MATH. SCAND. 6 (1958), 80–86

IDENTITIES INVOLVING THE PARTITION FUNCTIONS $q(n)$ AND $q_0(n)$

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1. In a previous paper [1] I have, among other results, proved the following three identities involving $p(n)$, the number of unrestricted partitions of n:

$$3\sum_{n=0}^{\infty} p(5n+1)x^n \sum_{n=0}^{\infty} p(5n+2)x^n - 2\sum_{n=0}^{\infty} p(5n)x^n \sum_{n=0}^{\infty} p(5n+3)x^n = x\left(\sum_{n=0}^{\infty} p(5n+4)x^n\right)^2, \tag{1.1}$$

$$\sum_{n=0}^{\infty} p(5n)x^n \sum_{n=0}^{\infty} p(5n+2)x^n + x\sum_{n=0}^{\infty} p(5n+3)x^n \sum_{n=0}^{\infty} p(5n+4)x^n = 2\left(\sum_{n=0}^{\infty} p(5n+1)x^n\right)^2, \tag{1.2}$$

$$\sum_{n=0}^{\infty} p(5n+1)x^n \sum_{n=0}^{\infty} p(5n+3)x^n + \sum_{n=0}^{\infty} p(5n)x^n \sum_{n=0}^{\infty} p(5n+4)x^n = 2\left(\sum_{n=0}^{\infty} p(5n+2)\,x^n\right)^2. \tag{1.3}$$

Let $q(n)$ denote the number of partitions of n into unequal parts (or, equivalently, the number of partitions into odd parts), and let $q_0(n)$ denote the number of partitions of n into odd and unequal parts (which is also the number of self-conjugate partitions). Thus

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{n=1}^{\infty} (1+x^n)\,,$$

$$\sum_{n=0}^{\infty} q_0(n)x^n = \prod_{n=1}^{\infty} (1+x^{2n-1})\,.$$

Tables of $q(n)$ and $q_0(n)$ up to $n=400$ have been computed by Watson [2]. Especially, we notice that $q_0(2)=0$.

Received December 10, 1957.

The purpose of the present paper is to prove the six identities

$$(1.4)\qquad \sum_{n=0}^{\infty} q(5n)x^n \sum_{n=0}^{\infty} q(5n+2)x^n = \left(\sum_{n=0}^{\infty} q(5n+1)x^n\right)^2 ,$$

$$(1.5)\qquad \sum_{n=0}^{\infty} q(5n)x^n \sum_{n=0}^{\infty} q(5n+4)x^n = \sum_{n=0}^{\infty} q(5n+1)x^n \sum_{n=0}^{\infty} q(5n+3)x^n ,$$

$$(1.6)\qquad \sum_{n=0}^{\infty} q(5n+2)x^n \sum_{n=0}^{\infty} q(5n+3)x^n = \sum_{n=0}^{\infty} q(5n+1)x^n \sum_{n=0}^{\infty} q(5n+4)x^n ,$$

$$(1.7)\qquad \sum_{n=0}^{\infty} q_0(5n+1)x^n \sum_{n=0}^{\infty} q_0(5n+7)x^n = \left(\sum_{n=0}^{\infty} q_0(5n+4)x^n\right)^2 ,$$

$$(1.8)\qquad \sum_{n=0}^{\infty} q_0(5n)x^n \sum_{n=0}^{\infty} q_0(5n+7)x^n = \sum_{n=0}^{\infty} q_0(5n+3)x^n \sum_{n=0}^{\infty} q_0(5n+4)x^n ,$$

$$(1.9)\qquad \sum_{n=0}^{\infty} q_0(5n+1)x^n \sum_{n=0}^{\infty} q_0(5n+3)x^n = \sum_{n=0}^{\infty} q_0(5n)x^n \sum_{n=0}^{\infty} q_0(5n+4)x^n .$$

