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ON THE HOMOTOPY STRUCTURE OF COVERINGS

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1. Introduction. The purpose of this paper is to develop a theory of coverings of topological spaces, which in some respects is very similar to the ordinary homotopy theory of spaces. The results obtained may be useful in the theory of critical points. The basic definition is:

DEFINITION 1. Let $\alpha=\{A_\nu: \nu\in\mathscr{I}\}$ and $\beta=\{B_\nu: \nu\in\mathscr{I}\}$ be coverings of the topological spaces X and Y, where $\mathscr{I}$ denotes an arbitrary index set. Then α and β are called *$\mathscr{I}$-homotopic* if there exist functions

$$f\colon\ X\to Y,\qquad g\colon\ Y\to X,\qquad F\colon\ X\times I\to X,\qquad G\colon\ Y\times I\to Y,$$

where I as usual denotes the unit interval, such that

1° $f|A_\nu$, $g|B_\nu$, $F|A_\nu\times I$ and $G|B_\nu\times I$ are continuous functions for each $\nu\in\mathscr{I}$ with images in B_ν, A_ν, A_ν and B_ν, respectively, and
2° $F(x,0)=gf(x)$, $G(y,0)=fg(y)$, $F(x,1)=x$, $G(y,1)=y$.

This definition gives an equivalence relation between coverings with the same index set $\mathscr{I}$. Of course two coverings will generally be called *homotopic* if a common index set can be chosen in such a way that the two coverings referred to the same index set $\mathscr{I}$ are $\mathscr{I}$-homotopic. But it is convenient to operate with the definition above, and throughout the paper we shall assume that $\mathscr{I}$ is a fixed index set.

If $\mathscr{I}$ consists of a single element, definition 1 reduces to the usual definition of homotopy type. Now the most interesting invariants of homotopy type of spaces can be dealt with in the following way:

To every space X is attached a semi-simplical complex $S(X)$ and a minimal subcomplex $M(X)$, and it can be shown that if X and Y are homotopic then $M(X)$ and $M(Y)$ are isomorphic complexes (see [2] or for a more algebraic treatment [6]). Therefore the semi-simplicial complex $M(X)$ is an invariant of homotopy type of spaces, and in fact it contains the singular homology groups, the homotopy groups and almost every other known combinatorial invariant of the space X. In order to get simpler invariants one can construct semi-simplicial complexes

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$$M^{(n)}(X), \qquad n = -1, 0, 1, \ldots, \infty ,$$

where $M^{(n)}(X)$ is obtained by identifying simplices with the same n-skeleton of $M(X)$. (See [6].)

Our aim is to give an analogous treatment of coverings with index set $\mathscr{I}$, which is equivalent to the theory mentioned above when $\mathscr{I}$ contains only one element. In section 2 the necessary algebraic tool is introduced and discussed. To a semi-simplicial complex K may be attached an index projection p; that is a semi-simplicial map

$$p\colon \; K \to \tilde{\mathscr{I}} ,$$

where $\tilde{\mathscr{I}}$ is a complete semi-simplicial barycentric subdivision of the abstract simplicial complex whose simplices are the finite, non void subsets of $\mathscr{I}$. Such pairs $\mathsf{K} = (K, p)$ are called $\mathscr{I}$-complexes, and they can be treated just like ordinary semi-simplicial complexes. A homotopy theory for $\mathscr{I}$-complexes is carried out, and theorems about the existence of minimal subcomplexes, which are invariants of homotopy type of $\mathscr{I}$-complexes, are proved. Section 3 is concerned with the $\mathscr{I}$-complex

$$\mathsf{S}(\alpha, X) = \big(S(\alpha, X), p\big) ,$$

which is an analogue of the total singular complex $S(X)$. The $\mathscr{I}$-complex $\mathsf{S}(\alpha, X)$ gives rise to a minimal $\mathscr{I}$-complex $\mathsf{M}(\alpha, X)$ which is an invariant of $\mathscr{I}$-homotopy type of coverings with index set $\mathscr{I}$. Then $\mathscr{I}$-complexes

$$\mathsf{M}^{(n)}(\alpha, X), \qquad n = -1, 0, 1, 2, \ldots, \infty ,$$

are introduced and discussed. Each of the $\mathscr{I}$-complexes $\mathsf{M}^{(n)}(\alpha, X)$ gives rise to a semi-simplicial complex $M^{(n)}(\alpha, X)$, and in section 4 the relations between the complexes $M^{(n)}(X)$ and $M^{(n)}(\alpha, X)$ are discussed, and an application of the whole theory is given.

2. $\mathscr{I}$-complexes. We shall use both s.s. and c.s.s. complexes (semi-simplicial and complete semi-simplicial complexes). If K is an s.s. or a c.s.s. complex, then K_n, $n = 0, 1, 2, \ldots$, denotes the set of its n-simplices; if $\sigma \in K_n$ then $\partial_i \sigma$, $i = 0, 1, \ldots, n$, is the i'th face of σ. If K is a c.s.s. complex and $\sigma \in K_n$ then $s_i \sigma$, $i = 0, 1, \ldots, n$, is the i'th degeneracy of σ. This notation follows [6]. In sections 2 and 3 only complete complexes are used, but the definitions of $\mathscr{I}$-complexes and $\mathscr{I}$-maps are easily adapted to s.s. complexes, and this is used in section 4. If K and L are c.s.s. complexes, then $K \times L$ is the c.s.s. complex defined by:

$$(K \times L)_n = K_n \times L_n, \quad \partial_i(\sigma, \tau) = (\partial_i \sigma, \partial_i \tau), \quad s_i(\sigma, \tau) = (s_i \sigma, s_i \tau) .$$

If $\sigma \in K$ then $\bar{\sigma}$ is the smallest subcomplex of K which contains σ, i.e. the smallest subset of K which is closed under application of face and degeneracy operators.

Δ_n is the standard geometrical n-simplex, the points of which are $(n+1)$-tuples of real numbers

$$\{t_0, t_1, \ldots, t_n\}, \qquad 0 \leqq t \leqq 1, \quad \sum_{i=0}^{n} t_i = 1 .$$

Δ_n is topologized by the metric

$$\varrho(\{t_0, t_1, \ldots, t_n\}, \{t_0', t_1', \ldots t_n'\}) = \left(\sum_{i=0}^{n} (t_i - t_i')^2\right)^{\frac{1}{2}} .$$

The vertices of Δ_n are:

$$v_0^n = \{1, 0, \ldots, 0\},\ v_1^n = \{0, 1, \ldots, 0\}, \ldots, v_n^n = \{0, 0, \ldots, 1\} .$$

There is a natural one to one correspondence between $(m+1)$-tuples of integers $\{a_0, a_1, \ldots, a_m\}$, $0 \leqq a_i \leqq n$, and simplicial maps $\zeta\colon \Delta_m \to \Delta_n$ given by $\zeta(v_i^m) = v_{a_i}^n$. An m-simplex of the c.s.s. complex $\Delta[n]$ is a simplicial map $\zeta\colon \Delta_m \to \Delta_n$ which corresponds to an $(m+1)$-tuple

$$\{a_0, a_1, \ldots, a_m\} ,$$

where $a_i \leqq a_j$ if $i < j$; the face and degeneracy operators are defined by their effect on the representatives of the simplices $\zeta \in \Delta[n]$ in the following way:

$$\partial_i\{a_0, a_1, \ldots, a_m\} = \{a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m\} .$$
$$s_i\{a_0, a_1, \ldots, a_m\} = \{a_0, \ldots, a_{i-1}, a_i, a_i, a_{i+1}, \ldots, a_m\} .$$

The identity map $\Delta_n \to \Delta_n$ denoted by Ω_n is represented by $\{0, 1, \ldots, n\}$. Therefore $\bar{\Omega}_n = \Delta[n]$. We shall use the notations

$$\Delta[n; k] = \overline{\partial_k \Omega_n} \qquad \text{and} \qquad \dot{\Delta}[n] = \bigcup_k \Delta[n; k] .$$

If σ is an n-simplex of some c.s.s. complex then $\chi_\sigma\colon \Delta[n] \to \bar{\sigma}$ will denote the c.s.s. map determined by $\chi_\sigma(\Omega_n) = \sigma$.
Notice that a simplex $\zeta_1 \in \Delta[n]_m$ and a simplex $\zeta_2 \in \Delta[m]_r$ can be multiplied, and that the product is a function $\zeta_2\zeta_1\colon \Delta_r \to \Delta_n$, i.e. $\zeta_2\zeta_1 \in \Delta[n]_r$.

In what follows $\mathscr{I}$ is supposed to be a fixed index set.

The c.s.s. *complex* $\tilde{\mathscr{I}}$ is defined as follows: $\tilde{\mathscr{I}}_n$ consists of all $(n+1)$-tuples $\{\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$, where each $\mathfrak{m}_i$ is a finite, non void subset of $\mathscr{I}$, and $\mathfrak{m}_i \subset \mathfrak{m}_j$ if $i < j$.

$$\partial_i\{\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_n\} = \{\mathfrak{m}_0, \ldots, \mathfrak{m}_{i-1}, \mathfrak{m}_{i+1}, \ldots, \mathfrak{m}_n\}.$$
$$s_i\{\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_n\} = \{\mathfrak{m}_0, \ldots, \mathfrak{m}_{i-1}, \mathfrak{m}_i, \mathfrak{m}_i, \mathfrak{m}_{i+1}, \ldots, \mathfrak{m}_n\}.$$

An $\mathscr{I}$-complex K is a pair (K, p), where K is a c.s.s. complex and p is a c.s.s. map $K \to \tilde{\mathscr{I}}$. K is augmented by putting $\partial_0\sigma = p\sigma$ if $\sigma \in K_0$.

We shall interpret the "index-projection" p of an $\mathscr{I}$-complex as an operator like ∂_i and s_i. Thus the projection is always denoted by p. A simplex $\sigma \in \mathsf{K}$ means a simplex $\sigma \in K$. $\bar{\mathscr{I}} = (\tilde{\mathscr{I}}, e)$ where e is the identity map $\tilde{\mathscr{I}} \to \tilde{\mathscr{I}}$. If K is an $\mathscr{I}$-complex then K is the corresponding c.s.s. complex.

An $\mathscr{I}$-map f: $\mathsf{K} \to \mathsf{L}$ is by definition a c.s.s. map f: $K \to L$ with the property that $p = pf$.

K *and* L *are called isomorphic* if there exist two $\mathscr{I}$-maps f: $\mathsf{K} \to \mathsf{L}$ and g: $\mathsf{L} \to \mathsf{K}$ such that fg and gf both are identity maps.

