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ON POLYGONS IN REAL PROJECTIVE n-SPACE

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Although Juel [2] made full use of the differentiability assumption in the proofs used in his theory of differentiable curves of bounded real order, many of the results seem to be true if the differentiability is weakened. This has already been pointed out by Hjelmslev (cf. [1], where polygons in the projective plane and polyhedra in the projective space are considered). The present paper develops a theory of polygons analogous to the simplest of Juel's curves, those of order n in n-space. The methods are related to those used by Kivikoski [3] [4] in his study of polygons in the projective plane. The principal result, given in 4.6, is that each such polygon may be obtained from a polygon connecting the vertices of a simplex by a succession of inscribed polygons defined in 3.1. Two applications of the results are given. The first of these, in 5.6, is a proof of a duality theorem for these polygons analogous to the one proved by Scherk [6] for differentiable curves. The second in 6.1, related to the first, proves a result for continuous but not necessarily differentiable curves. This result for $n=3$ follows from a result of Marchaud [5] on which he based his analysis of continuous plane curves of real order 3.

1. Definitions and notation.

A polygon π in real projective n-space is defined by its sides A_1A_2, A_2A_3, $\ldots$, $A_{r-1}A_r$ and in the case in which it is closed A_rA_1. The consecutive endpoints $A_1, A_2, \ldots, A_r$ of the sides are the vertices of π. If the polygon is closed it is useful to define the vertex subscripts i modulo r. Thus A_1, A_{1+r} both represent the same vertex.

A point common to a linear k-space L, $0 \leqq k \leqq n-1$, and a polygon π is defined to be an *intersection point* of L and π if it is a vertex of π or if it is the only point of a side of π which is within L.

A polygon π_n with vertices $A_1, A_2, \ldots, A_r$ is *defined* to be a *polygon of order n* in n-space provided it satisfies

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(1) the *dimension condition* that it is not contained in a hyperplane, and

(2) the *order condition* that no hyperplane intersect π_n in more than n points.

The symbol π_n will be used exclusively for polygons which satisfy these two conditions.

If $Q_1, Q_2, \ldots, Q_r$ are points in real projective n-space then the linear space generated by these points will be designated by the symbol $[Q_1, Q_2, \ldots, Q_r]$.

2. Preliminary lemmas.

2.1. *If* $B_0, B_1, \ldots, B_k$, $0 \leqq k \leqq n$, *are vertices of a polygon* π_n *then* $[B_0, B_1, \ldots, B_k]$ *has dimension* k.

PROOF. If $[B_0, B_1, \ldots, B_k]$ had dimension less than k then, as π_n satisfies the dimension condition, vertices $B_{k+1}, B_{k+2}, \ldots, B_m$, $m > n$, would exist so that $[B_0, B_1, \ldots, B_m]$ had dimension $n-1$. As this would contradict the order condition, 2.1 is proved.

This result implies that the vertices of a polygon π_n must be distinct.

2.2. *If, for* $n \geqq 2$, $\dot{A}_j$, $\dot{A}_{j-1}\dot{A}_j$ *be the projections of the vertex* A_j *and the side* $A_{j-1}A_j$, $i \neq j$, $i \neq j-1$, *of a closed polygon* π_n *from a vertex* A_i *onto a hyperplane, then the projection of* π_n *from* A_i *is an open polygon* π_{n-1} *with sides* $\dot{A}_1\dot{A}_2, \ldots, \dot{A}_{i-2}\dot{A}_{i-1}, \dot{A}_{i+1}\dot{A}_{i+2}, \ldots, \dot{A}_r\dot{A}_1$.

Where $\dot{A}_{i-1}\dot{A}_{i+1}$ *is defined to be the projection of the third side of the odd triangle with sides* $A_{i-1}A_i$, A_iA_{i+1} *then* π_{n-1} *may be closed by the addition of* $\dot{A}_{i-1}\dot{A}_{i+1}$ *without increase of order.*

PROOF. If $A_i \in [A_{j-1}, A_j]$ then by 2.1 either $i=j$ or $i=j-1$. Hence the projection of the side $A_{j-1}A_j$ of π_n from the vertex A_i is a line segment $\dot{A}_{j-1}\dot{A}_j$ if $i \neq j$, $i \neq j-1$. Let $A_{i-1}A_{i+1}$ be the third side of the odd triangle t which contains the two sides $A_{i-1}A_i$, A_iA_{i+1}. Where $\dot{A}_{i-1}\dot{A}_{i+1}$ is defined as the projection of $A_{i-1}A_{i+1}$ from the vertex A_i, let π be the closed polygon which consists of the segments $\dot{A}_1\dot{A}_2, \ldots, \dot{A}_{i-2}\dot{A}_{i-1}, \dot{A}_{i-1}\dot{A}_{i+1}, \dot{A}_{i+1}\dot{A}_{i+2}, \ldots, \dot{A}_r\dot{A}_1$. π is not included in a hyperplane of the projected space, for otherwise π_n would be included in a hyperplane of the original space.

To complete the proof it is sufficient to show π has order $n-1$. To do this let $\dot{H}$ be any hyperplane of the projected space and H the hyperplane of the original space, the projection of which from A_i is $\dot{H}$. Suppose first that H contains no interior point of $A_{i-1}A_{i+1}$. In this case each intersection of $\dot{H}$ and π is the projection of an intersection point of H

and π_n different from A_i. As π_n has order n and A_i is an intersection point there are at most $n-1$ such intersection points. Hence $\dot{H}$ intersects π in at most $n-1$ points. We now consider the case in which H does contain an interior point of $A_{i-1}A_{i+1}$. H now supports the triangle t at A_i as t is odd. Let L be the linear subspace generated by the vertices of π_n within H which are different from A_i. It follows from 2.1 that $A_i \notin L$. Hence H may be subjected to a displacement to a position $\bar{H}$ so that the points of L remain fixed but so that $\bar{H}$ intersects $A_{i-1}A_i$ in an interior point. If this displacement of H is sufficiently small the original points of intersection of H and π_n will each move in arbitrarily small neighborhoods. As H supports t at A_i, $\bar{H}$ will also intersect A_iA_{i+1} in an interior point. Thus $\bar{H}$ contains exactly one more intersection of π_n than H. Therefore H intersects π_n in at most $n-1$ points including A_i. The hyperplane $\dot{H}$ intersects $\dot{A}_{i-1}\dot{A}_{i+1}$ in an interior point as H is assumed to intersect $A_{i-1}A_{i+1}$ in an interior point. All the other intersection points of $\dot{H}$ and π are projections of intersection points of H and π_n different from A_i. As we have proved there are at most $n-2$ such points it follows that $\dot{H}$ intersects π in at most $n-1$ points. Thus π has order $n-1$. The proof is now complete.

