MATH. SCAND. 6 (1958), 160–174

ON ENTIRE FUNCTIONS ALMOST PERIODIC IN TWO DIRECTIONS

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Introduction.

We shall study entire functions $f(z) = f(x+iy)$ which are almost periodic in every strip $|x| < a$ and in every strip $|y| < b$. A non-constant function with this property was constructed by R. Petersen [4]. This function is limit periodic in both directions, i.e. it can be approximated uniformly with any given accuracy by a function with a purely imaginary period in every strip $|x| < a$ and by a function with a real period in every strip $|y| < b$. The function has two Fourier-Dirichlet series of the forms

$$\sum_r a_r e^{r\alpha z} \qquad \text{and} \qquad \sum_r a_r' e^{ir\alpha' z}$$

where r runs through all rational numbers. It is easy to vary R. Petersen's method such that α and α' assume any given values although $\alpha = \alpha' = \pi$ in the actual example.

The class $\mathscr{D}$ of entire functions $f(z) = f(x+iy)$ almost periodic in every strip $|x| < a$ and in every strip $|y| < b$ is obviously closed with respect to addition, multiplication, derivation, and with respect to limit processes which are uniform in every strip $|x| < a$ and in every strip $|y| < b$. If we start from a finite number of entire functions limit periodic with the rationally independent basis numbers $\alpha_1, \ldots, \alpha_n$ in every strip $|x| < a$ and limit periodic with the rationally independent basis numbers $\alpha_1', \ldots, \alpha_{n'}'$ in every strip $|y| < b$, we can obviously combine these by addition or multiplication such that we obtain functions with exponents

$$r_1 \alpha_1 + \ldots + r_n \alpha_n$$

in the strips $|x| < a$ and with exponents

$$i(r_1 \alpha_1' + \ldots + r_n \alpha_{n'}')$$

in the strips $|y| < b$.

A basis $\beta_1, \ldots, \beta_n$ is called integral if all the coefficients r_ν which actually occur in the expressions for the exponents, are integers. The

Received September 15, 1958.

object of the present paper is to prove the existence of non-constant functions of the class $\mathscr{D}$ with integral basis in one or in both directions.

If an integral basis consists of only one number, the function is periodic in the corresponding direction. If it is almost periodic in every finite strip in the other direction, it is bounded in the period strip and, hence, everywhere, and it follows by Liouville's theorem that it is a constant. We have thus proved the following statement:

A finite basis of a non-constant function from $\mathscr{D}$ contains at least two elements.

We shall construct a non-constant function of the class $\mathscr{D}$ which has an integral basis in each direction. Each basis consists of two elements. We shall also construct a function of the class $\mathscr{D}$ which is limit periodic in every strip $|x| < a$ and has an integral basis consisting of two numbers in every strip $|y| < b$.

R. Petersen's example was constructed from an elliptic function, the poles of which were removed by displacements by the method first applied by C. Runge [5] in his proof of the theorem on approximation of analytic functions by polynomials. Our construction is carried out by a similar method, but we must start with an almost periodic, meromorphic function and the displacements of the poles must proceed according to a rather complicated strategy.

We shall first construct a class of sequences with almost periodic properties. These sequences will serve as the skeletons over which we shall build a class of meromorphic function. The displacements of the poles will be carried out in steps, which can be interpreted as operations on the skeletons. We can therefore, finally, develop the proper displacement strategy by a study of the skeletons.

§ 1. The sequences.

We shall first introduce a simple sequence depending on the following numbers and functions:

1) Two real numbers β_1 and β_2 with $\beta_1\beta_2^{-1}$ irrational.
2) A positive number d and a positive integer m.
3) A continuous mapping $\boldsymbol{v}(s) = \big(v_1(s), \ldots, v_m(s)\big)$ of the real s-axis into the m-dimensional complex $\boldsymbol{v}$-space C^m such that $\boldsymbol{v}(s) = \mathbf{0}$ when $|s| \geqq d$.

For the points $\boldsymbol{v} = (v_1, \ldots, v_m)$ of C^m we introduce the norm

$$\|\boldsymbol{v}\| = \max_{\mu} |v_\mu| .$$

We put $\beta = (\beta_1^2 + \beta_2^2)^{\frac{1}{2}} > 0$ and for every pair (p, q) of odd integers we define

$$(1) \qquad t_{pq} = (p\beta_1 + q\beta_2)\beta^{-2}\pi \qquad \text{and} \qquad s_{pq} = (q\beta_1 - p\beta_2)\beta^{-1}\pi\ .$$

Lemma 1. *The pairs (t_{pq}, s_{pq}) for which $|s_{pq}| < d$ can be arranged as a sequence (t_ν, s_ν) where ν runs through the odd integers, the sequence t_ν is increasing, and $t_{-\nu} = -t_\nu$, $s_{-\nu} = -s_\nu$.*

Proof. From (1) follows

$$(2) \qquad q = p\beta_1^{-1}\beta_2 + \beta\beta_1^{-1}\pi^{-1}s_{pq}\ ,$$

$$(3) \qquad t_{pq} = p\beta_1^{-1}\pi + \beta^{-1}\beta_1^{-1}\beta_2 s_{pq}\ .$$

For a fixed value of p at most $\beta\beta_1^{-1}\pi^{-1}d + 1$ values of s_{pq} in the interval $|s_{pq}| < d$ will give q as an odd integer in (2). Hence (3) yields only a bounded number of values t_{pq} with $|s_{pq}| < d$ in every interval of unit length. This proves the first part of the lemma, and the second part is a trivial consequence of the fact that $t_{-p,-q} = -t_{pq}$ and $s_{-p,-q} = -s_{pq}$.