The cartesian product $\mathsf{K} \times L$, where K is an $\mathscr{I}$-complex and L is a c.s.s. complex, is defined to be the $\mathscr{I}$-complex $(K \times L, p\pi)$ where π is the projection $K \times L \to K$.

K *is said to satisfy the $\mathscr{I}$-extension condition* if given

$$\sigma_0, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_{n+1} \in K_n, \qquad k \neq n+1,$$

and $\xi \in \tilde{\mathscr{I}}$ such that

(a) $\partial_i\sigma_j = \partial_{j-1}\sigma_i$ for $i < j$; $i, j \neq k$,
(b) $p\sigma_i = \partial_i\xi$, $i \neq k$,
(c) $s_k\partial_k\xi = \xi$,

then there exists a $\sigma \in \mathsf{K}_{n+1}$ such that $p\sigma = \xi$ and $\partial_i\sigma = \sigma_i$, $i \neq k$.

The extension condition for c.s.s. complexes is due to Kan (see [4] or [6]). Actually $\mathsf{S}(\alpha, X)$ and the associated complexes satisfy a stronger extension condition, but the one above is sufficient for the existence of minimal complexes.

f: $\mathsf{K} \to \mathsf{L}$ *and* g: $\mathsf{L} \to \mathsf{K}$ *are called $\mathscr{I}$-homotopic* if there exists an $\mathscr{I}$-map F: $\mathsf{K} \times \Delta[1] \to \mathsf{L}$ such that

$$F(\sigma, s_0^n\partial_0\Omega_1) = f(\sigma), \quad F(\sigma, s_0^n\partial_1\Omega_1) = g(\sigma), \quad \sigma \in \mathsf{K}_n, \quad n = 0, 1, 2, \ldots.$$

K *and* L *are called $\mathscr{I}$-homotopic* if there exist $\mathscr{I}$-maps f: $\mathsf{K} \to \mathsf{L}$ and g: $\mathsf{L} \to \mathsf{K}$ such that fg and gf are $\mathscr{I}$-homotopic to the identity.

The last definition is the algebraic analogue of definition 1. In order to handle this definition one must prove the lemmas $1(l)$, $l = 1, 2, 3$. The symbol $\bar{\Delta}[n]$ denotes an $\mathscr{I}$-complex $(\Delta[n], p)$,

$$\Gamma_1 = \dot{\bar{\Delta}}[n] \times \Delta[1] \cup \bar{\Delta}[n] \times \Delta[1; 1],$$
$$\Gamma_2 = \dot{\bar{\Delta}}[n] \times \Delta[2] \cup \bar{\Delta}[n] \times \Delta[2; 0] \cup \bar{\Delta}[n] \times \Delta[2; 2],$$
$$\Gamma_3 = \dot{\bar{\Delta}}[n] \times \Delta[2] \cup \bar{\Delta}[n] \times \Delta[2; 1] \cup \bar{\Delta}[n] \times \Delta[2; 2],$$

$\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and $\bar{\Gamma}_3$ are the corresponding subcomplexes of $\bar{\Delta}[n]\times\Delta[1]$ and $\bar{\Delta}[n]\times\Delta[2]$.

LEMMA 1 (l). *If* K *satisfies the* $\mathscr{I}$*-extension condition then each* $\mathscr{I}$*-map* f: $\bar{\Gamma}_l \to \mathsf{K}$ *can be extended to* $\bar{\Delta}[n]\times\Delta[1]$ *or* $\bar{\Delta}[n]\times\Delta[2]$. $l=1, 2, 3$.

PROOFS. The method of the proof is first shown in the simpler case lemma 1(1). Then the two other cases are treated simultaneously in the same way.

Let $\tau=(\zeta_1, \zeta_2)$ be a non degenerate simplex of $\bar{\Delta}[n]\times\Delta[1]$ such that $s_k\partial_k\zeta_1=\zeta_1$, and suppose that f has been extended to $\partial_i\tau$, $i\neq k$, but not to $\partial_k\tau$ and τ. Then

(a) $\partial_i(f(\partial_j\tau)) = \partial_{j-1}(f(\partial_i\tau)),\ \ i<j,\ \ i,j\neq k\,,$

(b) $p(f(\partial_i\tau)) = \partial_i(p\tau),\ \ i\neq k$

(c) $s_k\partial_k(p\tau) = p\tau\,,$

and there exists a $\sigma\in\mathsf{K}$ such that $p\sigma=p\tau$ and $\partial_i\sigma=f(\partial_i\tau)$, $i\neq k$. f can then be extended to $\bar{\tau}$ by the definition $f(\tau)=\sigma$.

The proof is based on successive application of this principle. First the family of non degenerate $(n+1)$-simplices of $\dot{\Delta}[n]\times\Delta[1]$ is ordered by a relation $\prec$, and then it is shown that if f has been extended to

$$J_{\tau_1} = \Gamma_1 \cup \left(\bigcup\{\bar{\tau}:\ \tau\prec\tau_1\}\right),$$

then it is possible to extend f to $J_{\tau_1}\cup\bar{\tau}_1$.

PROOF OF LEMMA 1(1). Each non degenerate $(n+1)$-simplex $\tau\in\bar{\Delta}[n]\times\Delta[1]$ can uniquely be written as

$$[\alpha] = (s_\alpha\Omega_n,\ s_0^\alpha s_1^{n-\alpha}\Omega_1), \qquad 0\leqq\alpha\leqq n\,.$$

The ordering $\prec$ is chosen as follows: $[\alpha_1]\prec[\alpha_2]$ if $\alpha_1>\alpha_2$.

$$\partial_i[\alpha]\begin{cases}\in\ \dot{\Delta}[n]\times\Delta[1] & \text{if}\quad i\neq\alpha,\alpha+1\,,\\ \in\ \overline{[\alpha+1]} & \text{if}\quad i=\alpha+1 \text{ and } \alpha<n\,,\\ \in\ \Delta[n]\times\Delta[1;1] & \text{if}\quad i=\alpha+1 \text{ and } \alpha=n\,,\\ \notin\ J_{[\alpha]} & \text{if}\quad i=\alpha\,.\end{cases}$$

The map $f|J_{[\alpha]}$ can be extended to $(\partial_\alpha[\alpha], [\alpha])$ because

$$s_\alpha\partial_\alpha(s_\alpha\Omega_n) = s_\alpha\Omega_n\,.$$

This proves lemma 1(1).

PROOF OF LEMMA 1(2) AND LEMMA 1(3). Each non degenerate $(n+2)$-simplex $\tau\in\bar{\Delta}[n]\times\Delta[2]$ can uniquely be written as

$$[\alpha, \beta] = (s_\alpha s_\beta \Omega_n, s_0^\alpha s_1^{\beta-\alpha} s_2^{n-\beta} \Omega_1), \qquad 0 \leqq \alpha \leqq \beta \leqq n .$$

The ordering chosen is:

$$[\alpha_1, \beta_1] \prec [\alpha_2, \beta_2] \quad \text{if} \quad \alpha_1 > \alpha_2 \quad \text{or} \quad \alpha_1 = \alpha_2 \quad \text{and} \quad \beta_1 > \beta_2 .$$

Further $J_{[\alpha,\beta]}$ means $\Gamma_l \cup \big(\bigcup\{\bar{\tau}\colon \tau \prec [\alpha, \beta]\}\big)$, $l=2, 3$.

I. $0<\alpha<\beta\leqq n$. In this case

$$\partial_i[\alpha, \beta] \begin{cases} \in \dot{\Delta}[n]\times\Delta[2] & \text{if} \quad i \neq \alpha, \alpha+1, \beta+1, \beta+2 , \\ \in \overline{[\alpha+1, \beta]} & \text{if} \quad i = \alpha+1 , \\ \in \overline{[\alpha, \beta+1]} & \text{if} \quad i = \beta+2 \text{ and } \beta < n , \\ \in \Delta[n]\times\Delta[2; 2] & \text{if} \quad i = \beta+2 \text{ and } \beta = n , \\ \notin J^2_{[\alpha,\beta]}, J^3_{[\alpha,\beta]} & \text{if} \quad i = \alpha, \beta+1 . \end{cases}$$

$$\partial_i\partial_\alpha[\alpha, \beta] \begin{cases} \in \dot{\Delta}[n]\times\Delta[2] & \text{if} \quad i \neq \beta, \beta+1 , \\ \in \overline{[\alpha, \beta+1]} & \text{if} \quad i = \beta+1 \text{ and } \beta < n , \\ \in \Delta[n]\times\Delta[2; 2] & \text{if} \quad i = \beta+1 \text{ and } \beta = n , \\ \notin J^2_{[\alpha,\beta]}, J^3_{[\alpha,\beta]} & \text{if} \quad i = \beta . \end{cases}$$

Both $f|J^2_{[\alpha,\beta]}$ and $f|J^3_{[\alpha,\beta]}$ can be extended to $\big(\partial_\beta\partial_\alpha[\alpha,\beta], \partial_\alpha[\alpha,\beta]\big)$ because

$$s_\beta\partial_\beta\big(\partial_\alpha(s_\alpha s_\beta \Omega_n)\big) = \partial_\alpha(s_\alpha s_\beta \Omega_n) ,$$

and to $\big(\partial_{\beta+1}[\alpha, \beta], [\alpha, \beta]\big)$ because

$$s_{\beta+1}\partial_{\beta+1}(s_\alpha s_\beta \Omega_n) = s_\alpha s_\beta \Omega_n .$$

II. $0<\alpha=\beta\leqq n$. In this case

$$\partial_i[\alpha, \beta] \begin{cases} \in \dot{\Delta}[n]\times\Delta[2] & \text{if} \quad i \neq \alpha, \beta+1, \beta+2 , \\ \in \overline{[\alpha, \beta+1]} & \text{if} \quad i = \beta+2 \text{ and } \beta < n , \\ \in \Delta[n]\times\Delta[2; 2] & \text{if} \quad i = \beta+2 \text{ and } \beta = n , \\ \in \Delta[n]\times\Delta[2; 1] & \text{if} \quad i = \beta+1 , \\ \notin J^2_{[\alpha,\beta]} & \text{if} \quad i = \beta+1 , \\ \notin J^2_{[\alpha,\beta]}, J^3_{[\alpha,\beta]} & \text{if} \quad i = \alpha . \end{cases}$$

$$\partial_i\partial_\alpha[\alpha, \beta] \begin{cases} \in \dot{\Delta}[n]\times\Delta[2] & \text{if} \quad i \neq \beta, \beta+1 , \\ \in \overline{[\alpha, \beta+1]} & \text{if} \quad i = \beta+1 \text{ and } \beta < n , \\ \in \Delta[n]\times\Delta[2; 2] & \text{if} \quad i = \beta+1 \text{ and } \beta = n , \\ \notin J^2_{[\alpha,\beta]} & \text{if} \quad i = \beta . \end{cases}$$

