MATH. SCAND. 6 (1958), 33–36

INVARIANT METRICS IN COSET SPACES

LEIF KRISTENSEN

It is a well-known fact that a topological group G is metrizable if and only if it satisfies the first countability axiom at the identity element (and thus at every element), and moreover that when the group is metrizable, there exists a left invariant metric, that is, a metric d satisfying

$$d(x, y) = d(gx, gy) \qquad \text{for all} \quad x, y, g \in G .$$

(Birkhoff [1], Kakutani [4]). It is also well known that if G is metrizable and H is a closed subgroup, then the coset space G/H is metrizable (Montgomery and Zippin [5, p. 36]).

In this note it is shown that if H is compact, then there exists an invariant metric in G/H, that is, a metric d satisfying

$$d(x, y) = d(gx, gy) \qquad \text{for all} \quad x, y \in G/H, \ g \in G .$$

For Lie groups this has already been proved by É. Cartan [2, p. 43]; in this case there exists an invariant Riemannian metric.

The basic idea of the proof given here is very similar to that of the existence of an invariant metric in a group G (that is the case $H=\{e\}$) given by Montgomery and Zippin [5, pp. 34–36].

Furthermore, a theorem is proved which implies that if the homogeneous space G/H is locally compact and connected, then there exists an invariant metric in G/H with the property that every bounded and closed subset of G/H is compact. This result is useful in the theory of discontinuous transformation groups. It is well known that a discret subgroup of a topological group G is discontinuous in every coset space G/H with compact group of stability H. This fact and the equivalence of the various definitions of discontinuity in this case (cf. Fenchel [3]) can easily be proved by means of an invariant metric in G/H with the property mentioned.

THEOREM I. *Let G be a topological group and H a compact subgroup of G. The coset space $M=G/H$ is metrizable if and only if it satisfies the*

Received April 2, 1958.

first countability axiom. Moreover, if M is metrizable, there exists an invariant metric.

PROOF. We first remark that the natural mapping $\nu: G \to M$ is open and, because of the compactness of H, also closed. To every neighborhood U of $o = \nu(e)$ we can find a symmetric neighborhood V of e such that

$$(HVH)^2 \subseteq \nu^{-1}(U) .$$

This means that there exists a neighborhood $U' = \nu(HVH)$ of o with the properties

$$\nu^{-1}(U')\,U' \subseteq U$$

and

$$hU' = U' \qquad \text{for every} \quad h \in H .$$

Assuming that there exists a countable fundamental system of neighborhoods at o, we can apply the remark above to construct in the usual way (see e.g. Montgomery and Zippin [5, p. 28]) a fundamental system $\{U_r\}$, $0 < r \leqq 1$, dyadic rational, of neighborhoods of o with the properties

1° $\nu^{-1}(U_{1/2^n})\,U_{m/2^n} \subseteq U_{(m+1)/2^n}$,
2° $hU_r = U_r$ for every $h \in H$,
3° $\nu^{-1}(U_1)$ spans the whole of G.

Although this is not necessary for the present proof, we shall enlarge the family $\{U_r\}$ defining U_r for all non-negative dyadic rationals r by

$$U_0 = \emptyset \ \text{(the empty set)}, \qquad U_r = \nu\big(\nu^{-1}(U_{r-[r]})\,\nu^{-1}(U_1)^{[r]}\big) .$$

For this enlarged system the properties 1°–3° remain valid, and in addition we have

$$\bigcap_{r>0} U_r = o , \qquad \bigcup_r U_r = M . \tag{1}$$

For every point $x \in M$ there is an element $g \in G$ such that $go = x$. Consequently, for every point $x \in M$ we can define a system of neighborhoods $\{U_r^x\}$ of x by $U_r^x = gU_r$. Because of 2°, this definition does not depend on the choice of g, and we have

$$gU_r^x = U_r^{gx}. \tag{2}$$

We can now define an auxiliary function $f_x(y)$ as follows:

$$f_x(y) = f(x, y) = \sup\{r \mid y \notin U_r^x\} .$$

Because of (1), this function is finite and

$$f(x, y) = 0 \iff x = y .$$

Because of (2)