Definition 1. *Corresponding to the previously given numbers and functions β_1, β_2, d, m, and $\boldsymbol{v}(s)$ we define the function*

$$\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(\beta_1, \beta_2; \boldsymbol{v}; t) = \begin{cases} \boldsymbol{v}(s_{pq}) & \text{when } t = t_{pq} \\ \mathbf{0} & \text{when } t \neq t_{pq} \text{ for all } (p, q)\ . \end{cases}$$

The pairs

$$\big(t_{pq}, \boldsymbol{v}(s_{pq})\big) \qquad \text{with} \qquad |s_{pq}| < d$$

arranged as a sequence $\big(t_\nu, \boldsymbol{v}(s_\nu)\big)$ according to lemma 1 is called the $(\beta_1, \beta_2, ; d, \boldsymbol{v})$-sequence.

The function $\boldsymbol{\varphi}$ does not depend on d. If, on the other hand, the value of d is changed such that the last condition in 3) remains satisfied, some pairs with $\mathbf{0}$ as the last member will be added to the $(\beta_1, \beta_2; d, \boldsymbol{v})$-sequence or removed from it.

Definition 2. *Let η denote a positive number. A real number is called a translation number corresponding to η for the $(\beta_1, \beta_2; d, \boldsymbol{v})$-sequence if*

$$\sup_t \|\boldsymbol{\varphi}(t+\tau) - \boldsymbol{\varphi}(t)\| \leqq \eta\ .$$

We observe that the set of differences $t_{p_2q_2} - t_{p_1q_1}$ is a subgroup of the additive group of real numbers. Hence, Kronecker's theorem implies that every translation number corresponding to an $\eta < \max_s \|\boldsymbol{v}(s)\|$ has the form $t_{p_2q_2} - t_{p_1q_1}$.

Lemma. 2. *Let η be a positive number and let $\delta > 0$ be chosen such that*

$$\|\boldsymbol{v}(s)-\boldsymbol{v}(s_0)\| \leqq \eta \qquad when \qquad |s-s_0| \leqq \delta .$$

If h_1, h_2 are two integers satisfying the condition

$$(4) \qquad |2h_2\beta_1 - 2h_1\beta_2| \leqq \beta\pi^{-1}\delta ,$$

then

$$(5) \qquad \tau = (2h_1\beta_1 + 2h_2\beta_2)\beta^{-2}\pi$$

is a translation number corresponding to η for the $(\beta_1, \beta_2; d, \boldsymbol{v})$-sequence.

PROOF. From (1) follows

$$t_{pq} + \tau = t_{p+2h_1,\, q+2h_2} ,$$

which implies that the mapping $t \to t+\tau$ maps the set of all t_{pq} onto itself. According to (1) we obtain

$$(6) \qquad \sigma = s_{p+2h_1,\, q+2h_2} - s_{pq} = (2h_2\beta_1 - 2h_1\beta_2)\beta^{-1}\pi$$

so that we get the estimate

$$|\sigma| \leqq \delta ,$$

which proves the lemma.

LEMMA 3. *Let η_1 be a positive number. With the notations of lemma 2 there exists a positive number Δ such that every simultaneous solution τ_1 of the inequalities*

$$(7) \qquad |\beta_1\tau_1| \leqq \Delta \pmod{2\pi}, \qquad |\beta_2\tau_1| \leqq \Delta \pmod{2\pi}$$

differs at most η_1 from one of the numbers τ which satisfy the conditions (4) *and* (5).

PROOF. We choose $\Delta = \min(\frac{1}{2}\delta,\ \frac{1}{2}\beta\eta_1)$. For τ_1 satisfying (7) we can find integers h_1, h_2 and real numbers Δ_1, Δ_2 satisfying $|\Delta_1|, |\Delta_2| \leqq \Delta$ such that

$$(8) \qquad \beta_1\tau_1 = 2h_1\pi + \Delta_1, \qquad \beta_2\tau_1 = 2h_2\pi + \Delta_2 .$$

These relations yield the following estimate for the difference (6)

$$|\sigma| = \beta^{-1}|\beta_2\Delta_1 - \beta_1\Delta_2| < 2\Delta \leqq \delta .$$

From (8) follows further

$$\begin{aligned} |\tau_1 - \tau| = |\tau_1 - \beta^{-2}(\beta_1{}^2\tau_1 - \beta_1\Delta_1 + \beta_2{}^2\tau_1 - \beta_2\Delta_2)| \\ = \beta^{-2}|\beta_1\Delta_1 + \beta_2\Delta_2| < 2\beta^{-1}\Delta \leqq \eta_1 , \end{aligned}$$

which proves the lemma.

