 0 \quad \text{on} \quad S_i^-$$

and

$$\text{(II.1.2)} \qquad \begin{vmatrix} l_{i1}{}^{n-i+1} & l_{i1}{}^{n-i+2} & \ldots\ldots & l_{i1}{}^n \\ \ldots & \ldots & \ldots & \ldots \\ l_{ii}{}^{n-i+1} & \ldots & \ldots & l_{ii}{}^n \end{vmatrix} \neq 0 \quad \text{on} \quad S_i^+ .$$

In fact, in the sequel we only need $l_{ik}{}^j \in C^1(S_i)$ for the coefficients occurring in (II.1.1) and (II.1.2). For the others we only need $l_{ik}{}^j \in C^0(S_i)$. For $u \in C_n{}^1(\bar{V})$,

$$\mathsf{L}u = (Lu, l_1 u, \ldots, l_n u)$$

defines a linear operator from $C_n{}^1(\bar{V})$ to the direct sum $C^0(\bar{V}, S) = (C_n{}^0(\bar{V}), C_1{}^0(S_1), \ldots, C_n{}^0(S_n))$.

The boundary problem to be treated in this part is to solve the equation $\mathsf{L}u = \mathsf{F}$.

II.2. Abstract formulation of the boundary problem. For vectors $u \in C_n{}^0(\bar{V})$ we define a norm by

$$\|u\|_1 = \left\{ \int_V \sum_{i=1}^{n} u_i^2 dV + \int_S \sum_{i=1}^{n} u_i^2 ds \right\}^{\frac{1}{2}}.$$

Here dV is the euclidean measure in V and ds is the element of arc on S. Let $\mathscr{H}_1$ be the Hilbert space obtained by completing $C_n{}^0(\bar{V})$ with respect to this norm.

For the elements $\mathsf{F} = (F, f_1, \ldots, f_n) \in C^0(\bar{V}, S)$ $(F = (F_1, \ldots, F_n),\ f_i = (f_{i1}, \ldots, f_{ii}))$ we define a norm by

$$\|\mathsf{F}\|_2 = \left\{ \int_V \sum_{i=1}^{n} F_i^2 dV + \sum_{i=1}^{n} \int_{S_i} \sum_{k=1}^{i} f_{ik}^2 ds \right\}^{\frac{1}{2}},$$

and let $\mathscr{H}_2$ be the Hilbert space obtained by completing $C^0(\overline{V}, S)$ with respect to this norm.

We now consider L as an operator from $\mathscr{H}_1$ to $\mathscr{H}_2$ with domain $D(\mathsf{L}) = C_n^1(\overline{V})$ and with range $R(\mathsf{L}) \subseteq C^0(\overline{V}, S)$. An operator from $\mathscr{H}_1$ to $\mathscr{H}_2$ is called *pre-closed*, if it has closed extensions. It is well known that a linear operator P from $\mathscr{H}_1$ to $\mathscr{H}_2$ is pre-closed if and only if

$$u^\mu \to 0 \quad \text{in} \quad \mathscr{H}_1, \qquad \mu = 1, 2, \ldots,$$

and

$$\mathsf{P}u^\mu \to \mathsf{F} \quad \text{in} \quad \mathscr{H}_2 \qquad \text{when} \quad \mu \to \infty$$

implies $\mathsf{F} = 0$.

Lemma II.1. *The operator* L *is pre-closed.*

Proof. *Let* $u^\mu = (u_1^\mu, \ldots, u_n^\mu)$ be a sequence of vectors in $C_n^1(\overline{V})$ such that

(II.2.1)
$$\|u^\mu\|_1 \to 0 \qquad \text{when} \quad \mu \to \infty$$

and

(II.2.2)
$$\|\mathsf{L}u^\mu - \mathsf{F}\|_2 \to 0 \qquad \text{when} \quad \mu \to \infty\,.$$

We shall prove that $\mathsf{F} = 0$. In virtue of (II.2.1), we have

$$\int_S \sum_{i=1}^n (u_i^\mu)^2 ds \to 0 \qquad \text{when} \quad \mu \to \infty\,,$$

and hence

$$\sum_{i=1}^n \int_{S_i} \sum_{k=1}^i \left(l_{ik}(u^\mu) - f_{ik}\right)^2 ds \to \sum_{i=1}^n \int_{S_i} \sum_{k=1}^i f_{ik}{}^2 ds \qquad \text{when} \quad \mu \to \infty\,.$$

Compared with (II.2.2) this implies $f_{ik} = 0$ for $k = 1, \ldots, i$; $i = 1, \ldots, n$. It remains to prove that $F_i = 0$ for $i = 1, \ldots, n$. Let φ be an arbitrary function in $C^1(\overline{V})$ which vanishes outside a compact subset of V and consider

$$\int_V L_i u^\mu\, \varphi\, dV = \int_V \left\{(D_t - D_x\alpha_i)u_i^\mu + u_i^\mu D_x\alpha_i + \sum_{k=1}^n a_{ik}u_k^\mu\right\} \varphi\, dV\,.$$

After an integration by parts we get

$$\int_V L_i u^\mu\, \varphi\, dV = \int_V \left\{-u_i^\mu(D_t\varphi - \alpha_i D_x\varphi) + (u_i^\mu D_x\alpha_i + \sum_{k=1}^n a_{ik}u_k^\mu)\varphi\right\} dV \to 0$$

when $\mu \to \infty$, because (II.2.1) implies

$$\int_V \sum_{i=1}^n (u_i^\mu)^2\, dV \to 0 \qquad \text{when} \quad \mu \to \infty\,.$$

Thus

$$\int_V F_i \varphi \, dV = 0 ,$$

and φ being arbitrary we conclude that $F_i = 0$.

Let $\bar{\mathsf{L}}$ be the closure of L. The main result in Part II is the following

THEOREM II.1. *The equation* $\bar{\mathsf{L}}u = \mathsf{F}$ *has a unique solution* $u = \bar{\mathsf{L}}^{-1}\mathsf{F} \in \mathscr{H}_1$ *for every* $\mathsf{F} \in \mathscr{H}_2$ *and*

$$\|\bar{\mathsf{L}}^{-1}\mathsf{F}\|_1 \leqq C \|\mathsf{F}\|_2 ,$$

where C *is a constant independent of* F.

Here, and in what follows, C means a positive constant independent of the particular elements in the Hilbert spaces considered, but it does not always mean the same constant even during the course of a proof. When necessary, we distinguish between different constants by using subscripts.