2.3. *If σ_n is a closed polygon not within a hyperplane with exactly $n+1$ sides and if a hyperplane exists which intersects σ_n in exactly n points each of which is interior to a side, then σ_n has order n.*

PROOF. We first show that a hyperplane which does not contain vertices of σ_n intersects σ_n in at most n points. By the hypothesis a hyperplane H exists which intersects σ_n in exactly n points each of which is an interior point of a side. Hence successive vertices $A_1, A_2, \ldots, A_{n+1}$ may be chosen so that H does not intersect the side $A_{n+1}A_1$. If P_i be the intersection point of H and the side A_iA_{i+1}, $1 \leqq i \leqq n$, then $[P_1, P_2, \ldots, P_n]$ is a hyperplane. Otherwise $[A_1, P_1, \ldots, P_n]$ would have dimension at most $n-1$ and contain all the vertices of σ_n as $P_1, P_2, \ldots, P_n$ are interior points of consecutive sides. Hence $\sigma_n \subseteq [A_1, P_1, \ldots, P_n]$, contrary to the hypothesis that σ_n is not in a hyperplane. This also proves that if P_i varies continuously but remains an interior point of A_iA_{i+1}, $1 \leqq i \leqq n$, then $[P_1, P_2, \ldots, P_n]$ is a hyperplane which varies continuously and can never enter a vertex of σ_n. Hence no position of this hyperplane can intersect the side $A_{n+1}A_1$. Consequently no hyperplane can intersect σ_n in $n+1$ points, none of which are vertices as such a hyperplane would be one of the hyperplanes $[P_1, P_2, \ldots, P_n]$.

Now let H be a hyperplane which intersects σ_n in at least one vertex. As σ_n is not contained in a hyperplane, not all of the vertices can be contained in H. Hence H contains at least one vertex, say A_1, so that a consecutive vertex, say A_2, is not contained in H. This means that H contains only the point A_1 of the line $[A_1, A_2]$. A_1 cannot be within the space generated by the other vertices of σ_n within H for then all vertices of σ_n would be within a hyperplane. Hence H may be displaced so that its displaced position $\bar{H}$ intersects the line $[A_1, A_2]$ in an interior point A while each vertex of σ_n within H other than A_1 remains within $\bar{H}$. If the displacement is sufficiently small any intersection point of H and σ_n is displaced into an intersection point of $\bar{H}$ and σ_n. This process may be repeated until a hyperplane is obtained which intersects σ_n only in interior points of its sides and which does not intersect σ_n in fewer points than H. As such a hyperplane intersects σ_n in at most n points by the result of the first paragraph, H intersects σ_n in at most n points. Hence σ_n has order n and the proof is complete.

3. The inscribed polygons $I(\pi_n)$.

3.1 DEFINITION. *If n consecutive sides $A_1A_2, A_2A_3, \ldots, A_nA_{n+1}$ of a closed polygon π_n with vertices A_i, $1 \leqq i \leqq r$, are each subdivided into segments A_iB_{i+1}, $B_{i+1}A_{i+1}$ by an interior point B_{i+1} and if B_iB_{i+1} is defined so that B_iB_{i+1}, B_iA_i, A_iB_{i+1} is an even triangle, then $I(\pi_n)$ is defined to be the polygon inscribed in π_n with sides $A_1B_2, B_2B_3, \ldots, B_nB_{n+1}, B_{n+1}A_{n+1}, A_{n+1}A_{n+2}, \ldots, A_rA_1$.*

3.2 *The polygons $I(\pi_n)$ satisfy the dimension condition and have order n.*

PROOF. As the vertices of π_n are linear combinations of the vertices of $I(\pi_n)$ it follows that if $I(\pi_n)$ is included in a hyperplane then this hyperplane also contains π_n. Hence, as π_n satisfies the dimension condition, $I(\pi_n)$ also satisfies this condition.

Let H be a given hyperplane. We define a one to one correspondence between the intersection points of H and $I(\pi_n)$ and intersection points of H and π_n. Let Q be an intersection point of H and $I(\pi_n)$. Suppose first that Q is a vertex B_i of $I(\pi_n)$ which is not an intersection point of H and π_n. This means that B_i is an interior point of a side $A_{i-1}A_i$ of π_n which is part of a polygon arc $A_mA_{m+1} \ldots A_{m+h}$, $1 \leqq m$, $m+h \leqq n+1$, entirely within H. This polygon arc may be assumed to have maximum length. $m=1$ and $m+h=n+1$ cannot be true simultaneously for otherwise H would contain $n+1$ vertices of π_n. If $m+h=n+1$ then $m \neq 1$ and we define $\bar{Q}$ to be A_{i-1}. If $m+h<n+1$, $\bar{Q}$ is defined to be A_i. In each case $\bar{Q}$ is a vertex of π_n within H which is not a point of $I(\pi_n)$. Now suppose Q is an interior point of a side B_iB_{i+1}, $2 \leqq i \leqq n$. In this case H contains exactly

one more point of the even triangle with sides B_iB_{i+1}, B_iA_i, A_iB_{i+1}. We define this additional point to be $\bar{Q}$. $\bar{Q}$ must be on either B_iA_i or A_iB_{i+1}. As B_i, $B_{i+1} \notin H$, $\bar{Q}$ is a point of intersection of H and π_n. $\bar{Q}$ may be the vertex A_i but in this case neither side $A_{i-1}A_i$ nor A_iA_{i+1} can be within H. Finally, any additional intersection point Q of H and $I(\pi_n)$ is also an intersection point of H and π_n and so in this case we define $\bar{Q}=Q$. As the correspondence $Q \to \bar{Q}$ is a one to one correspondence, H cannot intersect $I(\pi_n)$ in a greater number of points than it intersects π_n. Hence $I(\pi_n)$ has order n and the result is proved.

4. The escribed polygons $E_k(\pi_n)$.

In this section it is shown that every polygon π_n is either a polygon with $n+1$ vertices or can be obtained from such a polygon by constructing a succession of inscribed polygons.

4.1 Definition of B_q^k. *If, for $r > n+1$, $A_1, A_2, \ldots, A_r$ are the vertices of a closed polygon π_n, then, for $1 \leqq q \leqq n+1$, B_q^0 is defined to be the vertex A_q and B_q^k the intersection*

$$[A_1, A_2, \ldots, A_q] \cap [A_{k+q}, A_{k+q+1}, \ldots, A_{k+n+1}] ,$$

$1 \leqq k \leqq r-n-1$.

4.2 (1) $B_1^k = A_1$, $B_{n+1}^k = A_{k+n+1}$, (2) B_q^k *is a point and* (3) $[B_q^k, B_{q+1}^k]$ *is a line which contains* B_{q+1}^{k-1}, $B_{q+1}^{k-1} \neq B_q^k$, $B_{q+1}^{k-1} \neq B_{q+1}^k$, $1 \leqq k \leqq r-n-1$, $1 \leqq q \leqq n$.

Proof. As π_n has order n any $n+1$ distinct vertices of π_n generate the full projective n-space. It follows, then, from the definitions

$$B_1^k = A_1 \cap [A_{k+1}, A_{k+2}, \ldots, A_{k+n+1}] ,$$
$$B_{n+1}^k = [A_1, A_2, \ldots, A_{n+1}] \cap A_{k+n+1} ,$$

that $B_1^k = A_1$ and $B_{n+1}^k = A_{k+n+1}$. Thus (1) is proved.