2. We use the notation

$$\varphi(x) = \prod_{n=1}^{\infty} (1-x^n) .$$

Then we have

$$(2.1)\qquad \sum_{n=0}^{\infty} p(n)x^n = \varphi(x)^{-1} ,$$

$$(2.2)\qquad \sum_{n=0}^{\infty} q(n)x^n = \varphi(x^2)\varphi(x)^{-1} ,$$

$$(2.3)\qquad \sum_{n=0}^{\infty} q_0(n)x^n = \varphi(-x)\varphi(x^2)^{-1} .$$

Putting

$$P_s = \sum_{n=0}^{\infty} p(5n+s)x^{5n+s}, \qquad s = 0, 1, 2, 3, 4 ,$$

we get

$$(2.4)\qquad \sum_{n=0}^{\infty} p(n)x^n = P_0+P_1+P_2+P_3+P_4 ,$$

where the power series has been divided into five parts, each part consisting of terms whose exponents are congruent (mod 5), the residue class being indicated by the index. The same procedure is used in the equations (2.5)–(2.8), (2.10) and (2.11) below. Let

$$\sum_{n=0}^{\infty} q(n)x^n = Q_0+Q_1+Q_2+Q_3+Q_4\,, \tag{2.5}$$

$$\sum_{n=0}^{\infty} q_0(n)x^n = R_0+R_1+R_2+R_3+R_4\,, \tag{2.6}$$

$$\sum_{n=0}^{\infty} p(n)\,x^{2n} = S_0+S_1+S_2+S_3+S_4\,. \tag{2.7}$$

From Euler's identity

$$\varphi(x) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(3n+1)}$$

it follows that the power series expansion of $\varphi(x)$ contains no terms with exponents congruent to 3 or 4 (mod 5). Hence we can write

$$\varphi(x) = g_0+g_1+g_2\,. \tag{2.8}$$

Between these quantities there is a well-known relation, viz. (see [3, p. 102] and [1, p. 84])

$$g_0 g_2 = -g_1{}^2\,. \tag{2.9}$$

Further, we put

$$\varphi(x^2) = a_0+a_2+a_4\,, \tag{2.10}$$

$$\varphi(-x) = b_0+b_1+b_2\,. \tag{2.11}$$

We also need the identities

$$\varphi(x^2)^2\varphi(x)^{-1} = \sum_{n=0}^{\infty} x^{\frac{1}{2}n(n+1)}\,, \tag{2.12}$$

$$\varphi(-x)^2\varphi(x^2)^{-1} = \sum_{n=-\infty}^{\infty} x^{n^2}\,, \tag{2.13}$$

which follow from Jacobi's formula

$$\prod_{n=1}^{\infty} (1-z^{2n})(1+yz^{2n-1})(1+y^{-1}z^{2n-1}) = \sum_{n=-\infty}^{\infty} y^n z^{n^2}\,.$$

3. In this section we shall prove the identities (1.4)–(1.6). From (2.2) we get, using (2.1), (2.4), (2.5) and (2.10),

$$Q_0+Q_1+Q_2+Q_3+Q_4 = (a_0+a_2+a_4)(P_0+P_1+P_2+P_3+P_4)\,,$$

and hence

$$Q_0 = a_0P_0+a_2P_3+a_4P_1 , \tag{3.1}$$

$$Q_1 = a_0P_1+a_2P_4+a_4P_2 , \tag{3.2}$$

$$Q_2 = a_0P_2+a_2P_0+a_4P_3 , \tag{3.3}$$

$$Q_3 = a_0P_3+a_2P_1+a_4P_4 , \tag{3.4}$$

$$Q_4 = a_0P_4+a_2P_2+a_4P_0 . \tag{3.5}$$

Further, by (2.2), (2.5), (2.10) and (2.12)

$$(a_0+a_2+a_4)(Q_0+Q_1+Q_2+Q_3+Q_4) = \sum_{n=0}^{\infty} x^{\frac{1}{2}n(n+1)} .$$