In the proof, which will be postponed to Section II.4, we shall use a family of operators $\mathsf{L}^\lambda (0 \leqq \lambda \leqq 1)$ defined in $D(\mathsf{L}^\lambda) = C_n^1(\bar{V})$ by

$$\mathsf{L}^\lambda u = (L^\lambda u, l_1^\lambda u, \ldots, l_n^\lambda u) ,$$

where $L^\lambda u = (L_1^\lambda u, \ldots, L_n^\lambda u)$ with

$$L_i^\lambda u = (D_t - \alpha_i D_x) u_i + \lambda \sum_{k=1}^{n} a_{ik} u_k$$

and $l_i^\lambda u = \big(l_{i1}^\lambda(u), \ldots, l_{ii}^\lambda(u)\big)$ with

$$l_{ik}^\lambda(u) = \sum_{j=1}^{i} l_{ik}{}^j u_j + \lambda \sum_{j=i+1}^{n} l_{ik}{}^j u_j \qquad \text{on} \quad S_i^-$$

and

$$l_{ik}^\lambda(u) = \sum_{j=n-i+1}^{n} l_{ik}{}^j u_j + \lambda \sum_{j=1}^{n-i} l_{ik}{}^j u_j \qquad \text{on} \quad S_i^+ .$$

The operator L^λ thus defined is an operator of the same kind as L and coincides for $\lambda = 1$ with L. For $\lambda = 0$, however, L^0 only contains the principal part of L and the boundary forms only contain those coefficients which occur in conditions (II.1.1) and (II.1.2). From the definition of L^λ we get the formula

$$\text{(II.2.3)} \qquad \mathsf{L}^\lambda = \mathsf{L}^{\lambda_0} + (\lambda - \lambda_0)(\mathsf{L}^1 - \mathsf{L}^0)$$

for all λ and λ_0.

To solve the equation $\mathsf{L}^0 u = \mathsf{F}$ is equivalent to solving the system

$$\text{(II.2.4)} \qquad (D_t - \alpha_i D_x) u_i = F_i$$

with the boundary conditions

(II.2.5) $$\sum_{j=1}^{i} l_{ik}{}^j u_j = f_{ik}, \quad k = 1, \ldots, i, \qquad \text{on} \quad S_i^-$$

and

(II.2.6) $$\sum_{j=n-i+1}^{n} l_{ik}{}^j u_j = f_{ik}, \quad k = 1, \ldots, i, \qquad \text{on} \quad S_i^+ .$$

Because of the conditions (II.1.1) and (II.1.2) and the definition of S_i, this is equivalent to solving each of the equations (II.2.4) when u_i is given on the part of S where $\nu_\tau - \alpha_i \nu_\xi < 0$. This is, however, Cauchy's problem for each of the equations (II.2.4). From the elementary theory of Cauchy's problem, it is therefore possible to obtain the solvability of the equation $\overline{\mathsf{L}^0} u = \mathsf{F}$.

The idea in the proof is to derive the results for the equation $\overline{\mathsf{L}} u = \mathsf{F}$ from the results for $\overline{\mathsf{L}^0} u = \mathsf{F}$ by means of a continuous variation of the parameter λ. For the proof we deduce in Section II.3 some lemmas.

II.3. Some lemmas.

Lemma II.2. *The operator $\mathsf{L}^1 - \mathsf{L}^0$ is bounded, that is, there is a constant C_1 independent of u such that*

$$\|(\mathsf{L}^1 - \mathsf{L}^0) u\|_2 \leqq C_1 \|u\|_1, \qquad u \in C_n^1(\overline{V}) .$$

The proof is trivial.

Lemma II.3. *There is a constant C_2 independent of u and λ such that*

$$\|u\|_1 \leqq C_2 \|\mathsf{L}^\lambda u\|_2, \qquad u \in C_n^1(\overline{V}) .$$

Proof. According to Theorem I in [13] there is for every λ, $0 \leqq \lambda \leqq 1$, a constant $C(\lambda)$ such that $\|u\|_1 \leqq C(\lambda) \|\mathsf{L}^\lambda u\|_2$ for all $u \in C_n^1(\overline{V})$. Put

$$\inf_{u \in C_n^1(\overline{V})} \frac{\|\mathsf{L}^\lambda u\|_2}{\|u\|_1} = g(\lambda) .$$

Then $g(\lambda) > 0$ for $0 \leqq \lambda \leqq 1$, and we shall prove that $g(\lambda)$ has a positive lower bound. This will follow if we can prove that $g(\lambda)$ is a continuous function of λ in the interval $0 \leqq \lambda \leqq 1$. Let λ_0 be an arbitrary point with $0 \leqq \lambda_0 \leqq 1$ and $\varepsilon > 0$ an arbitrary number. Choose $\delta = \varepsilon / C_1$ where C_1 is the constant in Lemma II.2. We then have for $|\lambda - \lambda_0| < \delta$ (cf. (II.2.3))

$$\|\mathsf{L}^{\lambda_0} u\|_2 - \varepsilon \|u\|_1 \leqq \|\mathsf{L}^\lambda u\|_2 \leqq \|\mathsf{L}^{\lambda_0} u\|_2 + \varepsilon \|u\|_1 ,$$

that is,

$$\frac{\|\mathsf{L}^{\lambda_0} u\|_2}{\|u\|_1} - \varepsilon \leqq \frac{\|\mathsf{L}^\lambda u\|_2}{\|u\|_1} \leqq \frac{\|\mathsf{L}^{\lambda_0} u\|_2}{\|u\|_1} + \varepsilon$$

and hence $g(\lambda_0) - \varepsilon \leqq g(\lambda) \leqq g(\lambda_0) + \varepsilon$ which proves the assertion.

LEMMA II.4. *$R(\overline{\mathsf{L}^\lambda})$ equals the closure $\overline{R(\mathsf{L}^\lambda)}$ of $R(\mathsf{L}^\lambda)$.*

PROOF. It follows from the definition of $\overline{\mathsf{L}^\lambda}$ that $R(\overline{\mathsf{L}^\lambda}) \subseteq \overline{R(\mathsf{L}^\lambda)}$. It remains to prove the opposite inclusion. Suppose that $\mathsf{F} \in \overline{R(\mathsf{L}^\lambda)}$. Then there exists a sequence u^μ $(\mu = 1, 2, \ldots)$ of elements in $C_n{}^1(\overline{V})$ such that

$$\|\mathsf{L}^\lambda u^\mu - \mathsf{F}\|_2 \to 0$$

when $\mu \to \infty$. According to Lemma II.3 we have

$$\|u^\mu - u^\nu\|_1 \leqq C_2 \|\mathsf{L}^\lambda u^\mu - \mathsf{L}^\lambda u^\nu\|_2 \to 0$$

when μ and $\nu \to \infty$. Thus u^μ converges to an element $u \in \mathscr{H}_1$, and $\overline{\mathsf{L}^\lambda} u = \mathsf{F}$.

LEMMA II.5. *Assume that $R(\overline{\mathsf{L}^{\lambda_0}}) = \mathscr{H}_2$. Then*

$$\|\overline{\mathsf{L}^{\lambda_0}}^{-1} \mathsf{F}\|_1 \leqq C_2 \|\mathsf{F}\|_2, \qquad \mathsf{F} \in \mathscr{H}_2,$$

and

$$\|\overline{\mathsf{L}^{\lambda_0}}^{-1} (\overline{\mathsf{L}^1 - \mathsf{L}^0}) u\|_1 \leqq C_3 \|u\|_1, \qquad u \in \mathscr{H}_1,$$

where $C_3 = C_1 C_2$ and C_1 and C_2 are the constants in Lemma II.2 *and Lemma* II.3.