DEFINITION 3. *Let $(t_\nu, \boldsymbol{v}(s_\nu))$ be a $(\beta_1, \beta_2; d, \boldsymbol{v})$-sequence and let α be a positive number. The set of pairs $(t_\nu + i\nu'\alpha^{-1}\pi, \boldsymbol{v}(s_\nu))$, where (ν, ν') runs through all pairs of odd integers, is called a skeleton of the first kind with the* *basis* $(\beta_1, \beta_2; \gamma)$, *where* γ *is an arbitrary rational multiple of* α. *The translation numbers of the* $(\beta_1, \beta_2; d, \boldsymbol{v})$*-sequence will also be called translation numbers of the skeleton.*

The skeletons of the first kind will be used for the construction of functions $f(z) \in \mathscr{D}$ limit periodic in the strips $|x| < a$ and with (β_1, β_2) as an integral basis in the strips $|y| < b$. The functions with integral basis in both kinds of strips are constructed by means of double sequences. These depend on the following numbers and functions:

1) Two pairs (β_1, β_2) and (β_1', β_2') of real numbers with $\beta_1\beta_2^{-1}$ and $\beta_1'\beta_2'^{-1}$ irrational.
2) Two positive d, d' and a positive integer m.
3) A continuous mapping $\boldsymbol{v}(s, s') = \big(v_1(s, s'), \ldots, v_m(s, s')\big)$ of the real (s, s')-plane into C^m such that $\boldsymbol{v}(s, s') = \mathbf{0}$ outside the rectangle $|s| < d$, $|s'| < d'$.

We put $\beta = (\beta_1^2 + \beta_2^2)^{\frac{1}{2}} > 0$, $\beta' = (\beta_1'^2 + \beta_2'^2)^{\frac{1}{2}} > 0$ and

$$t_{pq} = (p\beta_1 + q\beta_2)\beta^{-2}\pi, \qquad s_{pq} = (q\beta_1 - p\beta_2)\beta^{-1}\pi$$
$$t'_{p'q'} = (p'\beta_1' + q'\beta_2')\beta'^{-2}\pi, \qquad s'_{p'q'} = (q'\beta_1' - p'\beta_2')\beta'^{-1}\pi$$

where (p, q) and (p', q') run through all pairs of odd integers. According to lemma 1, the numbers (t_{pq}, s_{pq}), $(t'_{p'q'}, s'_{p'q'})$ where $|s_{pq}| < d$ and $|s'_{p'q'}| < d'$ can be arranged as sequences (t_ν, s_ν), $(t'_{\nu'}, s'_{\nu'})$ where ν and ν' run through all odd integers, the sequences t_ν and $t'_{\nu'}$ are increasing, and $t_{-\nu} = -t_\nu$, $s_{-\nu} = -s_\nu$, $t'_{-\nu'} = -t'_{\nu'}$, $s'_{-\nu'} = -s'_{\nu'}$.

Definition 4. *Corresponding to* $\beta_1, \beta_2, \beta_1', \beta_2', d, d', m$, *and* $\boldsymbol{v}(s, s')$ *we define the function*

$$\boldsymbol{\varphi}(t + it') = \begin{cases} \boldsymbol{v}(s_{pq}, s'_{p'q'}) & \text{when } t + it' = t_{pq} + it'_{p'q'} \\ \mathbf{0} & \text{when } t + it' \neq t_{pq} + it'_{p'q'} \text{ for all } (p,q),\ (p',q') \,. \end{cases}$$

The pairs $(t_\nu + it'_{\nu'}, \boldsymbol{v}(s_\nu, s'_{\nu'}))$ *are called the* $(\beta_1, \beta_2, \beta_1', \beta_2'; d, d', \boldsymbol{v})$*-sequence. It is also called a skeleton of the second kind with the basis* $(\beta_1, \beta_2, \beta_1', \beta_2')$. *The numbers* τ *and* $i\tau'$ *where* τ *and* τ' *are real are called translation numbers of the* $(\beta_1, \beta_2, \beta_1', \beta_2'; d, d', \boldsymbol{v})$*-sequence corresponding to the positive number* η *if*

$$\sup_{t, t'} \|\boldsymbol{\varphi}(t + \tau + it') - \boldsymbol{\varphi}(t + it')\| \leqq \eta$$

and

$$\sup_{t, t'} \|\boldsymbol{\varphi}\big(t + i(t' + \tau')\big) - \boldsymbol{\varphi}(t + it')\| \leqq \eta \,.$$

The two kinds of translation numbers are quite analogous and we shall, therefore, work only with the real translation numbers.

Lemma 4. *Let η_1 be a positive number. There exists a positive number Δ such that every simultaneous solution τ_1 of the inequalities* (7) *differs at most η_1 from one of the real translation numbers introduced in definition* 4.