$f|J^2_{[\alpha,\beta]}$ is treated as in case I; $f|J^3_{[\alpha,\beta]}$ can immediately be extended to $(\partial_\alpha[\alpha,\beta],\ [\alpha,\beta])$ because

$$s_\alpha \partial_\alpha(s_\alpha s_\beta \Omega_n) = s_\alpha s_\beta \Omega_n .$$

III. $0=\alpha<\beta\leqq n$. In this case

$$\partial_i[\alpha,\beta] \begin{cases} \in \dot{\Delta}[n]\times\Delta[2] & \text{if} \quad i \neq \alpha,\ \alpha+1,\ \beta+1,\ \beta+2 , \\ \in \overline{[\alpha+1,\beta]} & \text{if} \quad i = \alpha+1 , \\ \in \overline{[\alpha,\beta+1]} & \text{if} \quad i = \beta+2 \text{ and } \beta < n , \\ \in \Delta[n]\times\Delta[2;\,2] & \text{if} \quad i = \beta+2 \text{ and } \beta = n , \\ \in \Delta[n]\times\Delta[2;\,0] & \text{if} \quad i = \alpha , \\ \notin J^3_{[\alpha,\beta]} & \text{if} \quad i = \alpha , \\ J^2_{[\alpha,\beta]},\ J^3_{[\alpha,\beta]} & \text{if} \quad i = \beta+1 . \end{cases}$$

$$\partial_i\partial_\alpha[\alpha,\beta] \begin{cases} \in \dot{\Delta}[n]\times\Delta[2] & \text{if} \quad i \neq \beta,\ \beta+1 , \\ \in \overline{[\alpha,\beta+1]} & \text{if} \quad i = \beta+1 \text{ and } \beta < n , \\ \in \Delta[n]\times\Delta[2;\,2] & \text{if} \quad i = \beta+1 \text{ and } \beta = n , \\ \notin J^3_{[\alpha,\beta]} & \text{if} \quad i = \beta . \end{cases}$$

$f|J^3_{[\alpha,\beta]}$ is treated as in case I; $f|J^2_{[\alpha,\beta]}$ can immediately be extended to $(\partial_{\beta+1}[\alpha,\beta],\ [\alpha,\beta])$ because

$$s_{\beta+1}\partial_{\beta+1}(s_\alpha s_\beta \Omega_n) = s_\alpha s_\beta \Omega_n .$$

IV. $0=\alpha=\beta\leqq n$. In this case

$$\partial_i[\alpha,\beta] \begin{cases} \in \dot{\Delta}[n]\times\Delta[2] & \text{if} \quad i \neq \alpha,\ \beta+1,\ \beta+2 , \\ \in [\alpha,\beta+1] & \text{if} \quad i = \beta+2 \text{ and } \beta < n , \\ \in \Delta[n]\times\Delta[2;\,2] & \text{if} \quad i = \beta+2 \text{ and } \beta = n , \\ \in \Delta[n]\times\Delta[2;\,1] & \text{if} \quad i = \beta+1 , \\ \notin J^2_{[\alpha,\beta]} & \text{if} \quad i = \beta+1 , \\ \in \Delta[n]\times\Delta[2;\,0] & \text{if} \quad i = \alpha , \\ \notin J^3_{[\alpha,\beta]} & \text{if} \quad i = \alpha . \end{cases}$$

$f|J^2_{[\alpha,\beta]}$ is treated as in case III; $f|J^3_{[\alpha,\beta]}$ is treated as in case II.

This proves lemma 1(2) and lemma 1(3).

THEOREM 1. *If L satisfies the $\mathscr{J}$-extension condition, then the relation*

f and g are $\mathscr{J}$-homotopic

is an equivalence relation in the family of $\mathscr{J}$-maps $\mathsf{K}\to\mathsf{L}$.

PROOF. (a) The reflexivity is obvious. The "deformation" F is defined by $F(\sigma, \zeta)=f(\sigma)$.

(b) The symmetry follows from lemma 1(3). There is given an $\mathscr{I}$-map $F\colon \mathsf{K}\times\Delta[1]\to\mathsf{L}$, and we have to show that there exists an $\mathscr{I}$-map $G\colon \mathsf{K}\times\Delta[1]\to\mathsf{L}$ such that

$$F(\sigma, s_0^n\partial_0\Omega_1) = G(\sigma, s_0^n\partial_1\Omega_1), \qquad F(\sigma, s_0^n\partial_1\Omega_1) = G(\sigma, s_0^n\partial_0\Omega_1), \qquad \sigma\in\mathsf{K}_n .$$

Let

$$\Phi\colon\ \mathsf{K}\times\Delta[2;1]\cup\mathsf{K}\times\Delta[2;2]\to\mathsf{L}$$

be defined by

$$\Phi\big(\sigma, (\partial_1\Omega_2)\zeta\big) = F(\sigma, s_0^n\partial_1\Omega_1) \quad\text{and}\quad \Phi\big(\sigma, (\partial_2\Omega_2)\zeta\big) = F(\sigma, \zeta),$$
$$\sigma\in\mathsf{K}_n, \quad \zeta\in\Delta[1]_n .$$

Now lemma 1(3) implies that Φ can be extended to $\mathsf{K}\times\Delta[2]$. In fact, Φ is defined on all $(\sigma,\zeta)\in\mathsf{K}\times\Delta[2]$, where $\sigma\in\bigcup\{\bar\tau\colon\ \tau\in\mathsf{K}_0\}$. Suppose, Φ has been extended to all $(\sigma,\zeta)\in\mathsf{K}\times\Delta[2]$, where $\sigma\in\bigcup\{\bar\tau\colon\ \tau\in\mathsf{K}_n\}$, and let σ_1 be a non degenerate n-simplex of K. We want to extend Φ to $\bar\sigma_1\times\Delta[2]$. Let $\bar\Delta[n]$ be the $\mathscr{I}$-complex $\big(\Delta[n], p\chi_{\sigma_1}\big)$ and let $\varphi\colon \bar\Gamma_3\to\mathsf{L}$ be defined by

$$\varphi(\zeta_1,\zeta_2) = \Phi\big(\chi_{\sigma_1}(\zeta_1),\ \zeta_2\big) .$$

φ is extendable to $\bar\Delta[n]\times\Delta[2]$, and we define

$$\Phi(\tau,\zeta) = \varphi\big(\chi_{\sigma_1}^{-1}(\tau),\zeta\big) \qquad\text{for}\quad \tau\in\bar\sigma_1 .$$

In this way Φ is extended to $\mathsf{K}\times\Delta[2]$ by induction, and G is defined by

$$G(\sigma,\zeta) = \Phi\big(\sigma, (\partial_0\Omega_2)\zeta\big) .$$

(c) The transitivity follows from lemma 1(2). Two $\mathscr{I}$-maps

$$F\colon\ \mathsf{K}\times\Delta[1]\to\mathsf{L} \qquad\text{and}\qquad G\colon\ \mathsf{K}\times\Delta[1]\to\mathsf{L}$$

are given such that

$$F(\sigma, s_0^n\partial_1\Omega_1) = G(\sigma, s_0^n\partial_0\Omega_1), \qquad \sigma\in\mathsf{K}_n ,$$

and we have to show the existence of a map $H\colon \mathsf{K}\times\Delta[1]\to\mathsf{L}$ such that

$$F(\sigma, s_0^n\partial_0\Omega_1)=H(\sigma, s_0^n\partial_0\Omega_1) \quad\text{and}\quad G(\sigma, s_0^n\partial_1\Omega_1)=H(\sigma, s_0^n\partial_1\Omega_1), \qquad \sigma\in\mathsf{K}_n .$$

The map

$$\Phi\colon\ \mathsf{K}\times\Delta[2;0]\cup\mathsf{K}\times\Delta[2;2]\to\mathsf{L}$$

is defined by

$$\Phi\big(\sigma, (\partial_0\Omega_2)\zeta\big) = F(\sigma,\zeta) \qquad\text{and}\qquad \Phi\big(\sigma, (\partial_2\Omega_2)\zeta\big) = G(\sigma,\zeta) .$$

Now 1(2) implies that Φ can be extended to $\mathsf{K} \times \Delta[2]$, and H is defined by

$$H(\sigma, \zeta) = \Phi\big(\sigma, (\partial_1 \Omega_2)\zeta\big).$$

Theorem 1 has the corollary:

THEOREM 2. *The relation*: K *and* L *are* $\mathscr{I}$*-homotopic, is an equivalence relation in the family of* $\mathscr{I}$*-complexes which satisfy the* $\mathscr{I}$*-extension condition.*

The rest of section 2 is concerned with minimal subcomplexes of $\mathscr{I}$-complexes. It is an obvious generalization to $\mathscr{I}$-complexes of the theory given in [2] and [6].

$\sigma_1, \sigma_2 \in \mathsf{K}_n$ *are called* $\mathscr{I}$*-compatible* if $p\sigma_1 = p\sigma_2$ and $\partial_i \sigma_1 = \partial_i \sigma_2$, $i = 0, 1, \ldots, n$. Evidently the relation:

$$\sigma_1 \text{ and } \sigma_2 \text{ are } \mathscr{I}\text{-compatible}$$

is an equivalence relation. The map $\chi_{\sigma_1}\colon \Delta[n] \to \sigma_1$ determines the $\mathscr{I}$-complex

$$\bar{\Delta}_{\sigma_1} = \big(\Delta[n], p\chi_{\sigma_1}\big),$$

and if σ_1 and σ_2 are $\mathscr{I}$-compatible then $\bar{\Delta}_{\sigma_1} = \bar{\Delta}_{\sigma_2}$.