PROOF. If F belongs to $R(\mathsf{L}^{\lambda_0})$, the first statement follows at once from Lemma II.3. For a general F it then follows from the continuity of the norms. The second statement then follows from Lemma II.2.

LEMMA II.6. *$R(\overline{\mathsf{L}^0}) = \mathscr{H}_2$, that is, the equation $\overline{\mathsf{L}^0} u = \mathsf{F}$ has a unique solution $u \in \mathscr{H}_1$ for every $\mathsf{F} \in \mathscr{H}_2$.*

PROOF. According to Lemma II.4, we need only prove that $\mathsf{L}^0 u = \mathsf{F}$ has a solution for every $\mathsf{F} = (F, f_1, \ldots, f_n)$ in a set which is dense in $\mathscr{H}_2$. As we already observed in Section II.2, solving the equation $\mathsf{L}^0 u = \mathsf{F}$ is equivalent to solving Cauchy's problem for each of the equations (II.2.4). The initial curves for these equations, which according to condition (c) are connected, consist of a finite number of smooth arcs. Now if $F \in C_n{}^1(\overline{V})$ and $f_i \in C_i{}^1(S_i)$ $(i = 1, \ldots, n)$ and certain compatibility relations are fulfilled in the endpoints of these arcs, then we can find $u \in C_n{}^1(\overline{V})$ such that $\mathsf{L}^0 u = \mathsf{F}$. We shall discuss these compatibility relations in some detail.

According to conditions (II.1.1) and (II.1.2), the boundary conditions (II.2.5) and (II.2.6) are, as already observed, equivalent to

$$u_k = f_{ik}', \quad k = 1, \ldots, i, \qquad \text{on} \quad S_i^-$$

and

$$u_k = f_{ik}', \quad k = n-i+1, \ldots, n, \qquad \text{on} \quad S_i^+$$

where f_{ik}' are certain linear combinations of $f_{ij} (j=1, \ldots, i)$. In order to ensure the continuity of u_k in $\overline{V}$, the functions f_{ik} must be such that in the points belonging to two of the sets S_i, u_k gets the same value. This gives in every such point one linear relation between the f_{ij} for each of the u_k which are given on both sets S_i. If the sets are S_i and S_k and $i<k$, we thus have i linear relations between the $f_{ij} (j=1, \ldots, i)$ and $f_{kj} (j=1, \ldots, k)$. In order to ensure the continuity of the derivatives of such a u_k, we must have one more condition in the same point, for (II.2.5) (or (II.2.6)) gives the derivatives of u_k in two directions and (II.2.4) gives the derivative of u_k in a third direction. In order to make these conditions consistent, we have therefore to impose one linear relation between the f_{ij}, their derivatives with respect to the arc length on S and F_k in the point in question. Obviously, these regularity and compatibility assumptions on F are also sufficient to ensure the existence of a solution in $C_n^1(\overline{V})$ of the equation $\mathsf{L}^0 u = \mathsf{F}$. The set of these F is, however, dense in $\mathscr{H}_2$, since point-wise conditions are non-essential when completing in the $\mathscr{H}_2$-norm. This proves Lemma II.6.

II.4. Proof of Theorem II.1. In this section we shall prove that the equation $\overline{\mathsf{L}}u = \overline{\mathsf{L}^1}u = \mathsf{F}$ has a solution $u \in \mathscr{H}_1$ for every $\mathsf{F} \in \mathscr{H}_2$. Combined with Lemma II.5 this proves Theorem II.1.

For the proof we consider the equation

$$\overline{\mathsf{L}^\lambda} u = \mathsf{F} . \tag{II.4.1}$$

From (II.2.3) and Lemma II.2 it follows that

$$\overline{\mathsf{L}^\lambda} = \overline{\mathsf{L}^{\lambda_0}} + (\lambda - \lambda_0)(\overline{\mathsf{L}^1 - \mathsf{L}^0}) .$$

Equation (II.4.1) can therefore be written

$$\overline{\mathsf{L}^{\lambda_0}} u + (\lambda - \lambda_0)(\overline{\mathsf{L}^1 - \mathsf{L}^0}) u = \mathsf{F} . \tag{II.4.2}$$

Now assume that we already know that $R(\overline{\mathsf{L}^{\lambda_0}}) = \mathscr{H}_2$. A solution of

$$u + (\lambda - \lambda_0) \overline{\mathsf{L}^{\lambda_0}}^{-1} (\overline{\mathsf{L}^1 - \mathsf{L}^0}) u = \overline{\mathsf{L}^{\lambda_0}}^{-1} \mathsf{F} \tag{II.4.3}$$

is then also a solution of (II.4.2) and therefore a solution of (II.4.1) as well. Let $|\lambda - \lambda_0| \leqq \varrho < 1/C_3$ where C_3 is the constant in Lemma II.6. With

$$B = (\lambda - \lambda_0) \overline{\mathsf{L}^{\lambda_0}}^{-1} (\overline{\mathsf{L}^1 - \mathsf{L}^0}) \qquad \text{and} \qquad g = \overline{\mathsf{L}^{\lambda_0}}^{-1} \mathsf{F} ,$$

(II.4.3) may be written

$$u + Bu = g . \tag{II.4.4}$$

Here

$$\|B\|_1 = \sup_{\mathscr{H}_1} \frac{\|Bu\|_1}{\|u\|_1} \leqq |\lambda - \lambda_0| C_3 < 1 .$$

The Neumann series

$$u = \sum_{k=0}^{\infty} (-B)^k g$$

is then a solution to equation (II.4.4).

We have thus proved that if $R(\overline{\mathsf{L}^{\lambda_0}}) = \mathscr{H}_2$ and $|\lambda - \lambda_0| \leqq \varrho$, then $R(\overline{\mathsf{L}^{\lambda}}) = \mathscr{H}_2$. Combined with the fact, proved in Lemma II.6, that $R(\overline{\mathsf{L}^0}) = \mathscr{H}_2$, this gives in a finite number of steps that $R(\overline{\mathsf{L}^1}) = \mathscr{H}_2$.

