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ON THE ALGEBRA GENERATED BY A CONTINUOUS FUNCTION

EBBE THUE POULSEN

Let A denote the algebra of continuous real-valued functions in the closed interval $0 \leqq x \leqq 1$. If $g \in A$ we shall let $A(g)$ denote the closed sub-algebra generated by g, that is, the set of all polynomials in g and their uniform limits.

It follows from the Stone-Weierstrass theorem (see for instance [1, p. 9]) that $A(g) = A$ if g distinguishes points of the interval, i.e. if g is monotone. Conversely, it is clear that $A(g) \neq A$ if g does not distinguish points.

Consequently, if g distinguishes points and $f \in L^1(0, 1)$, we have that

$$\int_0^1 f(x) g^n(x) dx = 0, \qquad n = 0, 1, 2, \ldots, \tag{1}$$

implies that $f(x) = 0$.

On the other hand, assume that g does not distinguish points, and let $L(g)$ denote the closure of $A(g)$ in $L^1(0, 1)$, then $L(g)$ is a closed subspace of $L^1(0, 1)$, and since it is easily seen that $L(g)$ is a proper subspace, there exists a non-zero function f, bounded and measurable in $0 \leqq x \leqq 1$, such that

$$\int_0^1 f(x) g^n(x) dx = 0 \qquad \text{for} \quad n = 0, 1, 2, \ldots .$$

The following question was communicated to the author by W. G. Bade (the original source is unknown to the author):

Can it happen that g does not distinguish points, but that (1) for a continuous function f implies $f = 0$?

The object of this paper is to construct an example that will show that the answer is affirmative.

LEMMA 1. *Let g be continuous in $0 \leqq x \leqq 1$, let $f \in L^1(0, 1)$, and assume that*

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$$\int_0^1 f(x)g^n(x)\,dx = 0 \qquad \text{for} \quad n = 0, 1, 2 \ldots .$$

Let α and $\beta > \alpha$ be real numbers, and let $E = \{x \mid \alpha \leqq g(x) \leqq \beta\}$. Then

$$\int_E f(x)\,dx = 0 .$$

Proof. Let $\alpha_0 \leqq y \leqq \beta_0$ be the range of g. It is obviously sufficient to consider the case $\alpha_0 \leqq \alpha < \beta \leqq \beta_0$. In the interval $\alpha_0 \leqq y \leqq \beta_0$ there exists a sequence of polynomials $P_n(y)$ such that

$$P_n(y) \to \begin{cases} 1 & \text{for } \alpha \leqq y \leqq \beta , \\ 0 & \text{otherwise} , \end{cases}$$

and the $P_n(y)$ are uniformly bounded. Then, if we consider the sequence $f_n(x) = f(x)P_n(g(x))$, we have

$$f_n(x) \to \begin{cases} f(x) & \text{for } x \in E , \\ 0 & \text{otherwise} , \end{cases}$$

and the f_n are majorized by a summable function. Since

$$\int_0^1 f_n(x)\,dx = 0$$

for each n, the assertion follows by Lebesgue's dominated convergence theorem.

Lemma 2. *There exists a continuous function g, monotonely increasing in $0 \leqq x \leqq \frac{1}{2}$, such that the upper right derivative of g is infinite at a dense set of points.*

Proof. Such a function g is easily constructed by superposition of suitable functions.

Theorem. *Let $g(x)$ be as decribed in lemma 2 with $g(0) = 0$, $g(\frac{1}{2}) = \frac{1}{2}$, and define*

$$g(x) = 1 - x \qquad \textit{in} \qquad \tfrac{1}{2} \leqq x \leqq 1 .$$

Then

$$\int_0^1 f(x)g^n(x)\,dx = 0, \qquad n = 0, 1, 2 \ldots ,$$

for a continuous function f implies that $f(x) = 0$ in $0 \leqq x \leqq 1$.

PROOF. Assume that $f(x)$ is not identically 0. By means of the continuity of f and lemma 1 we construct two intervals

$$0 \leqq a_1 \leqq x \leqq b_1 \leqq \tfrac{1}{2} \quad \text{and} \quad \tfrac{1}{2} \leqq b_2 \leqq x \leqq a_2 \leqq 1$$

such that $g(a_1) = g(a_2)$, $g(b_1) = g(b_2)$ and such that $f(x) \neq 0$ in both intervals. Assume for definiteness that $f(x) < 0$ in $a_1 \leqq x \leqq b_1$, then $f(x) > 0$ in $b_2 < x < a_2$. Now choose x_0 with $a_1 \leqq x_0 < b_1$ such that the upper right derivative of g is infinite at x_0, and let $\{x_n\}$ be a sequence of points with $x_0 < x_n \leqq b_1$ such that

$$\text{(2)} \qquad \frac{g(x_n) - g(x_0)}{x_n - x_0} \to \infty \quad \text{for} \quad n \to \infty .$$

By lemma 1 we have for each n:

$$\int_{x_0}^{x_n} f(x)\,dx + \int_{1-g(x_n)}^{1-g(x_0)} f(x)\,dx = 0$$

or

$$\int_{1-g(x_n)}^{1-g(x_0)} f(x)\,dx = \int_{x_0}^{x_n} |f(x)|\,dx .$$

Now let

$$m = \min_{b_2 \leqq x \leqq a_2} f(x), \qquad M = \max_{a_1 \leqq x \leqq b_1} |f(x)| ,$$

then the above equation yields

$$m\big(g(x_n) - g(x_0)\big) \leqq M(x_n - x_0) ,$$

which is in contradiction with (2).

The theorem follows.

REFERENCE

1. L. H. Loomis, *An introduction to abstract harmonic analysis*, New York, 1953.

UNIVERSITY OF CALIFORNIA, BERKELEY, U.S.A.
AND
UNIVERSITY OF AARHUS, DENMARK