$$f(x, y) = f(gx, gy) \qquad \text{for every} \quad g \in G .$$

Let now

$$d(x, y) = \sup_{u \in M} |f(x, u) - f(y, u)| .$$

The inequality

$$0 \leqq f(x, y) \leqq d(x, y) \leqq f(x, y) + 2 , \tag{3}$$

which is easily proved, shows that $d(x, y)$ is finite. Obviously $d(x, y) = 0$, if and only if $x = y$. Further $d(x, y) = d(y, x)$, and

$$\begin{aligned} d(x, z) &= \sup_{u \in M} |f(x, u) - f(z, u)| \\ &\leqq \sup_{u \in M} |f(x, u) - f(y, u)| + \sup_{u \in M} |f(y, u) - f(z, u)| \\ &= d(x, y) + d(y, z) \end{aligned}$$

shows that the triangle inequality holds. Thus d is a metric. Since

$$\begin{aligned} d(gx, gy) &= \sup_{u \in M} |f(gx, u) - f(gy, u)| \\ &= \sup_{u \in M} |f(x, g^{-1}u) - f(y, g^{-1}u)| \\ &= d(x, y) , \end{aligned}$$

d is invariant. The last equality holds because g^{-1} maps M onto itself.

It only remains to be shown that the topology induced by d coincides with the original topology of M. If $S(x, r)$ denotes the open r-sphere with center x, then

$$S(x, r) \subseteq U_r^x , \qquad r \text{ dyadic rational.}$$

For if $d(x, y) < r$, then $f(x, y) < r$ by (1), and from the definition of $f(x, y)$ it follows that $y \in U_r^x$. Conversely, if $y \in U_{1/2^{n+1}}^x$, by property 1° we have for every $u \in M$

$$|f(x, u) - f(y, u)| \leqq 1/2^n$$

which shows that

$$U_{1/2^{n+1}}^x \subseteq S(x, 1/2^n) ,$$

and the proof is complete.

THEOREM II. *If M is metrizable and if G contains a compact neighborhood of e generating the whole group G, then there exists an invariant metric in M with the property that a subset of M is compact if and only if it is bounded and closed.*

REMARKS. The assumption that G is generated by a compact neighborhood of e is easily seen to be equivalent to the assumption that M contains a compact neighborhood U of o such that $\nu^{-1}(U)$ generates G.

If M is locally compact and connected, then $\nu^{-1}(U)$, U a neighborhood of o, generates an open and closed subgroup of G which, because of the connectedness of M, is the whole group G. This shows that the assumption of the theorem is satisfied in this case.

PROOF. Making use of the properties of G assumed in the theorem, the system $\{U_r\}$ can be constructed in such a manner that all the sets U_r are compact. The metric d constructed by means of this system has the property that every bounded set is contained in a compact set. In fact, if $B \subseteq M$ is bounded, then for a fixed $a \in M$ and a certain dyadic rational N

$$d(a, x) < N \qquad \text{for all} \quad x \in B .$$

By (3) we also have $f(a, x) < N$, and by the definition of f

$$x \in U_N^a \qquad \text{for all} \quad x \in B .$$

Consequently, d has the properties claimed in the theorem.

REFERENCES

1. G. Birkhoff, *A note on topological groups*, Compositio Math. 3 (1936), 427–430.
2. É. Cartan, *La théorie des groupes finis et continus et l'analysis situs* (Mémorial des Sciences Mathématiques 42), Paris, 1930 [= Œuvres complètes, partie I, vol. 2, 1165–1225].
3. W. Fenchel, *Bemerkungen zur allgemeinen Theorie der diskontinuierlichen Transformationsgruppen*, 13e Congr. Math. Scand. Helsinki 1957, Helsingfors, 1958, 77–85.
4. S. Kakutani, *Über die Metrization der topologischen Gruppen*, Proc. Imp. Acad. Japan 12 (1936), 82–84.
5. D. Montgomery and L. Zippin, *Topological transformation groups* (Interscience Tracts in Pure and Applied Mathematics 1), New York, 1955.

UNIVERSITY OF AARHUS, DENMARK