II.5. Estimates for the higher order derivatives. In this section we shall extend the fundamental estimate, which is the essential tool in the existence proof above, to the higher order derivatives. Let q be an arbitrary positive integer. For $u \in C_n^q(\overline{V})$ we define

$$\|u\|_R^{(q)} = \left\{ \int_R \sum_{i=1}^{n} \sum_{|p| \leqq q} (D^p u_i)^2 dR \right\}^{\frac{1}{2}}, \qquad R = V, S ,$$

$\left(p = (p_t, p_x);\ |p| = p_t + p_x;\ D^p = D_t^{p_t} D_x^{p_x}\right)$, and for

$$\mathsf{F} \in C^q(\overline{V}, S) = \left(C_n^q(\overline{V}), C_1^q(S_1), \ldots, C_n^q(S_n)\right)$$

we define

$$\text{(II.5.1)} \qquad \|\mathsf{F}\|_{V,S}^{(q)}$$

$$= \left\{ \int_V \sum_{i=1}^{n} \sum_{|p| \leqq q} (D^p F_i)^2 dV + \int_S \sum_{i=1}^{n} \sum_{|p| \leqq q-1} (D^p F_i)^2 ds + \sum_{i=1}^{n} \int_{S_i} \sum_{j=1}^{i} \sum_{l \leqq q} (D_s^l f_{ij})^2 ds \right\}^{\frac{1}{2}} .$$

Here D_s stands for differentiation with respect to the arc length on S. In (II.5.1) the term

$$\int_S \sum_{i=1}^{n} \sum_{|p| \leqq q-1} (D^p F_i)^2 ds$$

is immaterial since it can be estimated by means of the term

$$\int_V \sum_{i=1}^{n} \sum_{|p| \leqq q} (D^p F_i)^2 dV .$$

This will, however, not be used in the sequel. Now let k be a fixed positve integer and assume that the boundary pieces S_i $(i = 1, \ldots, n)$ are k times continuously differentiable, that is, $\nu \in C_2^{k-1}(S_i)$, and that L satisfies the following differentiability assumptions, namely

$$\alpha_i \in C^{k+1}(\overline{V}), \qquad a_{ij} \in C^k(\overline{V}), \qquad l_{il}^j \in C^k(S_i) .$$

If $q \leqq k$, L can be considered as an operator from $C_n^{q+1}(\overline{V})$ to $C^q(\overline{V}, S)$ and we have the following

THEOREM II.2. *There is a constant C such that*

$$\|u\|_R^{(q)} \leqq C\,\|\mathsf{L}u\|_{V,S}^{(q)}, \qquad R = V,S\,, \tag{II.5.2}$$

for all $u \in C_n^{q+1}(\overline{V})$ and $q \leqq k$.

PROOF. We accomplish the proof by induction, noting that (II.5.2) for $q=0$ reduces to the inequalities (1.6) and (1.7) in [13]. Now suppose that the inequalities (II.5.2) are already proved for the integer $q<k$. We shall then prove that they are valid also for $q+1$.

We consider $D_x u_j$. On S_i there exist functions A^j and $B^j \in C^{k-1}(S_i)$ such that

$$D_x u_j = A^j D_s u_j + B^j (D_t - \alpha_j D_x) u_j \qquad \text{on} \quad S_i \tag{II.5.3}$$

with $A^j \neq 0$; for $D_s u_j$ and $(1+\alpha_j^2)^{-\frac{1}{2}}(D_t - \alpha_j D_x) u_j$ are in virtue of condition (*b*) derivatives of u_j in two different directions.

We consider an auxiliary operator L' of the same kind as L but with slightly modified coefficients, namely,

$$\begin{aligned} a_{il}' &= a_{il} \quad \text{for} \quad i \neq l\,, \\ a_{ii}' &= a_{ii} - D_x \alpha_i\,, \\ l'^{\,j}_{il} &= l_{il}^j / A^j\,. \end{aligned} \tag{II.5.4}$$

The operator L' then satisfies all assumptions made on the operator L in Section II.1 and has the same regularity properties as L. In particular (II.1.1) and (II.1.2) follow easily from the corresponding conditions for L. Let $u \in C_n^{q+2}(\overline{V})$. Then

$$D_x u = (D_x u_1, \ldots, D_x u_n) \in C_n^{q+1}(\overline{V})$$

and we get from the induction assumption

$$\|D_x u\|_R^{(q)} \leqq C\,\|\mathsf{L}' D_x u\|_{V,S}^{(q)}\,. \tag{II.5.5}$$

After a simple computation, we get from (II.5.3) and (II.5.4)

$$\begin{aligned} L_i' D_x u &= D_x L_i u - \sum_{k=1}^{n} (D_x a_{ik}) u_k\,, \\ l_{ik}'(D_x u) &= D_s l_{ik}(u) + \sum_{j=1}^{n} f_{ik}^j L_j u + \sum_{j=1}^{n} g_{ik}^j u_j\,, \end{aligned}$$

where f_{ik}^j and $g_{ik}^j \in C^{k-1}(S_i)$. This gives

$$\|\mathsf{L}' D_x u\|_{V,S}^{(q)} \leqq C\,\|\mathsf{L}u\|_{V,S}^{(q+1)}$$

and with (II.5.5)

(II.5.6) $$\|D_x u\|_R^{(q)} \leqq C \|\mathsf{L}u\|_{V,S}^{(q+1)} .$$

From

$$D_t u_i = \alpha_i D_x u_i - \sum_{k=1}^{n} a_{ik} u_k + L_i u$$

and (II.5.6) it follows that

$$\|D_t u\|_R^{(q)} \leqq C \|\mathsf{L}u\|_{V,S}^{(q+1)}$$

and using the assumptions (II.5.2) and (II.5.5) it follows that

$$\|u\|_R^{(q+1)} \leqq C \|\mathsf{L}u\|_{V,S}^{(q+1)} .$$

II.6. An auxiliary result. In this section we assume that the coefficients in the operator L satisfy somewhat stronger regularity conditions than i Sections II.1–II.4, namely,

$$\alpha_i \in C^2(\bar{V}), \quad a_{ik} \in C^0(\bar{V}), \quad l_{ik}^{\,j} \in C^2(S_i) .$$

Further we assume that the boundary forms $l_{ik}(u)$ like $l_{ik}^{\,0}(u)$ in Lemma II.6 only contain the coefficients occurring in conditions (II.1.1) and (II.1.2). For F in the equation $\mathsf{L}u = \mathsf{F}$ it is then possible, just as in the proof of Lemma II.6, to give compatibility relations, necessary in order that u_i be continuous in the points of the boundary which belong to two of the sets S_i. If a point belongs to S_i and S_k and $i < k$, we thus have i linear relations between the f_{ij} $(j = 1, \ldots, i)$ and f_{kj} $(j = 1, \ldots, k)$ in that point. Let $C^1(\bar{V}, S)^+$ be the set of $\mathsf{F} \in C^1(\bar{V}, S)$ satisfying these compatibility relations and define for $\mathsf{F} \in C^1(\bar{V}, S)^+$