The spaces $[A_1, A_2, \ldots, A_q]$, $[A_{k+q}, A_{k+q+1}, \ldots, A_{k+n+1}]$, and

$$[A_1, A_2, \ldots, A_q, A_{k+q}, \ldots, A_{k+n+1}]$$

contain q, $n+2-q$, and $n+2$ distinct vertices of π_n, respectively, as $k \leqq r-n-1$. It follows, then, from 2.1 that these spaces have dimension $q-1$, $n+1-q$, and n respectively. If d is the dimension of the intersection

$$[A_1, A_2, \ldots, A_q] \cap [A_{k+q}, A_{k+q+1}, \ldots, A_{k+n+1}] = B_q^k$$

it follows from the incidence relation $(q-1)+(n+1-q) = n+d$ that $d=0$. Hence B_q^k, $1 \leqq k \leqq r-n-1$, $1 \leqq q \leqq n+1$, is a point and (2) is proved.

It follows from 4.1 that $B_q^k \in [A_1, A_2, \ldots, A_q]$ and

$$B_{q+1}^k \in [A_{k+q+1}, A_{k+q+2}, \ldots, A_{k+n+1}].$$

By the method used in the previous paragraph the intersection

$$[A_1, A_2, \ldots, A_q] \cap [A_{k+q+1}, A_{k+q+2}, \ldots, A_{k+n+1}], \qquad 1 \leqq q \leqq n,$$

has dimension -1 and so is empty. Therefore $B_q^k \neq B_{q+1}^k$ and consequently $[B_q^k, B_{q+1}^k]$ is a straight line, $1 \leqq k \leqq r-n-1$, $1 \leqq q \leqq n$. It follows from 4.1 that

$$[B_q^k, B_{q+1}^k] \subseteq [A_1, A_2, \ldots, A_{q+1}] \cap [A_{k+q}, A_{k+q+1}, \ldots, A_{k+n+1}].$$

This intersection has dimension 1 by the method of the previous paragraph. Hence

$$[B_q^k, B_{q+1}^k] = [A_1, A_2, \ldots, A_{q+1}] \cap [A_{k+q}, A_{k+q+1}, \ldots, A_{k+n+1}],$$

$1 \leqq k \leqq r-n-1$, $1 \leqq q \leqq n$. Consequently, for $k=1$,

$$B_{q+1}^{k-1} = B_{q+1}^0 = A_{q+1} \in [B_q^k, B_{q+1}^k].$$

If $k>1$, B_{q+1}^{k-1} is defined to be

$$B_{q+1}^{k-1} = [A_1, A_2, \ldots, A_{q+1}] \cap [A_{k+q}, A_{k+q+1}, \ldots, A_{k+n}]$$

which point is likewise contained in $[B_q^k, B_{q+1}^k]$. Thus

$$B_{q+1}^{k-1} \in [B_q^k, B_{q+1}^k], \qquad 1 \leqq k \leqq n-r-1, \quad 1 \leqq q \leqq n.$$

It remains to prove

$$B_{q+1}^{k-1} \neq B_q^k, \qquad B_{q+1}^{k-1} \neq B_{q+1}^k.$$

Suppose for $k>1$,

$$\begin{aligned} B_{q+1}^{k-1} &= [A_1, A_2, \ldots, A_{q+1}] \cap [A_{k+q}, A_{k+q+1}, \ldots, A_{k+n}] \\ &= [A_1, A_2, \ldots, A_q] \cap [A_{k+q}, A_{k+q+1}, \ldots, A_{k+n+1}] = B_q^k. \end{aligned}$$

Then the intersection $[A_1, A_2, \ldots, A_q] \cap [A_{k+q}, A_{k+q+1}, \ldots, A_{k+n}]$ cannot be empty as it contains B_q^k. However, this intersection is the intersection of two spaces of dimension $q-1$ and $n-q$ which together generate the whole space. If d be its dimension it follows from the incidence relation $(q-1)+(n-q)=n-d$ that $d=-1$ which means the intersection is empty. This contradiction proves $B_{q+1}^{k-1} \neq B_q^k$, $k>1$. We now suppose, for $k>1$, that

$$\begin{aligned} B_{q+1}^{k-1} &= [A_1, A_2, \ldots, A_{q+1}] \cap [A_{k+q}, A_{k+q+1}, \ldots, A_{k+n}] \\ &= [A_1, A_2, \ldots, A_{q+1}] \cap [A_{k+q+1}, A_{k+q+2}, \ldots, A_{k+n+1}] = B_{q+1}^k. \end{aligned}$$

It follows from 2.1 that

$$[A_{k+q}, A_{k+q+1}, \ldots, A_{k+n}] \cap [A_{k+q+1}, A_{k+q+2}, \ldots, A_{k+n+1}] = [A_{k+q+1}, A_{k+q+2}, \ldots, A_{k+n}] .$$

Hence

$$B_{q+1}^{k-1} = B_{q+1}^{k} \in [A_1, A_2, \ldots, A_{q+1}] \cap [A_{k+q+1}, A_{k+q+2}, \ldots, A_{k+n}] .$$

This intersection is the intersection of two spaces of dimension q and $n-q-1$ which together generate the whole space. As before it follows from the incidence relation that it is empty and so $B_{q+1}^{k-1} \neq B_{q+1}^{k}$. Finally let $k=1$. In this case $B_{q+1}^{0} = A_{q+1} = B_q^1$ would imply $A_{q+1} \in [A_1, A_2, \ldots, A_q]$ which is impossible because of 2.1. Likewise $B_{q+1}^{0} = B_{q+1}^{1}$ would imply $A_{q+1} \in [A_{2+q}, A_{3+q}, \ldots, A_{n+k+1}]$ which is also impossible because of 2.1. This completes the proof.

These results enable us to introduce

4.3 Definition of $E_k(\pi_n)$. *If $A_1, A_2, \ldots, A_r$, $r > n+1$, are the vertices of a closed polygon π_n and B_q^k, $1 \leqq k \leqq r-n-1$, $1 \leqq q \leqq n+1$, are defined in 4.1 then $B_q^k B_{q+1}^k$, $1 \leqq q \leqq n$, is defined to be the line segment in $[B_q^k, B_{q+1}^k]$ which contains B_{q+1}^{k-1}.*

$E_k(\pi_n)$ is defined to be π_n if $k=0$ and the polygon with sides $B_1^k B_2^k$, $B_2^k B_3^k, \ldots, B_n^k B_{n+1}^k$ $(= B_n^k A_{n+k+1})$, $A_{n+k+1} A_{n+k+2}, \ldots, A_r A_1$ $(= A_r B_1^k)$ if $1 \leqq k \leqq r-n-1$.