This power series contains no terms with exponents congruent to 2 or 4 (mod 5), and therefore we have

$$a_0Q_2+a_2Q_0+a_4Q_3 = 0 , \tag{3.6}$$

$$a_0Q_4+a_2Q_2+a_4Q_0 = 0 . \tag{3.7}$$

The identities (1.1)–(1.3) can now be written

$$3P_1P_2-2P_0P_3- P_4^2 = 0 , \tag{3.8}$$

$$P_0P_2+ P_3P_4-2P_1^2 = 0 , \tag{3.9}$$

$$P_1P_3+ P_0P_4-2P_2^2 = 0 . \tag{3.10}$$

Finally, by (2.8)–(2.10) we get

$$a_0a_4 = -a_2^2 . \tag{3.11}$$

Now the identities (1.4)–(1.6) can be deduced from the system (3.1)–(3.11) by elimination of the quantities a_s and P_s. We proceed as follows:
From (3.1)–(3.3) we find, using (3.11)

$$\begin{aligned} Q_0Q_2-Q_1^2 = {} & a_0^2(P_0P_2-P_1^2) + a_2^2(P_1P_2-P_4^2) + a_4^2(P_1P_3-P_2^2) + \\ & + a_0a_2(P_0^2+P_2P_3-2P_1P_4) + a_2a_4(P_3^2+P_0P_1-2P_2P_4) . \end{aligned} \tag{3.12}$$

Inserting the expressions (3.1) and (3.3)–(3.5), we get from (3.6) and (3.7)

$$a_0^2P_2+2a_0a_2P_0-a_2^2P_3 = -a_4^2P_4-2a_2a_4P_1 , \tag{3.13}$$

$$a_4^2P_1+2a_2a_4P_3-a_2^2P_0 = -a_0^2P_4-2a_0a_2P_2 . \tag{3.14}$$

Multiplication of (3.13) and (3.14) yields

$$\begin{aligned} 0 = {} & -a_0^2P_0P_2 + a_2^2(5P_1P_2-3P_0P_3-P_4^2) - a_4^2P_1P_3 + \\ & +2a_0a_2(P_1P_4-P_2P_3-P_0^2) + 2a_2a_4(P_2P_4-P_0P_1-P_3^2) . \end{aligned} \tag{3.15}$$

Further we multiply (3.13) and (3.14) by P_0 and P_3 respectively, and add. Thus we obtain

$$0 = a_0^2(P_0P_2+P_3P_4) - 2a_2^2P_0P_3 + a_4^2(P_1P_3+P_0P_4) + \\ + 2a_0a_2(P_0^2+P_2P_3) + 2a_2a_4(P_3^2+P_0P_1)\,,$$

which by (3.9) and (3.10) reduces to

$$(3.16)\qquad 0 = a_0^2P_1^2 - a_2^2P_0P_3 + a_4^2P_2^2 + a_0a_2(P_0^2+P_2P_3) + \\ + a_2a_4(P_3^2+P_0P_1)\,.$$

Adding the equations (3.12), (3.15) and (3.16), we get

$$Q_0Q_2-Q_1^2 = 2a_2^2(3P_1P_2-2P_0P_3-P_4^2)\,.$$

Hence, by (3.8)

$$(3.17)\qquad Q_0Q_2 = Q_1^2\,,$$

which proves (1.4).

From (3.1), (3.2), (3.4) and (3.5) we find

$$(3.18)\quad Q_0Q_4-Q_1Q_3 = a_0^2(P_0P_4-P_1P_3) + a_2^2(2P_2P_3-P_1P_4-P_0^2) + \\ + a_4^2(P_0P_1-P_2P_4) + a_0a_2(P_0P_2-P_1^2) + a_2a_4(P_0P_3-P_4^2)\,.$$

We multiply the equations (3.13) and (3.14) by P_2 and P_0 respectively and subtract. Thus we obtain

$$(3.19)\quad 0 = a_0^2(P_2^2-P_0P_4) + a_2^2(P_0^2-P_2P_3) + a_4^2(P_2P_4-P_0P_1) + \\ + 2a_2a_4(P_1P_2-P_0P_3)\,.$$

Multiplying (3.19) by 2 and adding it to (3.18) we get, using (3.8) and (3.10)

$$(3.20)\quad Q_0Q_4-Q_1Q_3 = a_2^2(P_0^2-P_1P_4) + a_4^2(P_2P_4-P_0P_1) + \\ + a_0a_2(P_0P_2-P_1^2) + a_2a_4(P_1P_2-P_0P_3)\,.$$