$\sigma_1, \sigma_2 \in \mathsf{K}_n$ *are called* $\mathscr{I}$*-homotopic* if σ_1 and σ_2 are $\mathscr{I}$-compatible and there exists an $\mathscr{I}$-map

$$F_{\sigma_1}^{\sigma_2}\colon \bar{\Delta}[n] \times \Delta[1] \to \mathsf{K},$$

where $\bar{\Delta}[n] = \bar{\Delta}_{\sigma_1} = \bar{\Delta}_{\sigma_2}$, such that

(a) $F_{\sigma_1}^{\sigma_2}(\partial_i \Omega_n, \zeta) = \partial_i \sigma_1 = \partial_i \sigma_2, \quad \zeta \in \Delta[1]_{n-1}, \quad i = 0, 1, \ldots, n,$

(b) $F_{\sigma_1}^{\sigma_2}(\Omega_n, s_0^n \partial_0 \Omega_1) = \sigma_1,$

(c) $F_{\sigma_1}^{\sigma_2}(\Omega_n, s_0^n \partial_1 \Omega_1) = \sigma_2.$

The method applied in the proof of theorem 1 shows that if the $\mathscr{I}$-complex K satisfies the $\mathscr{I}$-extension condition, then the definition above gives an equivalence relation in the families of $\mathscr{I}$-compatible simplices.

A minimal subcomplex M of an $\mathscr{I}$-complex K, which satisfies the $\mathscr{I}$-extension condition, is defined by the properties:

(a) If $\sigma_1, \sigma_2 \in \mathsf{M}$ are $\mathscr{I}$-homotopic in K then $\sigma_1 = \sigma_2$.

(b) If $\sigma_1 \in \mathsf{K}$ and $\partial_i \sigma_1 \in \mathsf{M}_{n-1}$, $i = 0, 1, \ldots, n$, then there exists a simplex $\sigma_2 \in \mathsf{M}_n$ such that σ_1 and σ_2 are $\mathscr{I}$-homotopic in K.

The existence of minimal subcomplexes of an $\mathscr{I}$-complex which satisfies the $\mathscr{I}$-extension condition is easily established. First one element out of each class of $\mathscr{I}$-homotopic 0-simplices is chosen, and these simplices constitute M_0. Now suppose M_{n-1} has been constructed; let $\mathfrak{M}_n$ be the family of classes of $\mathscr{I}$-homotopic n-simplices such that $\sigma \in \psi \in \mathfrak{M}_n$ implies that $\partial_i \sigma \in \mathsf{M}_{n-1}$, $i=0, 1, \ldots, n$. The set M_n consists of one element out of each class $\psi \in \mathfrak{M}_n$; if a class contains a degenerate simplex then this is chosen. (A class $\psi \in \mathfrak{M}_n$ never contains more than one degenerate element.)

THEOREM 3. *If* M *is a minimal subcomplex of* K *and* $i\colon \mathsf{M} \to \mathsf{K}$ *is the inclusion map, then there exists an* $\mathscr{I}$*-map* $h\colon \mathsf{K} \to \mathsf{M}$ *such that* hi *is the identity and* ih *is homotopic to the identity.*

PROOF. A "deformation" $H\colon \mathsf{K} \times \Delta[1] \to \mathsf{K}$ such that

$$H(\sigma, s_0 \partial_1 \Omega_1) = \sigma, \quad H(\sigma, s_0^n \partial_0 \Omega_1) \in \mathsf{M}, \quad \sigma \in \mathsf{K}_n,$$

and $H(\sigma, \zeta) = \sigma$ if $\sigma \in \mathsf{M}$ is constructed in the following way:

If $\sigma^0 \in \mathsf{K}_0$ then $H(\sigma^0, \partial_1 \Omega_1) = \sigma^0$ and $H(\sigma^0, \partial_0 \Omega_1) = \tau^0$ where τ^0 is the unique 0-simplex of M such that σ^0 and τ^0 are homotopic; generally, if $\sigma \in \overline{\sigma^0}$ then

$$H(\sigma, \zeta) = F_{\tau^0}^{\sigma^0}\big(\chi_{\sigma^0}^{-1}(\sigma), \zeta\big).$$

Assume that $H(\sigma, \zeta)$ has been defined for all $\sigma \in \bigcup \{\bar{\tau}\colon \tau \in \mathsf{K}_{n-1}\}$, and let σ^n be a non degenerate simplex of $\mathsf{K}_n \setminus \mathsf{M}_n$. Using the notation $\bar{\Delta}[n] = \bar{\Delta}_{\sigma^n}$, we have that the function

$$\eta(\zeta_1, \zeta_2) = H\big(\chi_{\sigma^n}(\zeta_1), \zeta_2\big), \qquad \zeta_1 \in \bar{\Delta}[n], \quad \zeta_2 \in \Delta[1],$$

is defined on $\bar{\Gamma}_1$, and lemma 1(1) implies that η can be extended to

$$\bar{\Delta}[n] \times \Delta[1].$$

Now

$$\partial_i \eta(\Omega_n, s_0^n \partial_0 \Omega_1) \in \mathsf{M},$$

and there exists a unique $\tau^n \in \mathsf{M}$ such that

$$\sigma_1 = \eta(\Omega_n, s_0^n \partial_0 \Omega_1) \quad \text{and} \quad \tau^n$$

are $\mathscr{I}$-homotopic in K. The function $\varphi\colon \bar{\Gamma}_2 \to \mathsf{K}$ is defined by

$$\begin{aligned}
\varphi(\zeta_1, \zeta_2) &= \eta\big(\zeta_1, (s_1 \Omega_1) \zeta_2\big) && \text{if} \quad (\zeta_1, \zeta_2) \in \dot{\Delta}[n] \times \Delta[2], \\
\varphi\big(\zeta_1, (\partial_0 \Omega_2) \zeta_2\big) &= F_{\tau^n}^{\sigma_1}(\zeta_1, \zeta_2) && \text{if} \quad (\zeta_1, \zeta_2) \in \Delta[n] \times \Delta[1], \\
\varphi\big(\zeta_1, (\partial_2 \Omega_2) \zeta_2\big) &= \eta(\zeta_1, \zeta_2) && \text{if} \quad (\zeta_1, \zeta_2) \in \Delta[n] \times \Delta[1].
\end{aligned}$$

Lemma 1(2) implies that φ can be extended to $\bar{\Delta}[n] \times \Delta[2]$, and we define

$$H(\tau, \zeta) = \varphi\big(\chi_{\sigma^n}^{-1}(\tau), (\partial_1 \Omega_2) \zeta\big), \qquad \tau \in \overline{\sigma^n}, \quad \zeta \in \Delta[1].$$

In this way H is extended to $\mathsf{K} \times \Delta[1]$ by induction, and

$$h(\sigma) = H(\sigma, s_0^n \partial_0 \Omega_1), \qquad \sigma \in \mathsf{K}_n .$$

The main result of this section is:

THEOREM 4. *If* M' *and* M'' *are minimal subcomplexes of two* $\mathscr{I}$*-homotopic* $\mathscr{I}$*-complexes* K' *and* K''*, then* M' *and* M'' *are isomorphic.*

The following two lemmas are necessary for the proof of theorem 4. Let $\bar{\Delta}[n]$ be an arbitrary $\mathscr{I}$-complex $(\Delta[n], p)$ and $\bar{\Lambda}$ the subcomplex of

$$\bar{\Delta}[n] \times \Delta[1] \times \Delta[1]$$

determined by

$$\Lambda = \dot{\Delta}[n] \times \Delta[1] \times \Delta[1] \cup \Delta[n] \times \dot{\Delta}[1] \times \Delta[1] \cup \Delta[n] \times \Delta[1] \times \Delta[1; 1] .$$

LEMMA 2. *Each* $\mathscr{I}$*-map* $\bar{\Lambda} \to \mathsf{K}$*, where* K *satisfies the* $\mathscr{I}$*-extension condition, can be extended to* $\bar{\Delta}[n] \times \Delta[1] \times \Delta[1]$.

PROOF. $\Delta[1] \times \Delta[1] = \bar{\theta}_1 \cup \bar{\theta}_2$, where θ_1 and θ_2 are defined as follows:

$$\theta_1 = (s_1 \Omega_1, s_0 \Omega_1) \qquad \text{and} \qquad \theta_2 = (s_0 \Omega_1, s_1 \Omega_1) .$$

Both θ_1 and θ_2 are non degenerate. The map $\varphi_1 \colon \bar{\Gamma}_2 \to \mathsf{K}$ defined by

$$\varphi_1(\zeta_1, \zeta_2) = f\big(\zeta_1, \chi_{\theta_1}(\zeta_2)\big)$$

can be extended to $\bar{\Delta}[n] \times \Delta[2]$, and f is defined on $\bar{\Delta}[n] \times \bar{\theta}_1$ by

$$f(\zeta_1, \tau) = \varphi_1\big(\zeta_1, \chi_{\theta_1}^{-1}(\tau)\big) .$$

The map $\varphi_2 \colon \bar{\Gamma}_3 \to \mathsf{K}$ defined by

$$\varphi_2(\zeta_1, \zeta_2) = f\big(\zeta_1, \chi_{\theta_2}(\zeta_2)\big)$$

is then extendable to $\bar{\Delta}[n] \times \Delta[2]$, and f is defined on $\bar{\Delta}[n] \times \bar{\theta}_2$ by

$$f(\zeta_1, \tau) = \varphi\big(\zeta_1, \chi_{\theta_2}^{-1}(\tau)\big) .$$

LEMMA 3. *If* M *is a minimal subcomplex of* K *and* $f \colon \mathsf{K} \to \mathsf{K}$ *is* $\mathscr{I}$*-homotopic to the identity, then* $f|\mathsf{M}$ *is an isomorphism of* M *onto another minimal subcomplex* $f(\mathsf{M})$ *of* K.

PROOF. We have to prove: (a) If $\sigma_1, \sigma_2 \in \mathsf{M}$ are $\mathscr{I}$-compatible and $f(\sigma_1)$ and $f(\sigma_2)$ are $\mathscr{I}$-homotopic, then $\sigma_1 = \sigma_2$. (b) If $\partial_i \varrho = f(\sigma_i)$, $\varrho \in \mathsf{K}_n$, $\sigma_i = \mathsf{M}_{n-1}$, $i = 0, 1, \ldots, n$, then there exists a $\sigma \in \mathsf{M}_n$ such that $\partial_i \sigma = \sigma_i$ and $f(\sigma)$ and ϱ are $\mathscr{I}$-homotopic.