4.4 *If, for $1 \leqq k \leqq r-n-1$ and $n \geqq 2$, t_i^k, $2 \leqq i \leqq n$, is the triangle consisting of the side $B_i^{k-1} B_{i+1}^{k-1}$ of $E_{k-1}(\pi_n)$ and the segments $B_i^{k-1} B_i^k$, $B_i^k B_{i+1}^{k-1}$ of the sides $B_{i-1}^k B_i^k$, $B_i^k B_{i+1}^k$ of $E_k(\pi_n)$, then t_i^k is an even triangle.*

Proof. Let $A_1, A_2, \ldots, A_r$ be the vertices of the closed polygon π_n. We first consider the projection of $E_k(\pi_n)$ from the vertex A_1 for $n \geqq 2$. If $\dot{A}_{i-1}$ be the projection of A_i from A_1, $i \neq 1$, then the $r-1$ points $\dot{A}_1, \dot{A}_2, \ldots, \dot{A}_{r-1}$ are the vertices of a closed polygon π_{n-1}. Each side $\dot{A}_i \dot{A}_{i+1}$, $1 \leqq i \leqq r-2$, of π_{n-1} is the projection of the side $A_{i+1} A_{i+2}$ of π_n while the side $\dot{A}_{r-1} \dot{A}_1$ is defined as in 2.2. If $\dot{B}_q^k$ is defined to be

$$\dot{B}_q^k = [\dot{A}_1, \dot{A}_2, \ldots, \dot{A}_q] \cap [\dot{A}_{k+q}, \dot{A}_{k+q+1}, \ldots, \dot{A}_{n+k}] ,$$

$1 \leqq k \leqq (r-1)-(n-1)-1 = r-n-1$, $1 \leqq q \leqq n$, then $\dot{B}_1^k, \dot{B}_2^k, \ldots, \dot{B}_n^k, \dot{A}_{n+k}, \ldots, \dot{A}_{r-1}$ are the vertices of $E_k(\pi_{n-1})$ in accordance with 4.3. It follows from the definition of B_q^k in 4.1 that its projection from A_1 is $\dot{B}_{q-1}^k$, $2 \leqq q \leqq n+1$. As $B_1^k = A_1$, the side $B_1^k B_2^k$ is projected into the single point $\dot{B}_1^k$. To consider the sides $B_q^k B_{q+1}^k$, $2 \leqq q \leqq n$, we note that, for $1 < k \leqq r-n-1$, the interior point

$$B_{q+1}^{k-1} = [A_1, A_2, \ldots, A_{q+1}] \cap [A_{k+q}, A_{k+q+1}, \ldots, A_{n+k}]$$

of $B_q^k B_{q+1}^k$ is projected into the interior point

$$\dot{B}_q^{k-1} = [\dot{A}_1, \dot{A}_2, \ldots, \dot{A}_q] \cap [\dot{A}_{k+q-1}, \dot{A}_{k+q}, \ldots, \dot{A}_{k+n-1}]$$

of $\dot{B}_{q-1}^k \dot{B}_q^k$ while for $k=1$ the interior point $B_{q+1}^0 = A_{q+1}$ of $B_q^1 B_{q+1}^1$ is projected into the interior point $\dot{B}_q^0 = \dot{A}_q$ of $\dot{B}_{q-1}^1 \dot{B}_q^1$. Thus $B_q^k B_{q+1}^k$ is projected into the line segment with endpoints $\dot{B}_{q-1}^k$, $\dot{B}_q^k$ which contains the point $\dot{B}_q^{k-1}$ in its interior which is, by the definition, the side $\dot{B}_{q-1}^k \dot{B}_q^k$ of $E_k(\pi_{n-1})$, $2 \leqq q \leqq n$, $1 \leqq k \leqq n-r-1$. Each side $A_i A_{i+1}$, $n+k+1 \leqq i \leqq r-1$, is projected into the side $\dot{A}_{i-1} \dot{A}_i$ of $E_k(\pi_{n-1})$ by 2.2. The remaining side $A_r A_1$ of $E_k(\pi_n)$ is projected into the single vertex $\dot{A}_{r-1}$. Thus the projection of $E_k(\pi_n)$ from A_1 is the polygon $E_k(\pi_{n-1})$ without the side $\dot{A}_{r-1} \dot{A}_1$.

Following the definition 4.1 the points B_2^k, $1 \leqq k \leqq r-n-1$, are on the line $[A_1, A_2]$. We now prove that the points A_1, A_2, B_2^{k-1}, B_2^k follow in order in the direction from A_1 to A_2 in the side $A_1 A_2$ of π_n. If $1 \leqq k \leqq r-n-1$, none of the three vertices A_r, A_1, A_2 can occur in the $n-2$-space

$$L = [A_{2+k}, A_{3+k}, \ldots, A_{n+k}]$$

and so the polygon arc of all vertices not in L has the form $A_{n+k+1} A_{n+k+2} \ldots A_r A_1 A_2 \ldots A_{1+k}$. If s is a point of this arc then $s \notin L$. This follows from 2.1 if s is a vertex. It is also true if s is not a vertex, for otherwise the space generated by L and the two endpoints of the side containing s would have at most dimension $n-1$ and contain $n+1$ vertices of π_n. This also proves that $[L,s]$ is a hyperplane which intersects π_n in s. As $[L,s]$ intersects π_n in the $n-1$ vertices in L, it follows from the order of π_n that two different positions of s on the polygon arc define different hyperplanes $[L,s]$. Hence, as s runs continuously and monotonously through the polygon arc from A_{n+k+1} through A_1, A_2 to A_{1+k}, $[L,s]$ moves continuously and monotonously within the hyperplane pencil through L. Therefore, as $[L,s]$ contains exactly one point of the line $[A_1, A_2]$, the intersection $[A_1, A_2] \cap [L,s]$ runs continuously and monotonously on $[A_1, A_2]$ from

$$[A_1, A_2] \cap [A_{2+k}, A_{3+k}, \ldots, A_{n+k+1}] = B_2^k$$

through A_1 and A_2 to

$$[A_1, A_2] \cap [A_{1+k}, A_{2+k}, \ldots, A_{n+k}] = B_2^{k-1} .$$

This proves that A_1, A_2, B_2^{k-1}, B_2^k follow on $[A_1, A_2]$ in the order defined by the direction A_1 to A_2 in the side $A_1 A_2$.

In particular the points A_1, A_2, B_2^1 follow in the order A_1 to A_2 on $A_1 A_2$. By 4.3 the side $B_1^1 B_2^1 = A_1 B_2^1$ of $E_1(\pi_n)$ is defined to contain $B_2^0 = A_2$ as an interior point. Therefore the segment $B_1^1 B_2^1$ contains the side $B_1^0 B_2^0 = A_1 A_2$ of $E_0(\pi_n) = \pi_n$ as a subsegment. More generally as, A_1, A_2,

B_2^{k-1}, B_2^k follow in the order A_1 to A_2 and B_2^{k-1} is an interior point of the side $B_1^kB_2^k$ of $E_k(\pi_n)$ it follows that $B_1^kB_2^k$ contains the side $B_1^{k-1}B_2^{k-1}$ of $E_{k-1}(\pi_n)$ as a subsegment, $1 \leqq k \leqq r-n-1$. If this result is applied to π_{n-1} it follows that the side $\dot{B}_1^k\dot{B}_2^k$ of $E_k(\pi_{n-1})$ contains the side $\dot{B}_1^{k-1}\dot{B}_2^{k-1}$ of $E_{k-1}(\pi_{n-1})$ as a subsegment.