(II.6.1) $$\|\mathsf{F}\|_2^{+} = \|\mathsf{F}\|_{V,S}^{(1)}$$

(cf. Section II.5). Let $\mathscr{H}_2^{+}$ be the Hilbert space obtained by completing $C^1(\bar{V}, S)^+$ with respect to the norm (II.6.1). The Hilbert space $\mathscr{H}_2^{+}$ depends essentially on the compatibility relations, and by completing $C^1(\bar{V}, S)$, we would have obtained a larger Hilbert space than $\mathscr{H}_2^{+}$. Let $\mathscr{H}_1^{+}$ be the Hilbert space obtained by completing $C_n^{\,1}(\bar{V})$ with respect to the norm

$$\|u\|_1^{+} = \{(\|u\|_V^{(1)})^2 + (\|u\|_S^{(0)})^2\}^{\frac{1}{2}} .$$

Because $\|u\|_V^{(1)} \leqq \|u\|_1^{+}$ it follows that the components of an element $u = (u_1, \ldots, u_n) \in \mathscr{H}_1^{+}$ have strong derivatives in $L^2(V)$. (The norms $\|u\|_1^{+}$ and $\|u\|_V^{(1)}$ are, in fact, equivalent). Because

$$\|\mathsf{F}\|_2 \leqq \|\mathsf{F}\|_2^{+} \quad \text{for} \quad \mathsf{F} \in C^1(\bar{V}, S)^+ ,$$

$\mathscr{H}_2^+$ can be considered as a subset of $\mathscr{H}_2$. It follows from the proof of Lemma II.6 that $C^1(\overline{V}, S)^+$ is dense in $\mathscr{H}_2$. Therefore, so is $\mathscr{H}_2^+$.

Let L^+ be the restriction of L to $C_n^2(\overline{V})$. Then L^+ can be considered as an operator from $\mathscr{H}_1^+$ to $\mathscr{H}_2^+$, and is then pre-closed. For

$$\|\mathsf{L}^+ u^\mu - \mathsf{F}\|_2^+ \to 0 \qquad \text{when} \quad \mu \to \infty$$

implies

$$\|\mathsf{L} u^\mu - \mathsf{F}\|_2 \to 0 \qquad \text{when} \quad \mu \to \infty$$

and the statement follows from Lemma II.1. If the closure of L^+ is denoted by $\overline{\mathsf{L}^+}$ we obtain

THEOREM II.3. *The equation*

$$\overline{\mathsf{L}^+} u = \mathsf{F}$$

has a unique solution $u \in \mathscr{H}_1^+$ *for every* $\mathsf{F} \in \mathscr{H}_2^+$.

The proof is analogous to the proof of Theorem II.1, and we shall only point out a few details.

Firstly, if we introduce the family $(\mathsf{L}^+)^\lambda$ defined in the same way as L^λ above, then a change in λ does not imply any change in the boundary forms. Therefore

$$(\mathsf{L}^+)^\lambda u \in C^1(\overline{V}, S)^+ \qquad \text{for} \qquad u \in C_n^2(\overline{V})$$

independently of λ. Further, $(\mathsf{L}^+)^1 - (\mathsf{L}^+)^0$ only contains the lower order terms of the differential operator and is easily seen to be a bounded operator from $\mathscr{H}_1^+$ to $\mathscr{H}_2^+$.

Secondly, the a priori estimate needed is contained in Theorem II.2 for $k=1$.

Observe, that in order to solve the equation $(\mathsf{L}^+)^0 u = \mathsf{F}$ we have to give stronger compatibility relations than in the proof of Lemma II.6, for we want the solution to be in $C_n^2(\overline{V})$. It does not, however, present any difficulties to find a set of such F which is dense in $\mathscr{H}_2^+$.

III. The hyperbolic equation of order n.

III.1. Notations and preliminaries. Suppose that

$$Mu = \sum_{|p| \leq n} a_p D^p u, \qquad u \in C^n(\overline{V}),$$

with $a_p \in C^2(\overline{V})$ for $|p| = n$ and $a_p \in C^0(\overline{V})$ for $|p| < n$ defines a linear hyperbolic differential operator from $C^n(\overline{V})$ to $C^0(\overline{V})$. The characteristic form associated with M is then, if $a_{(n,0)} = 1$

$$\sum_{|p|=n} a_p \tau^{p_t} \xi^{p_x} = \prod_{i=1}^{n} (\tau - \alpha_i \xi)$$

with $\alpha_i \in C^2(\bar{V})$. We assume that $\alpha_1 < \alpha_2 < \ldots < \alpha_n$.

This characteristic form gives rise to the same division of the boundary S of V as in Part II. We assume again that the boundary is piece-wise smooth and satisfies conditions (a), (b) and (c) (see p. 13). Further, we assume that

(d) *S_0 and S_n do not have a common tangent in a possible common point.*

Together with conditions (a) and (b), condition (d) implies that V satisfies condition (R) in Part I. It will therefore be possible to apply the results of Part I for the region V.

On S_i $(i=1, \ldots, n-1)$ we define a boundary operator $m_i u = \big(m_{i1}(u), \ldots, m_{ii}(u)\big)$ from $C^{n-1}(\bar{V})$ to $C_i^0(S_i)$, the components of which are linear forms in $D^p u$ $(|p| \leqq n-1)$ with coefficients in $C^2(S_i)$ for $|p| = n-1$ and in $C^0(S_i)$ for $|p| < n-1$, namely

$$m_{ik}(u) = \sum_{|p| \leqq n-1} m_{ik,p} D^p u \; .$$

On S_n we put

$$m_{nk}(u) = D_t^{k-1} D_x^{n-k} u, \quad k = 1, \ldots, n, \qquad u \in C^{n-1}(\bar{V}) \; ,$$

and $m_n u = \big(m_{n1}(u), \ldots, m_{nn}(u)\big)$. Thus $m_n u$ is an operator from $C^{n-1}(\bar{V})$ to $C_n^0(S_n)$. On S_n we do not only give the derivatives of u of order $n-1$ but also the totality of derivatives of lower order and we put

$$\tilde{m}_n u = (D^p u; \; |p| < n-1) \; .$$

Of course $\tilde{m}_n u$ is uniquely determined on the connected set S_n (cf. condition (c)) by $m_n u$ up to integration constants by means of integration with respect to the arc length. We consider $\tilde{m}_n u$ as an operator from $C^{n-1}(\bar{V})$ to the set $\tilde{C}(S_n)$ of restrictions to S_n of arrays

$$\big(\tilde{g}_p = D^p \tilde{g}; \; |p| < n-1, \; \tilde{g} \in C^{n-1}(\bar{V})\big) \; .$$

We introduce n differential operators

$$M_i u = \sum_{k=1}^{n} b_{ik} D_t^{k-1} D_x^{n-k} u$$

with the characteristic forms

$$\sum_{k=1}^{n} b_{ik} \tau^{k-1} \xi^{n-k} = \prod_{j \neq i} (\tau - \alpha_j \xi) \; .$$