(a) Since f is $\mathscr{I}$-homotopic to the identity, there exists an $\mathscr{I}$-map

$$F \colon \ \mathsf{K} \times \Delta[1] \to \mathsf{K}$$

such that

$$F(\sigma, s_0^n \partial_1 \Omega_1) = f(\sigma), \quad F(\sigma, s_0^n \partial_0 \Omega_0) = \sigma, \quad \sigma \in \mathsf{K}_n .$$

Now $\bar{\Delta}[n] = \bar{\Delta}_{\sigma_1} = \bar{\Delta}_{\sigma_2} = \bar{\Delta}_{f(\sigma_1)} = \bar{\Delta}_{f(\sigma_2)}$, and $\varphi\colon \bar{\Lambda} \to \mathsf{K}$ is defined by

$$\begin{aligned}
\varphi(\eta, \zeta_1, \zeta_2) &= F(\chi_{\sigma_1}(\eta), \zeta_2) && \text{if} \quad \eta \in \dot{\Delta}[n],\ \zeta_1 \in \Delta[1],\ \zeta_2 \in \Delta[1],\\
\varphi(\eta, \zeta_1, \zeta_2) &= F_{f(\sigma_1)}^{f(\sigma_2)}(\eta, \zeta_1) && \text{if} \quad \eta \in \Delta[n],\ \zeta_1 \in \Delta[1],\ \zeta_2 \in \Delta[1;1],\\
\varphi(\eta, \zeta_1, \zeta_2) &= F(\chi_{\sigma_1}(\eta), \zeta_2) && \text{if} \quad \eta \in \Delta[n],\ \zeta_1 \in \Delta[1;0],\ \zeta_2 \in \Delta[1],\\
\varphi(\eta, \zeta_1, \zeta_2) &= F(\chi_{\sigma_2}(\eta), \zeta_2) && \text{if} \quad \eta \in \Delta[n],\ \zeta_1 \in \Delta[1;1],\ \zeta_2 \in \Delta[1].
\end{aligned}$$

Lemma 2 implies that φ can be extended to $\bar{\Delta}[n] \times \Delta[1] \times \Delta[1]$, and the function

$$F_{\sigma_1}^{\sigma_2}\colon \ \bar{\Delta}[n] \times [1] \to \mathsf{K}$$

defined by

$$F_{\sigma_1}^{\sigma_2}(\eta, \zeta) = \varphi(\eta, \zeta, s_0^m \partial_0 \Omega_1), \qquad \eta \in \Delta[n]_m, \quad \zeta \in \Delta[1]_m ,$$

shows that σ_1 and σ_2 are $\mathscr{I}$-homotopic.

(b) It follows (a) that $f|\mathsf{M}$ is one to one and has an inverse function g. $f(\mathsf{M})$ can be extended to a minimal subcomplex M' of K, and the method of proof of theorem 3 shows that g can be extended to

$$g\colon \ \mathsf{M}' \to \mathsf{M} ,$$

such that g is homotopic to the inclusion map $\mathsf{M}' \to \mathsf{K}$. We have to prove that $f(\mathsf{M}) = \mathsf{M}'$. Suppose that $f(\mathsf{M}_{n-1}) = \mathsf{M}_{n-1}$; if $\varrho \in \mathsf{M}_n{}'$ then (a) implies that $fg(\varrho)$ and ϱ are $\mathscr{I}$-homotopic, and therefore $\varrho \in f(\mathsf{M}_n)$, and since it is obvious that $f(\mathsf{M}_0) = \mathsf{M}_0{}'$, the proof is completed by induction.

Proof of theorem 4. Let $f\colon \ \mathsf{K}' \to \mathsf{K}''$ and $g\colon \ \mathsf{K}'' \to \mathsf{K}$ be inverse homotopy equivalences. If

$$h\colon \ \mathsf{K}'' \to \mathsf{M}''$$

is the $\mathscr{I}$-map described in theorem 3, then lemma 3 implies that $ghf(\mathsf{M}')$ is a minimal subcomplex of K, and there exists an $\mathscr{I}$-map

$$k\colon \ \mathsf{K}' \to ghf(\mathsf{M}')$$

such that ki is the identity and ik is homotopic to the identity (i is the inclusion map $\mathsf{M}' \to \mathsf{K}'$). Now

$$h(f|\mathsf{M}')\colon \ \mathsf{M}' \to \mathsf{M}''$$

must be one to one because otherwise M' and $ghf(\mathsf{M}')$ would not be isomorphic. And $h(f|\mathsf{M}')$ must be onto M'' because $fk(g|\mathsf{M}'')$ is $\mathscr{I}$ homotopic to the inclusion map $\mathsf{M}'' \to \mathsf{K}''$ and therefore an isomorphism onto.

Theorem 4 implies that the minimal complexes are invariants of $\mathscr{I}$-homotopy type of $\mathscr{I}$-complexes which satisfy the $\mathscr{I}$-extension condition. The same holds for the derived complexes $\mathsf{M}^{(n)}$, $n = -1, 0, 1, \ldots, \infty$, defined in the following way:

An m-simplex of $\mathsf{M}^{(n)}$ is a class ϱ^m of m-simplices $\sigma \in \mathsf{M}$ such that $\sigma_1, \sigma_2 \in \varrho^m$ implies (1) that $p\sigma_1 = p\sigma_2$ and (2) if $n \geqq 0$, that χ_{σ_1} and χ_{σ_2} are identical on

$$\bigcup \{\bar{\tau}\colon\ \tau \in \Delta[m]_n\}$$

or on $\Delta[m]$ if $n = \infty$. The face and degeneracy operators are induced by the face and degeneracy operators of M in the following way: If $\sigma \in \varrho^m$ then $\partial_i \varrho$ and $s_i \varrho$ are the $(m-1)$- and $(m+1)$-simplices of $\mathsf{M}^{(n)}$ which contain $\partial_i \sigma$ and $s_i \sigma$. The complexes $\mathsf{M}^{(\infty)}$ and M are isomorphic and will be identified.

The natural maps

$$\omega_r^s\colon\ \mathsf{M}^{(s)} \to \mathsf{M}^{(r)}, \qquad r \leqq s\,,$$

are all $\mathscr{I}$-maps. (If $\sigma \in \varrho \in \mathsf{M}^{(s)}$ then $\omega_r^s(\varrho)$ is the simplex of $\mathsf{M}^{(r)}$ which contains σ.)

3. $\mathsf{S}(\alpha, X)$ and the associated complexes. As a generalization of the fact that definition 1 reduces to the usual definition of homotopy type if $\mathscr{I}$ contains only one element, we have the following obvious theorem:

THEOREM 5. *If $\mathfrak{m}$ is a subset of $\mathscr{I}$ and the coverings $\{A_\nu\colon\ \nu \in \mathscr{I}\}$ and $\{B_\nu\colon\ \nu \in \mathscr{I}\}$ are $\mathscr{I}$-homotopic, then the subspaces $\bigcap \{A_\nu\colon\ \nu \in \mathfrak{m}\}$ and $\bigcap \{B_\nu\colon\ \nu \in \mathfrak{m}\}$ are of the same homotopy type.*

Theorem 5 implies that the usual homotopy types of the intersections of elements of a covering are $\mathscr{I}$-homotopy invariants of the covering, but it is easily seen that this does not yield a complete set of invariants. Consider for example the torus and the Klein bottle; these two spaces can both be covered by two compact cylinders (topological products of circles and unit intervals) such that the corresponding intersections are homotopic, although the coverings are not $\mathscr{I}$-homotopic because this would imply that the torus and the Klein bottle were of the same homotopy type. Therefore it is necessary to take into consideration how the homotopy types of the intersections are "linked" together. This is done in the construction of $\mathsf{S}(\alpha, X)$.

If $\xi = \{\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_n\} \in \bar{\mathscr{I}}$ we shall use the notation $\langle \xi \rangle = \mathfrak{m}_0$.

If X is a topological space then the total singular complex $S(X)$ is the c.s.s. complex defined by:

$S(X)_n$ consists of all continuous maps $\sigma^n\colon\ \Delta_n \to X$, the face and degeneracy operators being determined by

$$\partial_i \sigma^n = \sigma^n(\partial_i \Omega_n) \qquad \text{and} \qquad s_i \sigma^n = \sigma^n(s_i \Omega_n)\,.$$

If $\alpha = \{A_\nu\colon \nu \in \mathscr{I}\}$ is a covering of the topological space X and $\sigma^n \in S(X)_n$, we shall use the notation

$$\langle \sigma^n \rangle_\alpha = \{\nu \in \mathscr{I}\colon \sigma^n(\Delta_n) \subset A_\nu\} .$$

Definition 2. If $\alpha = \{A_\nu\colon \nu \in \mathscr{I}\}$ is a covering of the topological space X then $\mathsf{S}(\alpha, X)$ is the subcomplex of the $\mathscr{I}$-comples $\bar{\mathscr{I}} \times S(X)$ defined thus:

$$(\xi, \sigma) \in (\bar{\mathscr{I}} \times S(X))_n$$

is an n-simplex of $\mathsf{S}(\alpha, X)$ if and only if

$$\langle \partial_0^i \xi \rangle \subset \langle \partial_0^i \sigma \rangle_\alpha, \qquad i = 0, 1, \ldots, n .$$

Notice that $\mathsf{S}(\alpha, X)$ is the largest subcomplex of $\bar{\mathscr{I}} \times S(X)$ with the property that if $(\xi, \sigma) \in \mathsf{S}(\alpha, X)$, then $\langle \xi \rangle \subset \langle \sigma \rangle_\alpha$.

$\mathsf{S}(\alpha, X) = \bar{\mathscr{I}} \times S(X)$ if and only if $A_\nu = X$, $\nu \in \mathscr{I}$.

$\mathsf{S}(\alpha, X)$ satisfies the following extension condition: If

$$\sigma_0, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_{n+1} \in \mathsf{S}(\alpha, X)_n \qquad \text{and} \qquad \xi \in \tilde{\mathscr{I}}$$

such that

(a) $\partial_i \sigma_j = \partial_{j-1} \sigma_i$ for $i < j$; $i, j \neq k$,

(b) $p\sigma_i = \partial_i \xi$ for $i \neq k$,

(c) $\langle \partial_k \xi \rangle = \langle \xi \rangle$,

then there exists a $\sigma \in \mathsf{S}(\alpha, X)_{n+1}$ such that $p\sigma = \xi$ and $\partial_i \sigma = \sigma_i$ for $i \neq k$.

This extension condition is stronger than the $\mathscr{I}$-extension condition of section 2, and it guarantees the existence of a minimal subcomplex $\mathsf{M}(\alpha, X)$ of $\mathsf{S}(\alpha, X)$. (Theorem 4 implies that all minimal subcomplexes are isomorphic.)