By the result of the first paragraph the projection of the segment $B_2^kB_3^{k-1}$ of the side $B_2^kB_3^k$ of $E_k(\pi_n)$ is the segment of the side $\dot{B}_1^k\dot{B}_2^k$ of $E_k(\pi_{n-1})$ with endpoints $\dot{B}_1^k = \dot{A}_1 = \dot{B}_1^{k-1}$ and $\dot{B}_2^{k-1}$. This by the paragraph above is the side $\dot{B}_1^{k-1}\dot{B}_2^{k-1}$ of $E_{k-1}(\pi_{n-1})$. As the projection of the side $B_2^{k-1}B_3^{k-1}$ of t_2^k is also $\dot{B}_1^{k-1}\dot{B}_2^{k-1}$ it follows that t_2^k is projected into this single segment. The centre of projection A_1 is not located on a side of t_2^k as it is on the line $[A_1, A_2]$ containing the side $B_2^{k-1}B_2^k$ but not within $B_2^{k-1}B_2^k$. Therefore t_2^k is even because the projection of an odd triangle from a point not on the triangle is the full projective line. In particular this proves the result for $n=2$.

We assume it true for all polygons π_{n-1}, $n>2$, and proceed by induction. We need now only consider triangles t_i^k with $i>2$. Let $B_i^{k-1}B_{i+1}^{k-1}$, $B_i^{k-1}B_i^k$, $B_i^kB_{i+1}^{k-1}$ be the sides of such a triangle t_i^k. Then by the result of the first paragraph the projection of t_i^k is a triangle $\dot{t}_i^k$ defined for the projection $E_k(\pi_{n-1})$. By the induction assumption $\dot{t}_i^k$ is even, hence t_i^k is even. This completes the proof.

4.5 *$E_{r-n-1}(\pi_n)$ is not within a hyperplane, has $n+1$ vertices and order n.*

Proof. We first show that

$$[B_{n+1-j}^k, B_{n+2-j}^k, \ldots, B_{n+1}^k] = [A_{n+k+1-j}, A_{n+k+2-j}, \ldots, A_{n+k+1}],$$

$0 \leqq k \leqq r-n-1$, $0 \leqq j \leqq n-1$. For $j=0$ this is an immediate consequence of 4.2. We assume the result to be true for $j-1$, $j>0$, and proceed by induction. According to the induction assumption

$$[B_{n+2-j}^k, B_{n+3-j}^k, \ldots, B_{n+1}^k] = [A_{n+k+2-j}, A_{n+k+3-j}, \ldots, A_{n+k+1}].$$

It follows from the definition 4.1 that

$$B_{n+1-j}^k \in [A_{n+k+1-j}, A_{n+k+2-j}, \ldots, A_{n+k+1}]$$

and so

$$[B_{n+1-j}^k, B_{n+2-j}^k, \ldots, B_{n+1}^k] \subseteq [A_{n+k+1-j}, A_{n+k+2-j}, \ldots, A_{n+k+1}].$$

However

$$B_{n+1-j}^k \notin [A_{n+k+2-j}, A_{n+k+3-j}, \ldots, A_{n+k+1}].$$

For otherwise, as by 4.1 $B_{n+1-j}^k \in [A_1, A_2, \ldots, A_{n+1-j}]$, the intersection

$$[A_1, A_2, \ldots, A_{n+1-j}] \cap [A_{n+k+2-j}, A_{n+k+3-j}, \ldots, A_{n+k+1}]$$

would not be empty. It follows from 2.1 that this intersection is an intersection of spaces of dimension $n-j$ and $j-1$ which together generate the projective n-space. Hence, if d is the dimension of the intersection, it follows from the incidence relation $(n-j)+(j-1)=n-d$ that $d=-1$ which means that the intersection is empty. This contradiction proves

$$B_{n+1-j} \notin [A_{n+k+2-j}, A_{n+k+3-j}, \ldots, A_{n+k+1}] .$$

It now follows that

$$[B^k_{n+1-j}, B^k_{n+2-j}, \ldots, B^k_{n+1}] = [A_{n+k+1-j}, A_{n+k+2-j}, \ldots, A_{n+k+1}]$$

as the left hand side is included in the right hand side and both sides have the same dimension. We now specialize j to be $n-1$. As in 4.1 we assume $r>n+1$ and so $r-n-2 \geqq 0$. We may therefore specialize k to be $r-n-2$. The result becomes

$$[B_2^{r-n-2}, B_3^{r-n-2}, \ldots, B_{n+1}^{r-n-2}] = [A_{r-n}, A_{r-n+1}, \ldots, A_{r-1}] .$$

The vertices of $E_{n-r-1}(\pi_n)$ are defined to be $B_1^{r-n-1}(=A_1)$, B_2^{n-r-1}, $\ldots$, $B_{n+1}^{r-n-1}(=A_r)$. Following the definition 4.3 each side $B_i^{r-n-1}B_{i+1}^{r-n-1}$ contains the interior point B_{i+1}^{r-n-2}, $1 \leqq i \leqq n$. By the previous paragraph,

$$[B_2^{r-n-2}, B_3^{r-n-2}, \ldots, B_{n+1}^{r-n-2}] = [A_{r-n}, A_{r-n+1}, \ldots, A_{r-1}] .$$

It follows, then, that the $n+1$ vertices of $E_{r-n-1}(\pi_n)$ generate the space $[A_{r-n}, A_{r-n+1}, \ldots, A_r]$, that is, the full projective n-space. Consequently $E_{r-n-1}(\pi_n)$ cannot be included in a hyperplane. Moreover the hyperplane $[B_2^{r-n-2}, B_3^{r-n-2}, \ldots, B_{n+1}^{r-n-2}]$ generated by n points, one interior to each of n consecutive sides of $E_{r-n-1}(\pi_n)$, cannot intersect the side A_rA_1 of $E_{r-n-1}(\pi_n)$ as it already intersects π_n in the n vertices A_{r-n}, A_{r-n-1}, $\ldots$, A_{r-1}. It follows, then, from 2.3 that $E_{r-n-1}(\pi_n)$ has order n. This completes the proof.

4.6 *For* $n \geqq 2$ *the polygon* $E_k(\pi_n)$ *has order* n. $E_{k-1}(\pi_n)$ *is inscribed in* $E_k(\pi_n)$ *in the sense of* 3.1, $1 \leqq k \leqq r-n-1$.