The proof is copied from the proofs of the lemmas 2 and 3, the only difference being that δ is chosen such that $\|\boldsymbol{v}(s, s')-\boldsymbol{v}(s_0, s')\| \leqq \eta$ for all s' when $|s-s_0| \leqq \delta$.

We shall occasionally speak of a skeleton $(z_{\nu\nu'}, \boldsymbol{v}_{\nu\nu'})$ or $\big(t_\nu+it'_{\nu'}, \boldsymbol{v}_{\nu'}(s_\nu)\big)$ without stating whether it is of the first or the second kind. From the proof of lemma 1 follows:

Lemma 5. *To every skeleton $(z_{\nu\nu'}, \boldsymbol{v}_{\nu\nu'})$ corresponds a positive integer N depending only on the basis and d (and d' for the second kind) such that every closed unit square of the complex plane contains at most N of the numbers $z_{\nu\nu'}$.*

§ 2. The meromorphic functions.

We shall use the skeletons introduced in § 1 for the construction of certain meromorphic functions of a complex variable

Definition 5. *With a vector $\boldsymbol{v}=(v_1, \ldots, v_m)$ of C^m we associate the singular polynomial*

$$P(\boldsymbol{v}; z) = v_1 z^{-3}+v_2 z^{-4}+\ldots+v_m z^{-m-2}.$$

With a skeleton $(z_{\nu\nu'}, \boldsymbol{v}_{\nu\nu'})$ where $\boldsymbol{v}_{\nu\nu'} \in C^m$ we associate the meromorphic function

$$F(z) = \sum_{\nu,\nu'} P(\boldsymbol{v}_{\nu\nu'}; z-z_{\nu\nu'}) \tag{9}$$

where (ν, ν') runs through all pairs of odd numbers.

To justify this definition we must prove that the development (9) converges. Let ϱ be a positive number and let z be a complex number such that $|z-z_{\nu\nu'}| \geqq \varrho$ for every (ν, ν'). We have then

$$\begin{aligned} |P(\boldsymbol{v}_{\nu\nu'}; z-z_{\nu\nu'})| &\leqq (|z-z_{\nu\nu'}|^{-3}+\ldots+|z-z_{\nu\nu'}|^{-m-2})\cdot \max\|\boldsymbol{v}\| \\ &\leqq (1+\varrho^{-1}+\ldots+\varrho^{-m+1})|z-z_{\nu\nu'}|^{-3}\cdot \max\|\boldsymbol{v}\|. \end{aligned}$$

Only the factor $|z-z_{\nu\nu'}|^{-3}$ depends on the indices. We shall prove that $\sum|z-z_{\nu\nu'}|^{-3}$ converges uniformly in z. For a fixed z we divide the complex plane into unit squares by lines parallel to the real and imaginary axes such that z is on a vertex. The points $z_{\nu\nu'}$ in the interior of a square which does not have z as a vertex are moved to the vertex nearest z. The points $z_{\nu\nu'}$ on an edge which does not have z as an endpoint are moved to the endpoint nearest z. The points $z_{\nu\nu'}$ in the squares and on the edges adjoining z are moved to within the distance ϱ from z. This process will not make the absolute value of any term of the series smaller and, according to lemma 5 the absolute value of the sum of the resulting series is at most

$$4N\varrho^{-3}+8N\sum_{h=1}^{\infty} h^{-3}+4N\sum_{h,k=1}^{\infty}(h^2+k^2)^{-\frac{3}{2}} < 4N(\varrho^{-3}+6)$$

where the last estimate is rather crude. For the function $F(z)$ we get the estimate

$$|F(z)| \leqq 4N(\varrho^{-3}+6)(1+\varrho^{-1}+\ldots+\varrho^{-m+1}) \max\|\boldsymbol{v}\| \tag{10}$$

Theorem 1. *If $z_{11}=a+ib$, the function $F(z)$ introduced in definition 5 is almost periodic in each of the strips $|x|\leqq a-\varrho$, $|y|\leqq b-\varrho$ for every positive value of ϱ. If the skeleton is of the first kind as introduced in definition 3, the function $F(z)$ will have the period $2\alpha^{-1}\pi i$ and the basis (β_1, β_2) as integral basis in the strip $|y|\leqq b-\varrho$. If the skeleton is of the second kind as introduced in definition 4, the function $F(z)$ will have (β_1', β_2') as integral basis in the strip $|x|\leqq a-\varrho$ and (β_1, β_2) as integral basis in the strip $|y|\leqq b-\varrho$.*

Proof. From the structure of the skeletons follows immediately that $F(z)$ is regular in each of the strips $|x|<a$, $|y|<b$, and that $F(z)$ has the period $2\alpha^{-1}\pi i$ if the skeleton is of the first kind. It will be convenient for the proof to add a number of zero terms to the series (9) such that terms corresponding to every t_{pq} (definitions 1, 4) are included. The skeleton may then be written $\left(t_{pq}+it'_{\nu'}, \boldsymbol{v}_{\nu'}(s_{pq})\right)$. We can then write

$$F(z) = \sum_{p,q,\nu'} P\left(\boldsymbol{v}_{\nu'}(s_{pq}), z-t_{pq}-it'_{\nu'}\right).$$