Theorem 6. *If $\alpha = \{A_\nu\colon \nu \in \mathscr{I}\}$ and $\beta = \{B_\nu\colon \nu \in \mathscr{I}\}$ are $\mathscr{I}$-homotopic coverings of the topological spaces X and Y, respectively, then $\mathsf{S}(\alpha, X)$ and $\mathsf{S}(\beta, Y)$ are $\mathscr{I}$-homotopic $\mathscr{I}$-complexes.*

Proof. Let $f\colon X \to Y$, $g\colon Y \to X$, $F\colon X \times I \to X$ and $G\colon Y \times I \to Y$ be the maps of definition 1, and let $i\colon \Delta_1 \to I$ be a continuous function, which maps $(\partial_0 \Omega_1)[\Delta_0]$ into 0 and $(\partial_1 \Omega_1)[\Delta_0]$ into 1. Now let

$$\begin{aligned} \bar{f}\colon &\ \mathsf{S}(\alpha, X) \to \mathsf{S}(\beta, y) , \\ \bar{g}\colon &\ \mathsf{S}(\beta, Y) \to \mathsf{S}(\alpha, X) , \\ \bar{F}\colon &\ \mathsf{S}(\alpha, X) \times \Delta[1] \to \mathsf{S}(\alpha, X) , \\ \bar{G}\colon &\ \mathsf{S}(\beta, Y) \times \Delta[1] \to \mathsf{S}(\beta, Y) \end{aligned}$$

be defined by

$$\begin{aligned}
\bar{f}(\xi, \sigma) &= (\xi, f\sigma)\,, && (\xi, \sigma) \in \mathsf{S}(\alpha, X)\,, \\
\bar{g}(\xi, \sigma) &= (\xi, g\sigma)\,, && (\xi, \sigma) \in \mathsf{S}(\beta, Y)\,, \\
\bar{F}((\xi, \sigma), \zeta) &= (\xi, F(\sigma(\ ), i\zeta(\ )))\,, && (\xi, \sigma) \in \mathsf{S}(\alpha, X), \quad \zeta \in \Delta[1]\,, \\
\bar{G}((\xi, \sigma), \zeta) &= (\xi, G(\sigma(\ ), i\zeta(\ )))\,, && (\xi, \sigma) \in \mathsf{S}(\beta, Y), \quad \zeta \in \Delta[1]\,.
\end{aligned}$$

It is easily verified that

$$F(\sigma, s_0^n \partial_0 \Omega_1) = \sigma, \quad F(\sigma, s_0^n \partial_1 \Omega_1) = \bar{g}\bar{f}(\sigma), \quad \sigma \in \mathsf{S}(\alpha, X)_n\,,$$

and

$$G(\sigma, s_0^n \partial_0 \Omega_1) = \sigma, \quad G(\sigma, s_0^n \partial_1 \Omega_1) = \bar{f}\bar{g}(\sigma), \quad \sigma \in \mathsf{S}(\beta, Y)_n\,.$$

Theorem 6 shows that $\mathsf{M}(\alpha, X)$ and the derived complexes

$$\mathsf{M}^{(n)}(\alpha, X), \qquad n = -1, 0, 1, \ldots,$$

are invariants of $\mathscr{I}$-homotopy type of coverings, and in fact they contain very detailed information. If, for example, $\{\mathfrak{m}\}$ is a 0-simplex of $\tilde{\mathscr{I}}$, then $P^{-1}(\overline{\{\mathfrak{m}\}})$, where p is the projection of $\mathsf{M}(\alpha, X)$, is isomorphic to the minimal c.s.s. complex of $\bigcap \{A_\nu \colon \nu \in \mathfrak{m}\}$. The "mixed" simplices of $\mathsf{M}(\alpha, X)$ tell something about the way in which the homotopy types of the intersections are related. We shall examine $\mathsf{M}^{(-1)}(\alpha, X)$ and $\mathsf{M}^{(0)}(\alpha, X)$ more closely.

$\underline{\mathsf{M}^{(-1)}(\alpha, X)}$. $\sigma_1, \sigma_2 \in \mathsf{M}(\alpha, X)$ are elements of the same simplex of $\mathsf{M}^{(-1)}(\alpha, X)$ if $p\sigma_1 = p\sigma_2$. Therefore $\mathsf{M}^{(-1)}(\alpha, X)$ is isomorphic with the subcomplex N of $\tilde{\mathscr{I}}$ determined by

$$\{\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_n\} \in \mathsf{N} \qquad \text{if} \qquad \bigcap \{A_\nu \colon \nu \in \mathfrak{m}_n\} \neq \emptyset\,.$$

This shows that N is the c.s.s. barycentric subdivision of the usual nerve of the covering α, and it is obvious that $\mathsf{M}^{(-1)}(\alpha, X)$ does not carry more information about the covering than the nerve. If $\mathscr{I}$ contains only one element, then $\mathsf{M}^{(-1)}(\alpha, X)$ is trivial; it tells only whether X is void or non void, and therefore $M^{(-1)}(X)$ is usually not defined in ordinary homotopy theory.

$\underline{\mathsf{M}^{(0)}(\alpha, X)}$. $(\xi_1, \sigma_1), (\xi_2, \sigma_2) = \mathsf{M}(\alpha, X)_n$ are elements of the same n-simplex of $\mathsf{M}^{(0)}(\alpha, X)$ if (a) $\xi_1 = \xi_2 = \xi$ and (b) $(\partial_0^k \partial_1^{n-k} \sigma_1)[\Delta_0]$ and $(\partial_0^k \partial_1^{n-k} \sigma_2)[\Delta_0]$ are in the same arc-component of

$$\bigcap \{A_\nu \colon \nu \in \langle \partial_0^k \xi \rangle\}, \quad k = 0, 1, \ldots, n\,.$$

This is true if (a) $\xi_1 = \xi_2 = \xi$ and (b) $(\partial_0^n \sigma_1)[\Delta_0]$ and $(\partial_0^n \sigma_2)[\Delta_0]$ are in the same arc-component of

$$\bigcap \{A_\nu \colon \nu \in \langle \partial_0^n \xi \rangle\}\,.$$

Therefore a simplex $\tau \in \mathsf{M}^{(0)}(\alpha, X)_n$ is determined by a simplex $\xi \in \tilde{\mathscr{I}}_n$ and an arc-component of

$$\bigcap \{A_\nu:\ \nu \in \langle \partial_0^n \xi \rangle\}\ .$$

This implies, which holds as well in the case of $\mathsf{M}^{(-1)}(\alpha, X)$, that if $\tau \in \mathsf{M}^{(0)}(\alpha, X)$ and $p\tau$ is degenerate, then τ is degenerate. A consequence of this is that every homology group

$$H_n\big(M^{(0)}(\alpha, X),\ G\big)$$

vanishes if $n+1$ is greater than or equal or the order of the covering α, i.e. the maximal number of intersecting elements of α.

Like $\mathsf{M}^{(-1)}(\alpha, X)$, $\mathsf{M}^{(0)}(\alpha, X)$ is isomorphic to the barycentric subdivision of a complex, but this complex is usually not simplicial although it has the property that all faces of a non degenerate simplex are different.

If $\mathscr{I}$ contains only one element, then $S(\alpha, X)$ and $S(X)$ as well as $M^{(n)}(\alpha, X)$ and $M^{(n)}(X)$, $n = -1, 0, 1, \ldots, \infty$, are isomorphic, and it is well known that if each continuous map of the n-sphere S^n, $(n = 0, 1, \ldots)$, into X is homotopic to a constant map, then

$$\omega_{n-1}^n:\ M^{(n)}(X) \to M^{(n-1)}(X)$$

is an isomorphism onto. This is generalized to the obvious

THEOREM 7. *If each continuous map*

$$f:\ S^n \to \bigcap \{A_\nu:\ \nu \in \mathfrak{m}\}$$

is homotopic in $\bigcap \{A_\nu:\ \nu \in \mathfrak{m}\}$ *to a constant map for all finite, non void subsets* $\mathfrak{m}$ *of* $\mathscr{I}$, *then the* $\mathscr{I}$*-map*

$$\omega_{n-1}^n:\ \mathsf{M}^{(n)}(\alpha, X) \to \mathsf{M}^{(n-1)}(\alpha, X)$$

is an isomorphism onto.

4. The c.s.s. complexes $M^{(n)}(\alpha, X)$. In general the c.s.s. complexes $S(\alpha, X)$ and $M^{(n)}(\alpha, X)$ do not satisfy the extension condition for c.s.s. complexes, and therefore they have no simple algebraic homotopy theory. Nevertheless homotopy results can be obtained by using the geometric realization of the complexes in question. (The results obtained in this way are equivalent to those obtained by applying some algebraic homotopy theory, see [4].) A c.s.s. complex has two different realizations; both are CW-complexes; the difference is that in one of the realizations there is a one to one correspondance between the open n-cells and the n-simplices of the c.s.s. complex (see [3]), and in the other there is a one to one correspondence between the open n-cells and the non degenerate n-simplices of the c.s.s. complex (see [5]). We shall use the first one, which does not involve the degeneracy operators and can therefore also be applied to s.s. complexes. If K is an s.s. complex, then $|K|$ can be defined as follows (each set K_n is regarded as having the discrete topology):

$|K|$ is the quotient space

$$\bigcup \{K_n \times \Delta_n :\ n = 0, 1, 2, \ldots\} / R ,$$

where the relation R corresponds to the identification of

$$(\partial_i \sigma^n, x) \quad \text{and} \quad \big(\sigma^n, (\partial_i \Omega_n)x\big), \qquad \sigma^n \in K_n, \quad x \in \Delta_{n-1}, \qquad i = 0, 1, \ldots, n; \quad n = 0, 1, \ldots .$$

This definition is an analogue of the one given in [5] for c.s.s. complexes, and it is not difficult to show that this realization and that of [5] are of the same homotopy type if K is a c.s.s. complex and not merely an s.s. complex. Using the definition above, we shall say that an s.s. map $f: K \to L$ is a homotopy equivalence if the induced map $\bar{f}: |K| \to |L|$ is a homotopy equivalence. This definition is consistent with those given for complexes which satisfy an extension condition. If $f: K \to L$ is a homotopy equivalence then the induced homomorphisms

$$H_n(K, G) \to H_n(L, G)$$

are isomorphisms onto for all n and G.

There exists a (non unique) c.s.s. map

$$\Psi:\ S(X) \to M(X) ,$$

such that Ψi is the identity and $i\Psi$ is homotopic to the identity, where i denotes the inclusion map (theorem 3). In the following Ψ is regarded as a fixed map chosen out of the possible ones.