PROOF. By 4.5 $E_{r-n-1}(\pi_n)$ has order n. We assume $E_k(\pi_n)$ has order n, $1<k \leqq r-n-1$, and proceed by induction. Each of the n consecutive sides $B_i^kB_{i+1}^k$, $1 \leqq i \leqq n$, of $E_k(\pi_n)$ contains the vertex B_{i+1}^{k-1} of the polygon $E_{k-1}(\pi_n)$ in accordance with 4.3. By 4.4 the triangle t_i^k which consists of the side $B_i^{k-1}B_{i+1}^{k-1}$ of $E_{k-1}(\pi_n)$ and the segments $B_i^{k-1}B_i^k$, $B_i^kB_{i+1}^{k-1}$ of the sides $B_{i-1}^kB_i^k$, $B_i^kB_{i+1}^k$ of $E_k(\pi_n)$, respectively, is even, $2 \leqq i \leqq n$. This means that the polygon $E_{k-1}(\pi_n)$ which is defined to have the sides $B_1^{k-1}B_2^{k-1}\,(=A_1B_2^{k-1})$, $B_2^{k-1}B_3^{k-1}$, $\ldots$, $B_n^{k-1}B_{n+1}^{k-1}$, $B_{n+1}^{k-1}B_{n+1}^k\,(=A_{n+k}A_{n+k+1})$, $A_{n+k+1}A_{n+k+2}$, $\ldots$, A_rA_1 is inscribed in $E_k(\pi_n)$ following 3.1. By the in-

duction assumption $E_k(\pi_n)$ has order n. Therefore by 3.2 $E_{k-1}(\pi_n)$ has order n and the proof is complete.

5. The osculating hyperplanes of π_n.

5.1 DEFINITION. *If, for $n \geqq 2$, A_i, A_{i+1}, $\ldots$, A_{i+n} are vertices of a polygon π_n and $A_{i+n}A_i$ is the straight line segment for which the sides A_iA_{i+1}, $A_{i+1}A_{i+2}$, $\ldots$, $A_{i+n-1}A_{i+n}$ of π_n and $A_{i+n}A_i$ form a closed polygon of order n, then the hyperplanes which contain A_{i+1}, A_{i+2}, $\ldots$, A_{i+n-1} but do not contain an interior point of $A_{i+n}A_i$ are defined to be osculating hyperplanes of π_n.*

It follows from 2.3 that the segment $A_{i+n}A_i$ can always be constructed. This segment is unique.

If π_n is closed, the hyperplanes

$$V_1 = [A_1, A_2, \ldots, A_n],\ V_2 = [A_2, A_3, \ldots, A_{n+1}], \ldots, \quad V_r = [A_r, A_1, \ldots, A_{n-1}]$$

satisfy the definition for the osculating hyperplanes. These hyperplanes will be called the *vertex hyperplanes* of π_n.

The osculating hyperplanes which contain

$$V_i \cap V_{i+1} = [A_{i+1}, A_{i+2}, \ldots, A_{i+n-1}]$$

and an interior point of the segment complementary to $A_{i+n}A_i$ form a linear segment of the hyperplane pencil through $[A_{i+1}, A_{i+2}, \ldots, A_{i+n-1}]$ bounded by the two hyperplanes V_i, V_{i+1}. The hyperplanes of this segment will be denoted by V_iV_{i+1}.

5.2 *If, for $n \geqq 2$, π_n is a closed polygon with vertices A_1, A_2, $\ldots$, A_r, H is an osculating hyperplane interior to the segment V_1V_2 and H' is a hyperplane which approaches H in such a way that it always contains a point interior to each side A_iA_{i+1}, $1 \leqq i \leqq n-1$, then H' intersects π_n in these points and also in an additional point interior to the side A_nA_{n+1} providing it is sufficiently close to H.*

PROOF. Let B_{i+1} be the point interior to A_iA_{i+1} which is contained within H', $1 \leqq i \leqq n-1$. It follows from the existence of these points that if H' contains one of the vertices A_1, A_2, $\ldots$, A_n then it contains all of them and so becomes the hyperplane $V_1 = [A_1, A_2, \ldots, A_n]$. By the hypothesis H is interior to the segment V_1V_2 and so $H \neq V_1$. Hence if H' is sufficiently close to H it cannot contain any vertex A_1, A_2, $\ldots$, A_n and so it intersects π_n in B_2, B_3, $\ldots$, B_n. If $A_{n+1}A_1$ be the segment defined in 5.1, let $-A_{n+1}A_1$ be the complement of $A_{n+1}A_1$ in the straight line $[A_{n+1}, A_1]$. As H is interior to V_1V_2, H intersects $-A_{n+1}A_1$ in an interior point and so H' intersects $-A_{n+1}A_1$ in an interior point if it is sufficiently close to H. If this position of H' did not intersect A_nA_{n+1} then by 2.3 the polygon with sides $-A_{n+1}A_1, A_1A_2, \ldots, A_nA_{n+1}$ would have order n. This is impossible as $A_{n+1}A_1$ is defined in 5.1 so that $A_{n+1}A_1, A_1A_2, \ldots, A_nA_{n+1}$ has order n. This completes the proof.

5.3 *If, for $n \geqq 2$, H is a hyperplane interior to the segment V_iV_{i+1} defined for a closed polygon π_n with vertices $A_1, A_2, \ldots, A_r$, then the two intersections*

$$\pi_n \cap H, \qquad \pi_n \cap V_i \cap V_{i+1},$$

$1 \leqq i \leqq r$, *are the same point set.*

PROOF. Without restriction in generality we may assume $i=1$ as, because π_n is closed, the vertex notation can be adjusted so that V_iV_{i+1} becomes V_1V_2. The vertex hyperplanes V_1, V_2 are defined, in 5.1, to be $[A_1, A_2, \ldots, A_n]$ and $[A_2, A_3, \ldots, A_{n+1}]$ respectively. Therefore $V_1 \cap V_2 = [A_2, A_3, \ldots, A_n]$. Consequently the set of points α of the polygon arc $A_2A_3 \ldots A_n$ of π_n is contained within $V_1 \cap V_2$. Hence, as H is defined to contain $V_1 \cap V_2$,

$$\alpha \subseteq \pi_n \cap V_1 \cap V_2 \subseteq \pi_n \cap H .$$

To prove the result it is therefore sufficient to show that

$$\pi_n \cap H \subseteq \alpha .$$

No point of the side A_1A_2 other than A_2 can be within H for otherwise H would be $[A_1, V_1 \cap V_2] = V_1$ contrary to the assumption that H is interior to the segment V_1V_2. Similarly no point of A_nA_{n+1} other than A_n can be within H. It remains to show that H cannot contain any point s interior to the polygon arc $A_{n+1}A_{n+2} \ldots A_rA_1$. Suppose such a point s exists. Then $s \notin [A_2, A_3, \ldots, A_n]$. This is clear if s is a vertex. If s is interior to a side A_jA_{j+1} of the arc containing s then it is also true for otherwise $[A_2, A_3, \ldots, A_n, A_j, A_{j+1}]$ would have at most dimension $n-1$ and contain $n+1$ distinct vertices in contradiction to 2.1. It follows then that $H = [s, A_2, A_3, \ldots, A_n]$ and also that H intersects π_n in s. Let B_{i+1} be a point interior to the side A_iA_{i+1}, $1 \leqq i \leqq n-1$. If $B_3, B_4, \ldots, B_n$ remain fixed but B_2 approaches A_2 then $[B_2, B_3, \ldots, B_n]$ approaches

$$[A_2, B_3, B_4, \ldots, B_n] = [A_2, A_3, \ldots, A_n] .$$

Hence $[s, B_2, B_3, \ldots, B_n]$ approaches H and, if B_2 is sufficiently close to A_2, is therefore a hyperplane which intersects π_n in s. By 5.2

$[s,B_2,B_3,\ldots,B_n]$, besides intersecting π_n in $B_2,B_3,\ldots,B_n$, also intersects π_n in a point B_{n+1} interior to the side A_nA_{n+1} again provided B_2 is sufficiently close to A_2. As $s \notin A_nA_{n+1}$, $[s,B_2,B_3,\ldots,B_n]$ intersects π_n in the $n+1$ distinct points s, B_2, B_3, $\ldots$, B_{n+1} which is impossible because of the order of π_n. Hence H cannot contain a point s interior to the arc $A_{n+1}A_{n+2}\ldots A_rA_1$. Therefore $\pi_n \cap H$ is contained in the arc α, and the result is proved.