Let η be a positive number. We choose δ as in the lemmas 2 or 4. If h_1, h_2, τ, and σ are defined as in the proof of lemma 2, we have

$$\begin{aligned} F(z+\tau) &= \sum_{p,q,\nu'} P\left(\boldsymbol{v}_{\nu'}(s_{pq}), z-(t_{pq}-\tau)-it'_{\nu'}\right) \\ &= \sum_{p,q,\nu'} P\left(\boldsymbol{v}_{\nu'}(s_{pq}), z-t_{p-2h_1,q-2h_2}-it'_{\nu'}\right) \\ &= \sum_{p,q,\nu'} P\left(\boldsymbol{v}_{\nu'}(s_{p+2h_1,q+2h_2}), z-t_{pq}-it'_{\nu'}\right) \\ &= \sum_{p,q,\nu'} P\left(\boldsymbol{v}_{\nu'}(s_{pq}+\sigma), z-t_{pq}-it'_{\nu'}\right). \end{aligned}$$

Since the polynomial P is linear in $\boldsymbol{v}$, it follows that

$$F(z+\tau)-F(z) = \sum_{p,q,\nu'} P\left(\boldsymbol{v}_{\nu'}(s_{pq}+\sigma)-\boldsymbol{v}_{\nu'}(s_{pq}), z-t_{pq}-it'_{\nu'}\right).$$

We can estimate this function by means of (10). Since the number of poles in a unit square is at most $2N$ (viz. N from $F(z)$ and N from $F(z+\tau)$, we get the estimate

$$|F(z+\tau)-F(z)| \leqq 8N(\varrho^{-3}+6)(1+\varrho^{-1}+\ldots+\varrho^{-m+1})\cdot\max_{s,\nu'}\|\boldsymbol{v}_{\nu'}(s+\sigma)-\boldsymbol{v}_{\nu'}(s)\|$$

when $|y| \leqq b-\varrho$. But we have chosen δ such that

$$\max_{s,\nu'}\|\boldsymbol{v}_{\nu'}(s+\sigma)-\boldsymbol{v}_{\nu'}(s)\| \leqq \eta\,.$$

If $\varepsilon>0$ is given and we take

$$\eta = 16^{-1}N^{-1}(\varrho^{-3}+6)^{-1}(1+\varrho^{-1}+\ldots+\varrho^{-m+1})^{-1}\varepsilon\,,$$

it follows that τ is a translation number of $F(z)$ corresponding to $\frac{1}{2}\varepsilon$ in the strip $|y| \leqq b-\varrho$.

Since $F(z)$ is analytic and bounded in the strip $|y| \leqq b-\frac{1}{2}\varrho$, the derivative $F'(z)$ is bounded in the strip $|y| \leqq b-\varrho$, and it follows that $F(z)$ is uniformly continuous in this strip. This implies that there exists a number $\eta'>0$ such that every real number τ_1 which differs at most η' from the translation number τ chosen above, is a translation number of $F(z)$ corresponding to ε. According to lemma 3 this implies that every solution τ_1 of the inequalities (7) is a translation number for $F(z)$ corresponding to ε in the strip $|y| \leqq b-\varrho$. It is proved in the classical theory of almost periodic functions that this property is characteristic for almost periodic functions with (β_1, β_2) as integral basis (see e.g. [1], pp. 111–125). The results concerning the strips $|x|<a$ for skeletons of the second kind is proved in the same way.

§ 3. Displacement of the poles.

We shall now use the displacement of the poles, first applied by C. Runge [5]. We need a certain uniformity and our basic lemma is therefore more elaborate than in the original theory. The same uniformity was quite essential, but trivial, in the case treated by R. Petersen.

Lemma 6. *Let $\varepsilon_1>0$, a complex number τ, and a positive integer m be given. To these numbers correspond a positive integer m_1 and a complex matrix $A^m_{m_1}=\{A^{\mu}_{\mu_1}\}$, $\mu=1,\ldots,m$; $\mu_1=1,\ldots,m_1$ with the following property: Let $\boldsymbol{v}=(v_1,\ldots,v_m)$ be a complex vector. Then (definition 5)*

$$|P(\boldsymbol{v};z) - P(A^m_{m_1}\boldsymbol{v};z-\tau)| \leqq \|\boldsymbol{v}\|\varepsilon_1|z-\tau|^{-3}$$

when $|z-\tau| \geqq 2\tau$.