If α is an arbitrary covering of X, then there exists a natural map

$$q:\ S(\alpha, X) \to S(X)$$

given by the projection

$$\tilde{\mathscr{I}} \times S(X) \to S(X) .$$

We shall use the following maps:

(1) $\psi = \Psi\big(q | M(\alpha, X)\big):\ M(\alpha, X) \to M(X)$.

(2) If $\varrho \in M^{(n)}(\alpha, X)$ is a class of simplices, then $\{\psi(\sigma):\ \sigma \in \varrho\}$ is a subclass of some class $\zeta \in M^{(n)}(X)$, and the induced map

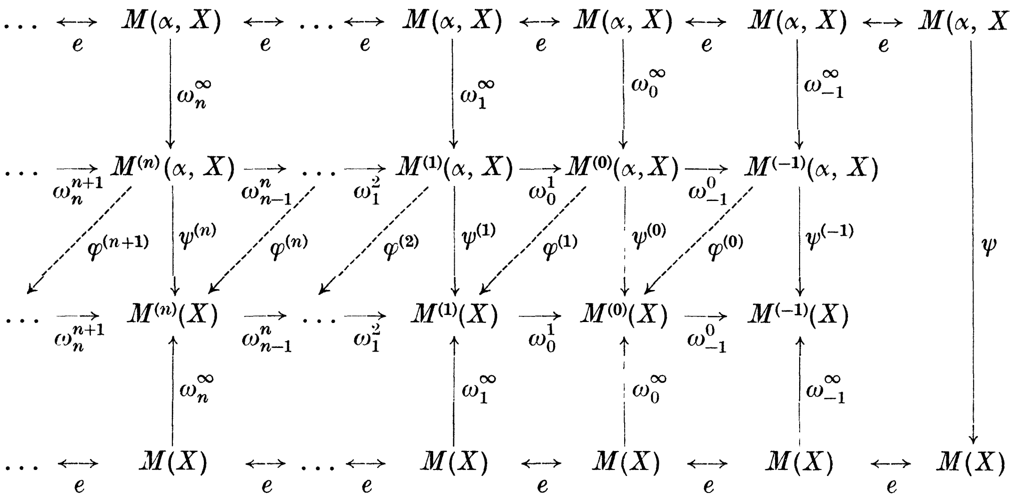
$$\psi^{(n)}:\ M^{(n)}(\alpha, X) \to M^{(n)}(X)$$

is a c.s.s. map.

(3) For $-1 \leqq s \leqq r \leqq \infty$ the maps

$$\omega_s^r:\ M^{(r)}(\alpha, X) \to M^{(s)}(\alpha, X) \quad \text{and} \quad \omega_s^r:\ M^{(r)}(X) \to M^{(s)}(X).$$

They form a system which connects the complexes $M^{(n)}(\alpha, X)$ and $M^{(n)}(X)$. It is useful to put up a diagram (where e is the identity map):



The diagram is easily verified to be commutative.

If every continuous map

$$f\colon \ S^n \to \bigcap \{A_\nu\colon \ \nu \in \mathfrak{m}\},$$

where $\mathfrak{m}$ is a finite subset of $\mathscr{I}$, is homotopic in

$$\bigcap \{A_\nu\colon \ \nu \in \mathfrak{m}\}$$

to a constant map, then

$$\omega_{n-1}^n\colon \ M^{(n)}(\alpha, X) \to M^{(n-1)}(\alpha, X)$$

has an inverse (theorem 7), and the diagram can be extended without losing the commutativity.

If every continuous map

$$f\colon \ S^n \to A_\nu, \qquad \nu \in \mathscr{I},$$

is homotopic in X to a constant map, then the diagram can be extended by adjoining a unique map

$$\varphi^{(n)}\colon \ M^{(n-1)}(\alpha, X) \to M^{(n)}(X)$$

without losing the commutativity.

This follows from the fact that if $\sigma_1, \sigma_2 \in M^{(n)}(\alpha, X)$ and $\omega_{n-1}^n(\sigma_1) = \omega_{n-1}^n(\sigma_2)$, then $\psi^{(n)}(\sigma_1) = \psi^{(n)}(\sigma_2)$.

The following theorem has a simpler related theorem (see [1, p. 196]).

THEOREM 8. *If α has an open refinement, then*

$$\psi: \quad M(\alpha, X) \to M(X)$$

is a homotopy equivalence.

PROOF: Consider the sequence of c.s.s. maps

$$M(\alpha, X) \to S(\alpha, X) \to \tilde{\mathscr{I}} \times S(X) \to S(X) \to M(X) ,$$

where the first and second are inclusion maps, the third is the projection, and the fourth is Ψ. The first and fourth maps are known to be homotopy equivalences, and we only have to prove that the inclusion map

$$S(\alpha, X) \to \tilde{\mathscr{I}} \times S(X)$$

is a homotopy equivalence, because this implies that the third map is a homotopy equivalence too. (One only has to utilise what has been proved in the special case, where α has a single non void element X and all other elements are void.)

Since α has an open refinement, the set

$$\alpha^0 = \{A_\nu^0 : \nu \in \mathscr{I}\} ,$$

where A_ν^0 is the interior of A_ν, is an open covering of X.

Now the idea is to construct a homotopy

$$h: \quad |\tilde{\mathscr{I}} \times S(X)| \times I \to |\tilde{\mathscr{I}} \times S(X)|$$

such that

(a) $h(x, 1) \in |S(\alpha^0, X)|$ for all $x \in |\tilde{\mathscr{I}} \times S(X)|$,
(b) $h(x, t) \in |S(\alpha^0, X)|$, $0 \leqq t \leqq 1$, if $x \in |S(\alpha^0, X)|$,
(c) $h(x, t) \in |S(\alpha, X)|$, $0 \leqq t \leqq 1$, if $x \in |S(\alpha, X)|$,
(d) $h(x, 0) = x$ for all $x \in |\tilde{\mathscr{I}} \times S(X)|$.

The existence of the map h proves theorem 8.

$\Delta_n \times I$ is a CW-complex with a natural cellular decomposition and a natural Euclidean metric

$$\varrho\big((x, t), (x', t')\big) = \big(\varrho(x, x')^2 + (t - t')^2\big)^{\frac{1}{2}} .$$

A s.s. complex Σ is called a Σ_n-complex if the simplices of Σ are ordered sets of points of $\Delta_n \times I$ such that the following conditions are satisfied: (1) The order of the point sets induces the maps

$$\partial_i: \quad \Sigma_p \to \Sigma_{p-1}, \qquad i = 0, 1, \ldots, p .$$

(2) The natural map

$$g: \quad |\Sigma| \to \Delta_n \times I$$

is a homeomorphism onto and determines a simplicial decomposition of $\Delta_n \times I$. (g maps each vertex of $|\Sigma|$ into the point of $\Delta_n \times I$ which determines the vertex, and is then extended linearely to $|\Sigma|$.) There exists a natural s.s. map

$$\gamma: \ \Sigma \to S(\Delta_n \times I) .$$

($\gamma(\sigma)$, $\sigma \in \Sigma$, is determined by the map $g\big||\sigma|$ of an ordered geometrical simplex into $\Delta_n \times I$.) If Σ is a Σ_n-complex then each map

$$d_i: \ \Delta_{n-1} \times I \to \Delta_n \times I$$

given by

$$d_i(x, t) = \big((\partial_i \Omega_n)x, t\big)$$

determines a Σ_{n-1}-complex $\partial_i \Sigma$ and an isomorphism δ_i: $\partial_i \Sigma$ into Σ.

With each n-simplex $\tau = (\xi, \sigma) \in \tilde{\mathscr{I}} \times S(X)$ is associated the continuous map

$$f_\tau: \ \Delta_n \times I \to X$$

defined by

$$f_\tau(x, t) = \sigma(x) ,$$

and the coverings

$$\alpha_\tau = \{f_\tau^{-1}[A_\nu]: \ \nu \in \mathscr{I}\} \qquad \text{and} \qquad \alpha_\tau^0 = \{f_\tau^{-1}[A_\nu^0]: \ \nu \in \mathscr{I}\}$$

of $\Delta_n \times I$. The map f_τ determines a c.s.s. map

$$\theta_\tau: \ \tilde{\mathscr{I}} \times S(\Delta_n \times I) \to \tilde{\mathscr{I}} \times S(X)$$

by $\theta_\tau(\xi, \sigma) = (\xi, f_\tau \sigma)$.

Let $\bar{\Sigma} = (\Sigma, p)$ be a non complete $\mathscr{I}$-complex, where Σ is a Σ_n-complex. Then there exist a natural $\mathscr{I}$-map

$$\varkappa: \ \bar{\Sigma} \to \tilde{\mathscr{I}} \times S(\Delta_n \times I)$$

given by $\varkappa(\sigma) = \big(p(\sigma), \ \gamma(\sigma)\big)$, and $\mathscr{I}$-complexes

$$\partial_i \bar{\Sigma} = (\partial_i \Sigma, p\delta_i), \qquad i = 0, 1, \ldots, n .$$

Let τ be an n-simplex of $\tilde{\mathscr{I}} \times S(X)$. An $\mathscr{I}$-complex $\bar{\Sigma}_\tau$ of the type described above will be called a *τ-complex* if

(a) $\varkappa(\sigma) \in S(\alpha_\tau^0, \Delta_n \times I)$ if $g(|\sigma|) \subset \Delta_n \times \{1\}$, $\sigma \in \bar{\Sigma}_\tau$,
(b) $\varkappa(\sigma) \in S(\alpha_\tau^0, \Delta_n \times I)$ if $\tau \in S(\alpha^0, X)$, $\sigma \in \bar{\Sigma}_\tau$,
(c) $\varkappa(\sigma) \in S(\alpha_\tau, \Delta_n \times I)$ if $\tau \in S(\alpha, X)$, $\sigma \in \bar{\Sigma}_\tau$,
(d) $v = \big((v_0^n, 0), (v_1^n, 0), \ldots, (v_n^n, 0)\big) \in \bar{\Sigma}_\tau$ (v_i^n is the i'th vertex of Δ_n), and $\theta_\tau \varkappa(v) = \tau$, $pv = p\tau$.

$\theta_\tau \varkappa$ determines a map

$$\overline{\theta_\tau \varkappa}: \ |\Sigma_\tau| \to |\tilde{\mathscr{I}} \times S(X)| ,$$

and we define

$$k_\tau = \big(\overline{\theta_\tau \varkappa}\big) g^{-1}: \ \Delta_n \times I \to |\tilde{\mathscr{I}} \times S(X)| .$$

LEMMA 4. *There exists a system* $\{\bar{\Sigma}_\tau\colon \tau \in \tilde{\mathscr{I}} \times S(X)\}$, *where* $\bar{\Sigma}_\tau$ *is a* τ-*complex and* $\partial_i \bar{\Sigma}_\tau = \bar{\Sigma}_{\partial_i \tau}$.