These n polynomials constitute a basis for all homogeneous polynomials in ξ and τ of degree $n-1$. The inverse matrix (c_{ik}) of (b_{ik}) is given by

$$c_{ik} = \alpha_k^{i-1} \prod_{j \neq k} (\alpha_k - \alpha_j)^{-1}$$

(cf. [13, p. 105]) and we have

$$D_t^{i-1} D_x^{n-i} u = \sum_{k=1}^{n} c_{ik} M_k u .$$

The principal part of the operator $m_{ik}(u)$ can then be expressed by means of $M_k u$ $(k=1, \ldots, n)$ and we obtain

$$\sum_{|p|=n-1} m_{ik,p} D^p u = \sum_{j=1}^{n} m_{ik}{}^j M_j u .$$

About the linear operators $m_{ik}(u)$ we now assume that

(III.1.1)
$$\begin{vmatrix} m_{i1}{}^1 & \ldots & m_{i1}{}^i \\ \ldots & \ldots & \ldots \\ m_{ii}{}^1 & \ldots & m_{ii}{}^i \end{vmatrix} \neq 0 \quad \text{on} \quad S_i^-$$

and

(III.1.2)
$$\begin{vmatrix} m_{i1}{}^{n-i+1} & \ldots & m_{i1}{}^n \\ \ldots & \ldots & \ldots \\ m_{ii}{}^{n-i+1} & \ldots & m_{ii}{}^n \end{vmatrix} \neq 0 \quad \text{on} \quad S_i^+ .$$

For $i=n$ this follows from the definition of $m_{nk}(u)$. An alternative formulation of these conditions can be found in [13], pp. 106ff.

The different operators considered can be condensed into

$$\mathsf{M}u = (Mu, m_1 u, \ldots, m_n u, \tilde{m}_n u) ,$$

which defines a linear operator from $C^n(\bar{V})$ to the set $\tilde{C}^0(\bar{V}, S)$ of elements

$$\mathsf{G} = (G, g_1, \ldots, g_n, \tilde{g}_n)$$

where $G \in C^0(\bar{V})$, $g_i = (g_{i1}, \ldots, g_{ii}) \in C_i^0(S_i)$ $(i=1, \ldots, n)$ and

$$\tilde{g}_n = (\tilde{g}_p = D^p \tilde{g};\ |p| < n-1) \in \tilde{C}(S_n)$$

and where

$$D_t^{k-1} D_x^{n-k} \tilde{g} = g_{nk}, \quad k = 1, \ldots, n, \quad \text{on} \quad S_n .$$

The boundary problem to be treated in this part is to solve the equation $\mathsf{M}u = \mathsf{G}$.

Observe that in this part we have made stronger regularity assumptions for the coefficients of M than for the corresponding coefficients of the operator L in Theorem II.1. We have done so in order to be able to use Theorem II.3 in the proof of Lemma III.6. The assumption $m_{ik}{}^j \in C^2(S_i)$ can be weakened a little. In fact we only use $m_{ik}{}^j \in C^2(S_i)$ for those coefficients which occur in conditions (III.1.1) and (III.1.2). For the others we only use $m_{ik}{}^j \in C^0(S_i)$.

III.2. Abstract formulation of the boundary problem. For functions $u \in C^{n-1}(\bar{V})$ we define a norm by

$$\|u\|_3 = \left\{ \int_V \sum_{|p| \leqq n-1} (D^p u)^2 dV + \int_S \sum_{|p| \leqq n-1} (D^p u)^2 ds \right\}^{\frac{1}{2}}$$

and let $\mathscr{H}_3$ be the Hilbert space obtained by completing $C^{n-1}(\bar{V})$ with respect to this norm.

For the elements $\mathsf{G} = (G, g_1, \ldots, g_n, \tilde{g}_n) \in \tilde{C}^0(\bar{V}, S)$ we define a norm by

$$\|\mathsf{G}\|_4 = \left\{ \int_V G^2 dV + \sum_{i=1}^{n} \int_{S_i} \sum_{k=1}^{i} g_{ik}{}^2 ds + \int_{S_n} \sum_{|p| < n-1} \tilde{g}_p{}^2 ds \right\}^{\frac{1}{2}}$$

and let $\mathscr{H}_4$ be the Hilbert space obtained by completing $\tilde{C}^0(\bar{V}, S)$ with respect to this norm. Note that also after the completion $\tilde{g}_n$ is uniquely determined by g_n up to the successive integration constants.

We now consider M as an operator from $\mathscr{H}_3$ to $\mathscr{H}_4$ with domain $D(\mathsf{M}) = C^n(\bar{V})$ and range $R(\mathsf{M}) \subseteq \tilde{C}^0(\bar{V}, S)$. As in Section II.2 we have

LEMMA III.1. *The operator* M *is pre-closed.*

The proof is analogous to the proof of Lemma II.1. Let $\bar{\mathsf{M}}$ be the closure of M. The main result in this part is the following

THEOREM III.1. *The equation* $\bar{\mathsf{M}}u = \mathsf{G}$ *has a unique solution* $u = \bar{\mathsf{M}}^{-1}\mathsf{G} \in \mathscr{H}_3$ *for every* $\mathsf{G} \in \mathscr{H}_4$ *and*

$$\|\bar{\mathsf{M}}^{-1}\mathsf{G}\|_3 \leqq C\|\mathsf{G}\|_4 ,$$

where C *is a constant independent of* G.

The idea in the proof is as in Part II to replace the equation $\mathsf{M}u = \mathsf{G}$ by an equation $\mathsf{M}^0 u = \mathsf{G}$, which is easier to handle. We therefore introduce a family of operators $\mathsf{M}^\lambda (0 \leqq \lambda \leqq 1)$ defined in $D(\mathsf{M}^\lambda) = C^n(\bar{V})$ by

$$\mathsf{M}^\lambda u = (M^\lambda u, m_1{}^\lambda u, \ldots, m_n{}^\lambda u, \tilde{m}_n{}^\lambda u) ,$$

where

$$M^\lambda u = \sum_{|p|=n} a_p D^p u + \lambda \sum_{|p| < n} a_p D^p u$$

and $m_i{}^\lambda u = \big(m_{i1}{}^\lambda(u), \ldots, m_{ii}{}^\lambda(u)\big)$ $(i = 1, \ldots, n)$ with

$$m_{ik}{}^\lambda(u) = \sum_{j=1}^{i} m_{ik}{}^j M_j u + \lambda \left(\sum_{j=i+1}^{n} m_{ik}{}^j M_j u + \sum_{|p| < n-1} m_{ik,p} D^p u \right) \quad \text{on} \quad S_i^-$$

and

$$m_{ik}{}^\lambda(u) = \sum_{j=n-i+1}^{n} m_{ik}{}^j M_j u + \lambda \left(\sum_{j=1}^{n-i} m_{ik}{}^j M_j u + \sum_{|p| < n-1} m_{ik,p} D^p u \right) \quad \text{on} \quad S_i^+.$$

Finally, we put $\tilde{m}_n{}^\lambda u = \tilde{m}_n u$.

The operator M^λ thus defined is, as well as M, an operator from $\mathscr{H}_3$ to $\mathscr{H}_4$ and coincides for $\lambda = 1$ with M. For $\lambda = 0$, however, M^0 only contains the principal part of M and $m_{ik}{}^0(u)$ only contains some terms of the principal part om $m_{ik}(u)$.

We then have the following lemmas.