5.4 *If, for $n \geqq 2$, $A_1, A_2, \ldots, A_r$ are the vertices of a closed polygon π_n and $B_{i+1} \in A_iA_{i+1}$, $1 \leqq i \leqq n$ are vertices of a polygon $I(\pi_n)$ inscribed in π_n, then the vertex hyperplanes of $I(\pi_n)$ are the vertex hyperplanes of π_n together with the hyperplane $[B_2,B_3,\ldots,B_{n+1}]$. All the osculating hyperplanes of π_n except those interior to the segment V_1V_2 are likewise osculating hyperplanes of $I(\pi_n)$.*

PROOF. As A_1, B_2, B_3, $\ldots$, B_{n+1}, A_{n+1}, $\ldots$, A_r are consecutive vertices of $I(\pi_n)$,

$$\begin{array}{l}[A_1,B_2,\ldots,B_n]\,,\\ [B_2,B_3,\ldots,B_{n+1}]\,,\\ [B_3,B_4,\ldots,B_{n+1},A_{n+1}]\,,\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ [A_r,A_1,B_2,\ldots,B_{n-1}]\end{array}$$

are the vertex hyperplanes of $I(\pi_n)$. As B_{i+1} by its definition is an interior point of the segment A_iA_{i+1}, $1 \leqq i \leqq n$, the above hyperplanes are, respectively,

$$\begin{array}{l}[A_1,A_2,\ldots,A_n]\,,\\ [B_2,B_3,\ldots,B_{n+1}]\,,\\ [A_2,A_3,\ldots,A_{n+1}]\,,\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ [A_r,A_1,A_2,\ldots,A_{n-1}]\,.\end{array}$$

These are exactly the vertex hyperplanes of π_n together with the additional hyperplane $[B_2,B_3,\ldots,B_n]$. Thus the first part of the result is established.

This means, for $2 \leqq i \leqq r$, that V_i, V_{i+1} besides being consecutive vertex hyperplanes of π_n are also consecutive vertex hyperplanes of $I(\pi_n)$. By 5.1 V_iV_{i+1} is one of the two segments of the hyperplane pencil through $V_i \cap V_{i+1}$ bounded by V_i and V_{i+1}. To prove that V_iV_{i+1} is the same whether it is defined for π_n or $I(\pi_n)$ it is sufficient to show a hyperplane H exists in the pencil which is different from V_i and V_{i+1} and is neither an osculating hyperplane of π_n nor of $I(\pi_n)$. If the side $A_{i+n}A_{i+n+1}$ is also a side of $I(\pi_n)$ let s be any interior point of this side. If $A_{i+n}A_{i+n+1}$ is not a side of $I(\pi_n)$ then the vertex B_{i+n+1} is an interior point of this side. In this case s is defined to be B_{i+n+1}.

$$s \notin [A_{i+1}, A_{i+2}, \ldots, A_{i+n-1}] = V_i \cap V_{i+1}$$

for otherwise $[A_{i+1}, A_{i+2}, \ldots, A_{i+n+1}]$ would have dimension at most $n-1$ and contain $n+1$ vertices of π_n in contradiction to 2.1. Therefore $[s, A_{i+1}, A_{i+2}, \ldots, A_{i+n-1}] = H$ is a hyperplane. As H intersects $A_{i+n}A_{i+n+1}$ in the interior points s, $H \neq V_i$, $H \neq V_{i+1}$. As $s \notin V_i \cap V_{i+1}$, $s \in \pi_n$, $s \in I(\pi_n)$, it follows from 5.3 that H is not an osculating hyperplane of either π_n or $I(\pi_n)$. This proves that V_iV_{i+1}, $2 \leqq i \leqq r$, is the same whether it is defined for π_n or $I(\pi_n)$ as its complementary segment consists of hyperplanes which are not osculating hyperplanes of either π_n or $I(\pi_n)$. The proof is now complete.

5.5 *No space point is within more than n vertex hyperplanes of a polygon π_n.*

PROOF. As the result is trivial for $n=1$ we assume $n \geqq 2$. Let $A_1, A_2, \ldots, A_r$ be the vertices of π_n. If r assumes its minimum value $n+1$ then, as π_n satisfies the dimension condition, $A_1, A_2, \ldots, A_{n+1}$ are the vertices of an n-simplex while the vertex hyperplanes

$$\begin{aligned} V_1 &= [A_1, A_2, \ldots, A_n], \\ V_2 &= [A_2, A_n, \ldots, A_{n+1}], \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ V_{n+1} &= [A_{n+1}, A_1, \ldots, A_{n-1}] \end{aligned}$$

of π_n are the face hyperplanes of this n-simplex. As no point can be within all the face hyperplanes of a simplex no point can be within the $n+1$ vertex hyperplanes of π_n. This proves the result if $r=n+1$. We assume it is true for all polygons π_n with $r-1$ sides, $r > n+1$, and proceed by induction.

Now let Q be a given space point. As $r > n+1$, $E_1(\pi_n)$ is defined as in 4.3. By 4.6 $E_1(\pi_n)$ has order n while $E_0(\pi_n)$, i.e. π_n, is inscribed in $E_1(\pi_n)$. As $E_1(\pi_n)$ has one vertex less than π_n the induction assumption may be applied to it. Hence Q is included in at most n vertex hyperplanes of $E_1(\pi_n)$. As $E_1(\pi_n)$ has at least $n+1$ vertices this means that at least one vertex hyperplane of $E_1(\pi_n)$ exists which does not contain Q. By 5.4 each vertex hyperplane of $E_1(\pi_n)$ is also a vertex hyperplane of π_n. Hence at least one vertex hyperplane of π_n exists which does not contain Q. By adjusting the vertex notation, if necessary, it may be assumed that

$[A_2, A_3, \ldots, A_{n+1}]$ is such a vertex hyperplane. As A_{i+1} is an interior point of the side $B_i^1 B_{i+1}^1$, $1 \leqq i \leqq n$, of $E_1(\pi_n)$ it follows from 5.4 that the vertex hyperplanes of the polygon π_n inscribed in $E_1(\pi_n)$ are the vertex hyperplanes of $E_1(\pi_n)$ together with $[A_2, A_3, \ldots, A_{n+1}]$. By the induction assumption Q is contained within at most n vertex hyperplanes of $E_1(\pi_n)$. As $Q \notin [A_2, A_3, \ldots, A_{n+1}]$, Q is within at most n vertex hyperplanes of π_n. This completes the proof.