PROOF. By means of the elementary development

$$(1-z)^{-2-\mu} = \sum_{\gamma=0}^{\infty} \binom{\mu+1+\gamma}{\mu+1} z^{\gamma}, \qquad |z| < 1$$

we obtain easily

$$z^{-2-\mu} = \sum_{\mu_1=\mu}^{\infty} \binom{\mu_1+1}{\mu+1} (-\tau)^{\mu_1-\mu}(z-\tau)^{-2-\mu_1}, \qquad |z-\tau| > |\tau| .$$

This series converges uniformly when $|z-\tau| \geqq 2|\tau|$. Hence, we can choose m_1 such that

$$\left| \sum_{\mu_1=m_1+1}^{\infty} \binom{\mu_1+1}{\mu+1} (-\tau)^{\mu_1-\mu}(z-\tau)^{1-\mu_1} \right| \leqq m^{-1}\varepsilon_1 \tag{11}$$

when

$$|z-\tau| \geqq 2|\tau|, \qquad \mu = 1, \ldots, m .$$

We define

$$A^{\mu}_{\mu_1} = \begin{cases} 0 & \text{when } \mu_1 < \mu \\ \binom{\mu_1+1}{\mu+1} (-\tau)^{\mu_1-\mu} & \text{when } \mu_1 \geqq \mu \end{cases}$$

and we introduce the m-dimensional basis vectors

$$\boldsymbol{e}^1 = (1, 0, \ldots, 0), \ldots, \boldsymbol{e}^m = (0, \ldots, 0, 1) .$$

The inequality (11) is then equivalent to the following:

$$|P(\boldsymbol{e}^{\mu}; z) - P(A^m_{m_1}\boldsymbol{e}^{\mu}; z-\tau)| \leqq m^{-1}\varepsilon_1 |z-\tau|^{-3} .$$

Since $P(\boldsymbol{v}; z)$ is linear in $\boldsymbol{v}$, this implies

$$|P(\boldsymbol{v}; z) - P(A^m_{m_1}\boldsymbol{v}; z-\tau)| \leqq m^{-1}\varepsilon_1 |z-\tau|^{-3} \sum |v_{\mu}| \leqq \|\boldsymbol{v}\| \varepsilon_1 |z-\tau|^{-3} ,$$

which proves the lemma.

DEFINITION 6. *Let $(z_{\nu\nu'}, \boldsymbol{v}_{\nu\nu'})$ be a skeleton while τ is a complex number and $A^m_{m^*} = \{A^{\mu}_{\mu^*}\}$, $\mu = 1, \ldots, m$; $\mu^* = 1, \ldots, m^*$ is a complex matrix. The set of pairs $(z_{\nu\nu'} + \tau, A^m_{m^*}\boldsymbol{v}_{\nu\nu'})$ will be called the $(\tau; A^m_{m^*})$-displacement of $(z_{\nu\nu'}, \boldsymbol{v}_{\nu\nu'})$ and the function*

$$F(\tau; A^m_{m^*}; z) = \sum_{\nu\nu'} P\big(A^m_{m^*}\boldsymbol{v}_{\nu\nu'}; z-(z_{\nu\nu'}+\tau)\big)$$

will be called the $(\tau; A^m_{m^*})$-displacement of the function

$$F(z) = \sum_{\nu\nu'} P(\boldsymbol{v}_{\nu\nu'}; z-z_{\nu\nu'}) .$$

The straight segment with endpoints $z_{\nu\nu'}$ and $z_{\nu\nu'}+\tau$ is called essential for the $(\tau, A^m_{m^*})$-displacement if $\boldsymbol{v}_{\nu\nu'} \neq \boldsymbol{0}$.

Lemma 7. *Let $\varepsilon > 0$, $\varrho > 0$, a complex number τ, and a skeleton $(z_{\nu\nu'}, \boldsymbol{v}_{\nu\nu'})$ be given. With the notations of definition 6 we can choose m^* and $\boldsymbol{A}^m_{m^*}$ such that*

$$|F(\tau;\, \boldsymbol{A}^m_{m^*};\, z) - F(z)| \leqq \varepsilon$$

when z has distance $\geqq \varrho$ from every straight segment essential for the $(\tau, \boldsymbol{A}^m_{m^*})$-displacement.

Proof. We shall first assume that $|\tau| \leqq \frac{1}{2}\varrho$. If we take $\boldsymbol{A}^m_{m^*} = \boldsymbol{A}^m_{m_1}$ introduced in lemma 6, we get

$$|F(z) \;-\; F(\tau;\, \boldsymbol{A}^m_{m_1};\, z)| \;\leqq\; \varepsilon_1 \sup \|\boldsymbol{v}_{\nu\nu'}\| \sum_{\nu\nu'} |z - (z_{\nu\nu'} + \tau)|^{-3}$$

when z has distance $\geqq 2\tau$ from every $z_{\nu\nu'} + \tau$ with $\boldsymbol{v}_{\nu\nu'} \neq \boldsymbol{0}$. The sum on the right was estimated in the remark following definition 5. It follows that

$$|F(z) - F(\tau;\, \boldsymbol{A}^m_{m_1};\, z)| \;\leqq\; 4\varepsilon_1 \sup \|\boldsymbol{v}_{\nu\nu'}\| N(\varrho^{-3} + 6)$$

which proves the lemma.