Lemma III.2. *The operator* $\mathsf{M}^1 - \mathsf{M}^0$ *is bounded, that is, there is a constant* C_1 *independent of* u *such that*

$$\|(\mathsf{M}^1 - \mathsf{M}^0)u\|_4 \leqq C_1 \|u\|_3, \qquad u \in C^n(\overline{V}) .$$

Lemma III.3. *There is a constant* C_2 *independent of* u *and* λ *such that*

$$\|u\|_3 \leqq C_2 \|\mathsf{M}^\lambda u\|_4, \qquad u \in C^n(\overline{V}) .$$

Lemma III.4. $R(\overline{\mathsf{M}^\lambda})$ *equals the closure* $\overline{R(\mathsf{M}^\lambda)}$ *of* $R(\mathsf{M}^\lambda)$.

Lemma III.5. Assume that $R(\overline{\mathsf{M}^{\lambda_0}}) = \mathscr{H}_4$. Then

$$\|\overline{\mathsf{M}^{\lambda_0}}^{-1} \mathsf{G}\|_3 \leqq C_2 \|\mathsf{G}\|_4, \qquad \mathsf{G} \in \mathscr{H}_4$$

and

$$\|\overline{\mathsf{M}^{\lambda_0}}^{-1} (\overline{\mathsf{M}^1 - \mathsf{M}^0}) u\|_3 \leqq C_3 \|u\|_3, \qquad u \in \mathscr{H}_3$$

where $C_3 = C_1 C_2$ *and* C_1 *and* C_2 *are the constants in Lemma* III.2 *and Lemma* III.3.

Lemma III.6. $R(\overline{\mathsf{M}^0}) = \mathscr{H}_4$, *that is, the equation* $\overline{\mathsf{M}^0} u = \mathsf{G}$ *has a unique solution* $u \in \mathscr{H}_3$ *for every* $\mathsf{G} \in \mathscr{H}_4$.

The proofs of Lemmas III.2–III.5 are analogous to the proofs of Lemmas II.2–II.5 and will not be carried through. In the proof of Lemma III.3, we use a result from [13], namely, that for every λ, $0 \leqq \lambda \leqq 1$, there is a constant $C(\lambda)$ such that

$$\|u\|_3 \leqq C(\lambda) \|\mathsf{M}^\lambda u\|_4 .$$

Lemma III.6 will be proved in the next section.

III.3. The equation $\overline{\mathsf{M}^0} u = \mathsf{G}$. In this section we shall prove that the equation

$$\overline{\mathsf{M}^0} u = \mathsf{G} \tag{III.3.1}$$

has a solution in $\mathscr{H}_3$ for every $\mathsf{G} \in \mathscr{H}_4$. According to Lemma III.4, we need only prove that (III.3.1) has a solution in $\mathscr{H}_3$ for every $\mathsf{G} = (G, g_1, \ldots, g_n, \tilde{g}_n)$ in a dense subset of $\mathscr{H}_4$.

The operators $(D_t - \alpha_i D_x) M_i u$ and $M^0 u$ have the same principal part and their difference only contains terms of order $n-1$. This implies that there are $a_{ik} \in C^1(\overline{V})$ such that we have the following identity:

$$(III.3.2) \qquad (D_t - \alpha_i D_x) M_i u + \sum_{k=1}^{n} a_{ik} M_k u = M^0 u .$$

Now suppose that $u \in C^n(\bar{V})$ is a solution of (III.3.1). Then (III.3.2) implies that $v = (v_1, \ldots, v_n) = (M_1 u, \ldots, M_n u)$ is a solution of

$$(D_t - \alpha_i D_x) v_i + \sum_{k=1}^{n} a_{ik} v_k = G, \qquad i = 1, \ldots, n ,$$

with the boundary conditions (cf. the definition of M^0)

$$\sum_{j=1}^{n} l_{ik}^{\,j} v_j = g_{ik}, \qquad k = 1, \ldots, i;\ i = 1, \ldots, n ,$$

where

$$l_{ik}^{\,j} = \begin{cases} m_{ik}^{\,j}, & j = 1, \ldots, i, \\ 0, & j = i+1, \ldots, n, \end{cases} \qquad \text{on} \quad S_i^-$$

and

$$l_{ik}^{\,j} = \begin{cases} 0, & j = 1, \ldots, n-i, \\ m_{ik}^{\,j}, & j = n-i+1, \ldots, n, \end{cases} \qquad \text{on} \quad S_i^+ .$$

The boundary problem for v thus defined is of exactly the same kind as the boundary problem considered in Section II.6. Therefore, let L^+ be the operator corresponding to this boundary problem. Then all conditions imposed in Section II.6 on the operator L^+ are satisfied. In particular, the conditions (II.1.1) and (II.1.2) are fulfilled according to (III.1.1) and (III.1.2).

Let Γ be the set of $\mathsf{G} = (G, g_1, \ldots, g_n, \tilde{g}_n) \in \mathscr{H}_4$ such that $\mathsf{G}^+ = (G^+, g_1, \ldots, g_n) \in \mathscr{H}_2^{\,+}$. Here G^+ is the vector $(G, \ldots, G)$ with n equal components. Then Γ is dense in $\mathscr{H}_4$ and it will be sufficient to solve (III.3.1) for $\mathsf{G} \in \Gamma$. Therefore we solve for $\mathsf{G} \in \Gamma$ the equation

$$(III.3.3) \qquad \overline{\mathsf{L}^+} v = \mathsf{G}^+ .$$

This is possible according to Theorem II.3. Let $v = (v_1, \ldots, v_n)$ be the solution of (III.3.3). Then v_i has generalized derivatives of the first order in $L^2(V)$. That is, v_i belongs to $L^{2,1}(V)$ and we shall prove that there exists a function $u \in L^{2,n}(V)$ with $M_i u = v_i$ $(i = 1, \ldots, n)$ in the generalized sense. Such a u must satisfy

$$D_t^{k-1} D_x^{n-k} u = \sum_{i=1}^{n} c_{ki} v_i .$$

We therefore construct $w_k = \Sigma_{i=1}^n c_{ki} v_i$ $(k = 1, \ldots, n)$. Then $w_k \in L^{2,1}(V)$ and we shall prove that

$$D_t w_{k-1} = D_x w_k \qquad \text{for} \quad k = 2, \ldots, n .$$

In fact,

$$\text{(III.3.4)}\quad D_t w_{k-1} - D_x w_k = D_t\Big(\sum_{i=1}^{n} c_{k-1,i}v_i\Big) - D_x\Big(\sum_{i=1}^{n} c_{ki}v_i\Big)$$
$$= \sum_{i=1}^{n} c_{k-1,i}D_t v_i - \sum_{i=1}^{n} c_{ki}D_x v_i + \sum_{i=1}^{n}(D_t c_{k-1,i} - D_x c_{ki})v_i$$
$$= \sum_{i=1}^{n} c_{k-1,i}(D_t - \alpha_i D_x)v_i + \sum_{i=1}^{n}(D_t c_{k-1,i} - D_x c_{ki})v_i$$
$$= a_0 G + \sum_{i=1}^{n} a_i v_i \,.$$

Here we have used $\alpha_i c_{k-1,i} = c_{ki}$ and the fact that v is a solution of (III.3.3). The functions a_i ($i=0, \ldots, n$) do not depend on G but only on the coefficients of L^+. Therefore, if G is of the form $M^0\varphi$ with $\varphi \in C^n(\overline{V})$, we have $v = (M_1\varphi, \ldots, M_n\varphi)$ and $w_k = D_t^{k-1}D_x^{n-k}\varphi$ and we get from (III.3.4) that for every $\varphi \in C^n(\overline{V})$

$$a_0 M^0\varphi + \sum_{i=1}^{n} a_i M_i\varphi = 0\,.$$

This implies, however, that $a_i = 0$ ($i=1, \ldots, n$) and thus that for a general G, we have $D_t w_{k-1} - D_x w_k = 0$ ($k=2, \ldots, n$).