Similarly we find

$$(3.21)\quad Q_2Q_3-Q_1Q_4 = a_0^2(P_1P_4-P_2P_3) + a_2^2(P_3^2-P_2P_4) + \\ + a_0a_2(P_1P_2-P_0P_3) + a_2a_4(P_1P_3-P_2^2)\,.$$

From (3.20) and (3.21) follows

$$a_0a_2^{-1}(Q_0Q_4-Q_1Q_3) + a_4a_2^{-1}(Q_2Q_3-Q_1Q_4) \\ = a_0^2(P_0P_2-P_1^2) + 2a_2^2(P_0P_3-P_1P_2) + a_4^2(P_1P_3-P_2^2) + \\ + a_0a_2(P_0^2+P_2P_3-2P_1P_4) + a_2a_4(P_3^2+P_0P_1-2P_2P_4) \\ = Q_0Q_2-Q_1^2\,,$$

by (3.12) and (3.8). Hence

$$a_0(Q_0Q_4-Q_1Q_3) + a_4(Q_2Q_3-Q_1Q_4) = 0 . \tag{3.22}$$

If we multiply this equation by Q_0 and replace Q_0Q_2 by Q_1^2, we get

$$(a_0Q_0-a_4Q_1)(Q_0Q_4-Q_1Q_3) = 0 .$$

Obviously, $a_0Q_0-a_4Q_1 \neq 0$. Hence

$$Q_0Q_4 = Q_1Q_3 , \tag{3.23}$$

and consequently, by (3.22) (or by (3.17))

$$Q_2Q_3 = Q_1Q_4 . \tag{3.24}$$

Thus the identities (1.5) and (1.6) are proved.

4. It remains to prove (1.7)–(1.9). From (2.3) we get, using (2.6), (2.7) and (2.11)

$$R_0+R_1+R_2+R_3+R_4 = (b_0+b_1+b_2)(S_0+S_1+S_2+S_3+S_4) ,$$

and hence

$$R_0 = b_0S_0+b_1S_4+b_2S_3 , \tag{4.1}$$

$$R_1 = b_0S_1+b_1S_0+b_2S_4 , \tag{4.2}$$

$$R_2 = b_0S_2+b_1S_1+b_2S_0 , \tag{4.3}$$

$$R_3 = b_0S_3+b_1S_2+b_2S_1 , \tag{4.4}$$

$$R_4 = b_0S_4+b_1S_3+b_2S_2 . \tag{4.5}$$

Further, by (2.3), (2.6), (2.11) and (2.13)

$$(b_0+b_1+b_2)(R_0+R_1+R_2+R_3+R_4) = \sum_{n=-\infty}^{\infty} x^{n^2} .$$

From this we conclude

$$b_0R_2+b_1R_1+b_2R_0 = 0 , \tag{4.6}$$

$$b_0R_3+b_1R_2+b_2R_1 = 0 . \tag{4.7}$$

Replacing x by x^2 in (3.8)–(3.10), we obtain

$$3S_2S_4-2S_0S_1- S_3^2 = 0 , \tag{4.8}$$

$$S_0S_4+ S_1S_3-2S_2^2 = 0 , \tag{4.9}$$

$$S_1S_2+ S_0S_3-2S_4^2 = 0 . \tag{4.10}$$

Finally, by (2.8), (2.9) and (2.11)

$$b_0 b_2 = -b_1^2 . \tag{4.11}$$

It is easily seen that the system (3.1)–(3.11) is changed into the system (4.1)–(4.11) by the substitutions

$P_0 \to S_0$	$Q_0 \to R_2$	$a_0 \to b_2$
$P_1 \to S_2$	$Q_1 \to R_4$	$a_2 \to b_1$
$P_2 \to S_4$	$Q_2 \to R_1$	$a_4 \to b_0$
$P_3 \to S_1$	$Q_3 \to R_3$	
$P_4 \to S_3$	$Q_4 \to R_0$	

Now, the equations (3.17), (3.23) and (3.24) were deduced from (3.1)–(3.11), and from these alone. Hence, the system (4.1)–(4.11) implies the equations

$$R_1 R_2 = R_4^2 ,$$
$$R_0 R_2 = R_3 R_4 ,$$
$$R_1 R_3 = R_0 R_4 ,$$

and thus the identities (1.7)–(1.9) are proved.

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