Let us assume lemma 4. Then, if $\tau \in (\tilde{\mathscr{I}} \times S(X))_n$,

(a) $k_\tau(\Delta_n \times \{1\}) \subset |S(\alpha^0, X)|$,
(b) $k_\tau(\Delta_n \times I) \subset |S(\alpha^0, X)|$ if $\tau \in S(\alpha^0, X)$,
(c) $k_\tau(\Delta_n \times I) \subset |S(\alpha, X)|$ if $\tau \in S(\alpha, X)$,
(d) k_τ maps the "interior" of $\Delta_n \times \{0\}$ homeomorphically onto the "interior" of $|\tau|$, $\tau \in \tilde{\mathscr{I}} \times S(X)$,
(e) $k_{\partial_i \tau} = k_\tau d_i\colon\ \Delta_{n-1} \times I \to |\tilde{\mathscr{I}} \times S(X)|$.

Now

$$h\colon\ |\tilde{\mathscr{I}} \times S(X)| \times I \to |\tilde{\mathscr{I}} \times S(X)|$$

is defined as follows: If $x \in |\tau|$, $\tau \in \tilde{\mathscr{I}} \times S(X)$ and $k_\tau(y, 0) = x$, then $h(x, t) = k_\tau(y, t)$. It is easily verified that h has the desired properties.

PROOF OF LEMMA 4. Suppose $\bar{\Sigma}_\tau$ has been constructed for each simplex $\tau \in \tilde{\mathscr{I}} \times S(X)$ of dimension less than n such that

(1) either $\varkappa(\zeta) \in S(\alpha_\tau^0, \Delta_n \times I)$ or $\langle p\zeta \rangle \subset \langle p\tau \rangle$, $\zeta \in \bar{\Sigma}_\tau$,
(2) $\partial_i \bar{\Sigma}_\tau = \bar{\Sigma}_{\partial_i \tau}$.

Then the complexes $\bar{\Sigma}_\tau$, $\tau \in (\tilde{\mathscr{I}} \times S(X))_n$ can be constructed independently. Let σ be a fixed n-simplex of $\tilde{\mathscr{I}} \times S(X)$. The $n+1$ complexes $\bar{\Sigma}_{\partial_i \sigma}$, $i = 0, 1, \ldots, n$, determine the subcomplex $\bar{\Sigma}_\sigma'$ of $\bar{\Sigma}_\sigma$ which corresponds to

$$\dot{\Delta}_n \times I \cup \Delta_n \times \{0\}\,.$$

A sequence of Σ_n-complexes $\Xi^{(i)}$, $i = 0, 1, 2, \ldots$, is defined inductively as follows:

Let a denote the barycenter of $\Delta_n \times \{1\}$ and b the barycenter of $\Delta_n \times I$. The simplices of $\Xi^{(0)}$ are

$$(b), \quad (a), \quad (b, a)$$

and

$$(v_0, v_1, \ldots, v_n), \quad (b, v_0, v_1, \ldots, v_n), \quad (a, v_0, v_1, \ldots, v_n), \quad (b, a, v_0, v_1, \ldots, v_n)\,,$$

where $(v_0, v_1, \ldots, v_n)$ is any simplex of $\bar{\Sigma}_\sigma'$.

Suppose $\Xi^{(i)}$ has been defined for $i < m$. Then $\Xi^{(m)}$ is defined in the following way. If $\zeta \in \Xi^{(m-1)}$ then $B(\zeta)$ denotes: (1) the barycenter of $g(|\zeta|)$ if $\zeta \notin \bar{\Sigma}_\sigma'$, (2) the point of $\Delta_n \times I$ which determines the leading vertex of ζ if $\zeta \in \bar{\Sigma}_\sigma'$. An n-simplex of $\Xi^{(m)}$ is an ordered n-tuple

$$\big(B(\zeta_0),\ B(\zeta_1),\ \ldots,B(\zeta_n)\big)\,,$$

where $B(\zeta_i)\neq B(\zeta_j)$ if $i\neq j$, and ζ_j is a face of ζ_i if $i<j$.

Σ_σ' is a subcomplex of each $\Xi^{(m)}$. If $\zeta\in\Xi^{(m)}$ then either (1) no face of ζ is an element of Σ_σ' or (2) there exists a face ζ' of ζ such that $\zeta'\in\Sigma_\sigma'$ and all faces of ζ, which are elements of Σ_σ', are faces of ζ'. To every $\varepsilon>0$ there exists an m such that $\zeta\in\Xi^{(m)}$ implies: (1) the diameter of $g(|\zeta|)$ is smaller than ε if no face of ζ is a simplex of Σ_σ'; otherwise (2)

$$\varrho(x,|\zeta'|)<\varepsilon \qquad \text{if} \qquad x\in|\zeta|\,.$$

This implies that if $\{D_\nu\colon\ \nu\in\mathfrak{M}\}$ is a covering with an open refinement of $\Delta_n\times I$, and $\{g(|\zeta|)\colon\ \zeta\in\Sigma_\sigma'\}$ is a refinement of

$$\{D_\nu^0\cap(\dot{\Delta}_n\times I\cup\Delta_n\times\{0\})\colon\ \nu\in\mathfrak{M}\}\,,$$

then there exists an m such that each

$$\mathfrak{m}(\zeta)=\{\nu\in\mathfrak{M}\colon\ D_\nu\subset g(|\zeta|)\},\qquad \zeta\in\Xi^{(m)}$$

is non void.

A covering $\{C_\nu\colon\ \nu\in\mathscr{I}\}$ of $\Delta_n\times I$ is defined by:

(1) $C_\nu=f_\sigma^{-1}[A_\nu]\cup\big([\Delta_n\times I\setminus\Delta_n\times\{1\}]\cap f_\sigma^{-1}[A_\nu]\big)$ if $\nu\notin\langle p\sigma\rangle$,

(2) $C_\nu=f_\sigma^{-1}[A_\nu]\cup[\Delta_n\times I\setminus\Delta_n\times\{1\}]$ if $\nu\in\langle p\sigma\rangle$.

Another covering $\{D_\nu\colon\ \nu\in\mathscr{I}\}$ of $\Delta_n\times I$ is defined by: $x\in D_\nu$ if (1) $x\in C_\nu$ and (2) $\nu\in\langle\partial_0^s p\zeta\rangle$ if $\zeta\in(\Sigma_\nu')_s$ and $x\in g(|\zeta|)$.

$\{D_\nu\colon\ \nu\in\mathscr{I}\}$ has an open refinement and, since $\Delta_n\times I$ is compact, there exists a finite set $\mathfrak{M}\subset\mathscr{I}$ such that (1) $\langle p\sigma\rangle\subset\mathfrak{M}$ and (2) $\{D_\nu\colon\ \nu\in\mathfrak{M}\}$ covers $\Delta_n\times I$ and has an open refinement. From the discussion above follows that there exists an m such that $\mathfrak{m}(\zeta)\neq\emptyset$, $\zeta\in\Xi^{(m)}$.

Now $\Sigma_\sigma=\Xi^{(m+1)}$, and $p\colon\ \Sigma_\sigma\to\tilde{\mathscr{I}}$ is defined by

$$p\big(B(\zeta_0),B(\zeta_1),\ \ldots,B(\zeta_s)\big)=\big(\mathfrak{m}(\zeta_0),\mathfrak{m}(\zeta_1),\ \ldots,\mathfrak{m}(\zeta_s)\big)\,.$$

$\bar{\Sigma}_\sigma$ is a σ-complex and

(1) either $\varkappa(\zeta)\in S(\alpha^0,\Delta_n\times I)$ or $\langle p\zeta\rangle\supset\langle p\sigma\rangle$,

(2) $\partial_i\Sigma_\sigma=\Sigma_{\partial_i\sigma}$, $i=0,1,\ \ldots,n$.

Since σ-complexes with these properties can be constructed for

$$\sigma\in(\tilde{\mathscr{I}}\times S(X))_0\,,$$

the lemma follows by induction.

Example. Finally we shall give an example which illustrates how the theory presented here may be used.

Suppose the covering $\alpha=\{A_\nu\colon\ \nu\in\mathscr{I}\}$ of X has an open refinement, and each map $f\colon S^1\to A_\nu$, $\nu\in\mathscr{I}$, is homotopic in X to a constant map. Then the diagram

$$\begin{array}{ccc} M(\alpha, X) & \xrightarrow[\omega_0^\infty]{} & M^{(0)}(\alpha, X) \\ \Big\downarrow \psi & & \Big\downarrow \varphi^{(1)} \\ M(X) & \xrightarrow[\omega_1^\infty]{} & M^{(1)}(X) \end{array}$$

is commutative, and the same is true of the derived diagram

$$\begin{array}{ccc} H_n\big(M(\alpha, X), G\big) & \xrightarrow[\bar\omega_0^\infty]{} & H_n\big(M^{(0)}(\alpha, X), G\big) \\ \Big\downarrow \bar\psi & & \Big\downarrow \bar\varphi^{(1)} \\ H_n\big(M(X), G\big) & \xrightarrow[\bar\omega_1^\infty]{} & H_n\big(M^{(1)}(X), G\big)\,. \end{array}$$

Now $\bar\psi$ has an inverse (theorem 8) and $\bar\varphi^{(1)}\bar\omega_0^\infty\bar\psi^{-1}=\bar\omega_1^\infty$. The homomorphism $\bar\omega_1^\infty$ is an invariant of the space X, and if, for some group G, $\bar\omega_1^\infty$ does not map $H_n\big(M(X),G\big)$ into 0 then $H_n\big(M^{(0)}(\alpha,X),G\big)$ contains an element different from 0, and this implies that the order of α is greater than n, because of the special structure of $M^{(0)}(\alpha, X)$.

In this way information can be obtained about the possible coverings of a space. The example given has an obvious generalization:

If α has an open refinement, $\varphi^{(n)}$ exists, $\bar\omega_{i-1}^i$, $i=1, 2, \ldots, n-1$, are isomorphisms and

$$\bar\omega_n^\infty\colon\ H_m\big(M(X), G\big)\to H_m\big(M^{(n)}(X), G\big)$$

is not constant, then α has at least order $m+1$.

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