According to Theorem I.2, the quantities $w_1, \ldots, w_n$ are therefore the generalized derivatives of order $n-1$ of a function $u \in L^{2,n}(V)$. This function u is uniquely determined up to the successive integration constants. Because of the consistency between g_n and $\tilde{g}_n$ it is possible to determine these constants so that $\tilde{m}_n u = \tilde{g}_n$, which determines u completely. We shall prove that u is a solution of (III.3.1). Since $u \in L^{2,n}(V)$ there exists in virtue of Theorem I.1 a sequence of functions $u^\mu \in C^n(\overline{V})$ ($\mu = 1, 2, \ldots$) such that for $|p| \leqq n$

$$D^p u^\mu \to D^p u \quad \text{in} \quad L^2(V) \qquad \text{when} \quad \mu \to \infty\,.$$

We shall prove that

$$\|\mathsf{M}^0 u^\mu - \mathsf{G}\|_4 \to 0 \qquad \text{when} \quad \mu \to \infty\,.$$

Put $V^\mu = (M_1 u^\mu, \ldots, M_n u^\mu)$ and let v^μ be a sequence of functions in $C_n^2(\overline{V})$ such that

$$v^\mu \to v \quad \text{in} \quad \mathscr{H}_1^{\,+} \qquad \text{and} \qquad \mathsf{L}^+ v^\mu \to \mathsf{G}^+ \quad \text{in} \quad \mathscr{H}_2^{\,+}$$

when $\mu \to \infty$. The existence of such a sequence is granted by the definition of $\overline{\mathsf{L}^+}$. From the definition of u^μ, we obtain

$$V^\mu \to v \qquad \text{in} \quad \mathscr{H}_1^{\,+}$$

when $\mu \to \infty$. We have

$\|\mathsf{M}^0 u^\mu - \mathsf{G}\|_4 \leqq$

$$\|\mathsf{L}^+ V^\mu - \mathsf{L}^+ v^\mu\|_2 + \|\mathsf{L}^+ v^\mu - \mathsf{G}^+\|_2{}^+ + \left\{\int\limits_{S_n} \sum_{|p|<n-1} (D^p u^\mu - \tilde{g}_p)^2 ds\right\}^{\frac{1}{2}} \to 0 .$$

The first term on the right side converges to 0 because L^+ is bounded, considered as an operator from $\mathscr{H}_1{}^+$ to $\mathscr{H}_2$ and $V^\mu \to v$, $v^\mu \to v$ in $\mathscr{H}_1{}^+$. The second term tends to 0 according to the definition of v^μ, and the third term tends to 0 because

$$D_t{}^{k-1} D_x{}^{n-k} u^\mu \to g_{nk} \quad \text{in} \quad L^2(S_n)$$

and because of the way in which we have determined the integration constants. This completes the proof of Lemma III.6

III.4. Estimates for the higher order derivatives. We shall complete the treatment of the hyperbolic equation of order n by stating a theorem corresponding to Theorem II.2 in Part II.

Let q be an arbitrary integer. For $u \in C^{n+q-1}(\bar{V})$ we define

$$\|u\|_R{}^{(q)} = \left\{\int\limits_R \sum_{|p| \leqq n+q-1} (D^p u)^2 dR\right\}^{\frac{1}{2}}, \qquad R = V, S ,$$

and for $\mathsf{G} \in \tilde{C}^q(\bar{V}, S) = (C^q(\bar{V}),\ C_1{}^q(S_1),\ \ldots,\ C_n{}^q(S_n),\ \tilde{C}^q(S_n))$, where $\tilde{C}^q(S_n)$ is the set of arrays $(\tilde{g}_p = D^p \tilde{g};\ |p| < n-1,\ \tilde{g} \in C^{n-1}(\bar{V}))$ in $\tilde{C}(S_n)$ such that

$$D_t{}^{k-1} D_x{}^{n-k} \tilde{g} = g_{nk} \qquad \text{on} \quad S_n .$$

Further we define

$$\|\mathsf{G}\|_{V,S}{}^{(q)} = \left\{\int\limits_V \sum_{|p| \leqq q} (D^p G)^2 dV + \int\limits_S \sum_{|p| \leqq q-1} (D^p G)^2 ds + \right.$$

$$\left. + \sum_{i=1}^{n} \int\limits_{S_i} \sum_{j=1}^{i} \sum_{l \leqq q} (D_s{}^l g_{ij})^2 ds + \int\limits_{S_n} \sum_{|p|<n-1} \tilde{g}_p{}^2 ds\right\}^{\frac{1}{2}} .$$

Here D_s stands for differentiation with respect to the arc length on S. Like in (II.5.1), the term

$$\int\limits_S \sum_{|p| \leqq q-1} (D^p G)^2 ds$$

is immaterial and can be estimated by means of

$$\int\limits_V \sum_{|p| \geqq q} (D^p G)^2 dV .$$

Let k be a fixed integer and assume that the boundary pieces S_i $(i=1, \ldots, n)$ are k times continuously differentiable, that is, $\nu \in C_2^{k-1}(S_i)$, and that M satisfies the following differentiability assumptions, namely

$$a_p \in C^{k+1}(\overline{V}) \quad \text{for} \quad |p| = n\,,$$
$$a_p \in C^{k}(\overline{V}) \quad \text{for} \quad |p| < n\,,$$
$$m_{ik,\,p} \in C^{k}(S_i)\,.$$

If $q \leqq k$, M can be considered as an operator from $C^{n+q}(\overline{V})$ to $\tilde{C}^{q}(\overline{V}, S)$ and we have the following

THEOREM III.2. *There is a constant C such that*

$$\|u\|_R^{(q)} \leqq C\,\|\mathsf{M}u\|_{V,S}^{(q)}, \qquad R = V, S\,,$$

for all $u \in C^{n+q}(\overline{V})$ and $q \leqq k$.

This theorem is proved from Theorem II.2 in the same way as (5.10) and (5.11) are proved from (1.6) and (1.7) in [13]. The proof will not be carried through.

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