5.6 (The duality theorem). *If, for* $n \geqq 2$, $V_1, V_2, \ldots, V_r$ *are the vertex hyperplanes of a closed polygon* π_n *then the dual of the system* Π *of all the hyperplanes of the segments* $V_1V_2, V_2V_3, \ldots, V_rV_1$ *is also a closed polygon of order n which satisfies the dimension condition.*

Proof. It follows from 5.5 that at most n of the hyperplanes $V_1, V_2, \ldots, V_r$ of π_n can pass through a given point. Because π_n satisfies the dimension condition, $r > n$ and so not all of the hyperplanes of Π can pass through a given point. Therefore the dual of Π satisfies the dimension condition.

To show that the dual of the system Π has order n we must show, in accordance with the definition of an intersection point in 1, that a given space point Q is contained in at most n hyperplanes which belong to either of the following two types. The first type consists of the vertex hyperplanes of π_n while the second type consists of hyperplanes of the segments V_iV_{i+1} for which $Q \notin V_i$, $Q \notin V_{i+1}$. This is done by constructing a polygon $\bar{\pi}_n$ of order n so that there is a one to one correspondence between the hyperplanes of the above two types which contain Q and vertex hyperplanes of $\bar{\pi}_n$ which contain Q. By applying 5.5 to $\bar{\pi}_n$ it follows that there are at most n vertex hyperplanes of $\bar{\pi}_n$ which contain Q. Hence there are at most n hyperplanes of the above two types which contain Q.

To complete the proof it only remains to construct a polygon $\bar{\pi}_n$. If no hyperplanes of the second type contain Q then $\bar{\pi}_n$ is defined to be π_n. Suppose then that a hyperplane H of the second type contains Q. The vertex notation may be adjusted so that H belongs to the segment V_1V_2. As H is of the second type $Q \notin V_1 \cap V_2$, and so $H = [V_1 \cap V_2, Q]$. Now let B_{i+1} be a point in the interior of the side A_iA_{i+1}, $1 \leqq i \leqq n-1$. If $B_3, B_4, \ldots, B_n$ remain fixed but B_2 approaches A_2 then $[B_2, B_3, \ldots, B_n, Q]$ approaches

$$[A_2, B_3, \ldots, B_n, Q] = [A_2, A_3, \ldots, A_n, Q] = [V_1 \cap V_2, Q] = H\ .$$

Therefore by 5.2 if B_2 is sufficiently close to A_2 then $[B_2, B_3, \ldots, B_n, Q]$ is a hyperplane which intersects π_n in a point B_{n+1} interior to A_nA_{n+1}

as well as in $B_2, B_3, \ldots, B_n$. Let $I(\pi_n)$ be the polygon inscribed in π_n defined as in 3.1 for the points $B_2, B_3, \ldots, B_{n+1}$. By 3.2 $I(\pi_n)$ has order n and so $[B_2, B_3, \ldots, B_{n+1}]$ is a hyperplane. Therefore $Q \in [B_2, B_3, \ldots, B_{n+1}]$. This hyperplane cannot be an osculating hyperplane of π_n as such hyperplanes intersect π_n only in vertices. All the osculating hyperplanes of π_n which do not belong to the segment $V_1 V_2$ are osculating hyperplanes of $I(\pi_n)$ by 5.4, and in particular a vertex hyperplane of π_n is likewise a vertex hyperplane of $I(\pi_n)$. We define H to correspond to the vertex hyperplane $[B_2, B_3, \ldots, B_{n+1}]$ of $I(\pi_n)$ and each of the other osculating hyperplanes of π_n which contains Q to correspond to the same osculating hyperplane of $I(\pi_n)$. As $[B_2, B_3, \ldots, B_{n+1}]$ cannot be an osculating hyperplane of π_n this is a one to one correspondence. Moreover each vertex hyperplane corresponds to a vertex hyperplane while one of the hyperplanes of the second type, namely H, corresponds to a vertex hyperplane. By 3.2 $I(\pi_n)$ has order n. Hence the above process may in turn be applied to $I(\pi_n)$ and then repeated until all the hyperplanes of the two types which contain Q are in a one to one correspondence with vertex hyperplanes of a polygon $\bar{\pi}_n$ of order n. With the construction of $\bar{\pi}_n$ the proof is complete.

6. An application to curves of order *n*.

The symbol C_n is used in this section to represent a curve in real projective n-space which is homeomorphic to a circle of circumference of length 1 such that no hyperplane contains more than n points of C_n. Let Q be a point of the circumference. The distance s measured along the circumference from a fixed point in a fixed direction to Q determines Q and so also the homeomorphic image of Q on C_n. Thus s serves as a coordinate to define the points of C_n. Accordingly the numbers s, com puted modulo 1, will be used to designate the points of C_n.

6.1 *If* $s_1, s_2, \ldots, s_{nq}$ *represent points on a curve* C_n *for which* $0 \leqq s_1 < s_2 < \ldots < s_{nq} < 1$, *and if the spaces*

$$[s_1, s_2, \ldots, s_n], \ \ldots, \ [s_{n(q-1)+1}, s_{n(q-1)+2}, \ldots, s_{nq}]$$

all pass through the same space point Q, *then* $q \leqq n$.

PROOF. If $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n$ are distinct points of C_n then it follows from the order of C_n that $[\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n]$ is a hyperplane and moreover $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n$ are the only points of C_n in this hyperplane. We now inscribe a polygon of order n in C_n with vertices $s_1, s_2, \ldots, s_{nq}$. We may assume $q > 1$ for otherwise the result is trivial. It is convenient to define

$s_{nq+1}=s_1$. Each arc of points s with $s_i \leqq s \leqq s_{i+1}$ $1 \leqq i \leqq nq$, can be realized in an affine space obtained by removing the hyperplane generated by n distinct points from the arc complementary to the arc $s_i \leqq s \leqq s_{i+1}$ from the projective space. We define the line segment $s_i s_{i+1}$ to be the straight line segment joining s_i and s_{i+1} in this affine space. π is now defined to be the closed polygon with sides $s_i s_{i+1}$, $1 \leqq i \leqq nq$. As $q > 1$ this polygon satisfies the dimension condition. To prove it has order n let H be any hyperplane of the original projective n-space. If H intersects π in a vertex s_j this vertex is also a point of C_n. If it intersects π in an interior point of side $s_i s_{i+1}$ then as this side together with the corresponding arc $s_i < s < s_{i+1}$ form a closed continuous curve within an affine space, H must intersect the arc $s_i < s < s_{i+1}$ at least once. Hence the number of intersection points of H and π cannot exceed the number of intersection points of H and C_n. As C_n has order n this proves that π has order n.

The q hyperplanes

$$[s_1, s_2, \ldots, s_n],\ [s_{n+1}, s_{n+2}, \ldots, s_{2n}],\ \ldots,\ [s_{n(q-1)+1}, s_{n(q-1)+2}, \ldots, s_{nq}]$$

are all vertex hyperplanes of π. If these all pass through a point Q, then by 5.5 there are at most n of them. Thus the result is proved.

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