In the general case we choose a positive integer h such that $\tau_1 = h^{-1}\tau$ satisfies the condition $|\tau_1| \leqq \frac{1}{2}\varrho$. From the special case of the lemma follows that we can choose the matrices $\boldsymbol{A}^{m_\varkappa}_{m_{\varkappa+1}}$; $\varkappa = 0, 1, \ldots, h-1$; $m_0 = m$, such that

$$|F(0;\, \boldsymbol{A}^{m_{\varkappa-1}}_{m_\varkappa} \ldots \boldsymbol{A}^{m_0}_{m_1};\, z) \;-\; F(\tau_1;\, \boldsymbol{A}^{m_\varkappa}_{m_{\varkappa+1}} \ldots \boldsymbol{A}^{m_0}_{m_1};\, z)| \;\leqq\; \varepsilon/h\,; \qquad \varkappa = 1, \ldots, h-1\,.$$

If we replace z by $z - \varkappa\tau_1$, the index 0 is replaced by $\varkappa\tau_1$ and the index τ_1 by $(\varkappa + 1)\tau_1$. If we add these inequalities, we find that the matrix

$$\boldsymbol{A}^m_{m^*} \;=\; \boldsymbol{A}^{m_{h-1}}_{m_h} \ldots \boldsymbol{A}^m_{m_1}$$

satisfies the condition of the lemma.

In the following we shall use the "injection matrices"

$$\boldsymbol{E}^m_{m^*} \;=\; \{E^\mu_{\mu^*}\}\,; \qquad \mu \;=\; 1, \ldots, m\,;\;\; \mu^* \;=\; 1, \ldots, m^*\,;\;\; m^* \geqq m\,,$$

where

$$E^\mu_{\mu^*} = \begin{cases} 1 & \text{when} \quad \mu^* = \mu \\ 0 & \text{when} \quad \mu^* \neq \mu\,. \end{cases}$$

The mapping $\boldsymbol{v}^* = \boldsymbol{E}^m_{m^*}\boldsymbol{v}$ is the "natural" mapping of C^m onto the subspace $\overset{*}{v}_{m+1} = \ldots = \overset{*}{v}_{m^*} = 0$ of C^{m^*}.

Theorem 2. *Let $\big(t_\nu + i\nu'\alpha^{-1}\pi,\, \boldsymbol{v}(s_\nu)\big)$ be a skeleton of the first kind, let $F(z)$ be the corresponding function* (definition 5), *and let $\varepsilon > 0$, $\varrho > 0$ be given. There exists a skeleton $\big(t_\nu + 3i\nu'\alpha^{-1}\pi,\, \boldsymbol{v}^*(s_\nu)\big)$ with the corresponding function $F^*(z)$ such that*

$$|F^*(z)-F(z)| \leqq \varepsilon$$

in each of the strips $|x| \leqq t_1-\varrho$ *and* $|y| \leqq \alpha^{-1}\pi-\varrho$.

PROOF. Let $F^l(z)$, $l=0, \pm 2$ be the functions corresponding to the skeletons $(t_\nu+i(3\nu'+l)\alpha^{-1}\pi,\ \boldsymbol{v}(s_\nu))$. According to lemma 7 with $\tau=2i\alpha^{-1}\pi$ there exist matrices ${}^+\boldsymbol{A}^m_{m*}$ and ${}^-\boldsymbol{A}^m_{m*}$ such that

$$|F^2(-\tau;{}^-\boldsymbol{A}^m_{m*};z)-F^2(z)| \leqq \tfrac{1}{2}\varepsilon; \qquad |F^{-2}(\tau;{}^+\boldsymbol{A}^m_{m*};z)-F^{-2}(z)| \leqq \tfrac{1}{2}\varepsilon$$

when z has distance $\geqq \varrho$ from every segment with endpoints

$$t_\nu + i(3\nu' \pm 2)\alpha^{-1}\pi .$$

We can choose the two matrices with the same number of columns since multiplication on the left by an injection matrix will have no influence on the corresponding functions. But the function

$$F^*(z) = F(z) + \big(F^2(-\tau;{}^-\boldsymbol{A}^m_{m*};z)-F^2(z)\big) + (F^{-2}(\tau;{}^+\boldsymbol{A}^m_{m*};z)-F^{-2}(z)$$

corresponds obviously to the skeleton

$$\big(t_\nu+3i\nu'\alpha^{-1}\pi,\ (\boldsymbol{E}^m_{m*}+{}^-\boldsymbol{A}^m_{m*}+{}^+\boldsymbol{A}^m_{m*})\,\boldsymbol{v}(s_\nu)\big)$$

and that proves the lemma.

§ 4. The strategy.

In this section we shall consider a skeleton of the first or second kind which can be written on any of the two forms

$$(12) \qquad (t_\nu+it'_{\nu'},\ \boldsymbol{v}_{\nu\nu'}) \qquad \text{or} \qquad \big(t_{pq}+it'_{\nu'},\ \boldsymbol{v}_{\nu'}(s_{pq})\big)\,.$$

We shall put $t_1+it'_1=a+ib$ such that the corresponding function is regular in each of the strips $|x|<a$, $|y|<b$. We shall also use another skeleton

$$(13) \qquad (t_\nu+it'_{\nu'},\ \boldsymbol{u}_{\nu\nu'}) \qquad \text{or} \qquad \big(t_{pq}+it'_{\nu'},\ \boldsymbol{u}_{\nu'}(s_{pq})\big)$$

which differs from (12) only by the choice of the vector function. The function $\boldsymbol{u}$ shall belong to the same space C^m as $\boldsymbol{v}$. We shall further use the integers h_1, h_2 and the numbers τ and σ such that (1), (5) and (6) are satisfied, but we shall not assume that τ is a particularly fine translation number. Let $G(z)$ be the function corresponding to the skeleton (13).

LEMMA 8. *Let* $\varepsilon>0$, $\varrho>0$ *be given. Let* $\boldsymbol{A}^m_{m*}$ *be the matrix which according to lemma* 7 *corresponds to* ε, ϱ, τ, *and the skeleton* (13). *The function*

$$G(\tau;\boldsymbol{A}^m_{m*};z) - G(z)$$

will then correspond to the skeleton

$$\left(t_{pq}+it'_{\nu'},\ A^m_{m*}u_{\nu'}(s_{pq}-\sigma)-E^m_{m*}u_{\nu'}(s_{pq})\right).$$

PROOF. The function $G(\tau;\, A^m_{m*};\, z)$ corresponds to the skeleton

$$\left(t_{pq}+\tau+it'_{\nu'},\ A^m_{m*}u_{\nu'}(s_{pq})\right) = \left(t_{p+2h_1,q+2h_2}+it'_{\nu'},\ A^m_{m*}u_{\nu'}(s_{pq})\right).$$

If we write $(p-2h_1,\, q-2h_2)$ instead of (p, q), this turns into

$$\left(t_{pq}+it'_{\nu'},\ A^m_{m*}u_{\nu'}(s_{p-2h_1,q-2h_2})\right) = \left(t_{pq}+it'_{\nu'},\ A^m_{m*}\,u_{\nu'}(s_{pq}-\sigma)\right)$$

which proves the lemma.

Theorem 2 may be considered as the "periodic case" of the following theorem.

THEOREM 3. *To the skeleton* (12), $\varrho>0$, $\varepsilon>0$, *and* $a_1>a$ *corresponds a new vector function* $v^*_{\nu'}(s)$ *such that the function* $F^*(z)$ *corresponding to the skeleton* $\left(t_{pq}+it'_{\nu'},\ v^*_{\nu'}(s_{pq})\right)$ *is regular in each of the strips* $|x|<a_1$, $|y|<b$, *and satisfies the condition*

$$|F^*(z)-F(z)| \leqq \varepsilon \tag{14}$$

in each of the strips

$$|x| \leqq a-\varrho, \qquad |y| \leqq b-\varrho. \tag{15}$$

PROOF. The condition that $F^*(z)$ is regular when $|x|<a_1$ will be satisfied if $v^*_{\nu'}(s)=\mathbf{0}$ when $|s|\geqq d_1$ where d_1 is a sufficiently small positive number. It will suffice to prove that we can determine $v^*_{\nu'}(s)$ corresponding to $d_1=\frac{2}{3}d$ such that the condition (14) is satisfied in the strips (15).

We shall use the following *standard operation* on the skeleton: We choose $\tau=t_{p_2q_2}-t_{p_1q_1}$ and the corresponding $\sigma=\sigma_{p_2q_2}-s_{p_1q_1}$. We choose a vector function $u_{\nu'}(s)$ of the same dimension m as $v_{\nu'}(s)$ and we choose a matrix A^m_{m*}. We replace v by the new vector function

$$E^m_{m*}\left(v_{\nu'}(s)-u_{\nu'}(s)\right) + A^m_{m*}u_{\nu'}(s-\sigma). \tag{16}$$

From lemma 7 and lemma 8 follows that *the matrix* A^m_{m*} *can be chosen such that the standard operation changes the corresponding function arbitrarily little in the strips* (15), *if no straight segment essential for the* $(\tau,\, A^m_{m*})$*-displacement enters the strip* $|x|<a$.

To prove the theorem we must find a particular strategy for the application of standard operations to the skeleton (12) such that the vector function after a finite number of steps becomes $\mathbf{0}$ when $|s|\geqq\frac{2}{3}d$.

For positive numbers $r_1>r_2>0$ we introduce a continuous function $\varphi_{r_1r_2}(s)$ which is 0 when $s\leqq r_2$ and 1 when $s\geqq r_1$. We shall always choose

$$u_{\nu'}(s) = \varphi_{r_1r_2}(s)\, v_{\nu'}(s) \qquad \text{or} \qquad u_{\nu'}(s) = \varphi_{r_1r_2}(-s)\, v_{\nu'}(s).$$

We choose an integer N such that the finite sequence $s_1, \ldots, s_N$ contains an element s_{ν_1} satisfying $0 < s_{\nu_1} \leqq \frac{1}{3}d$ and an element $s_{\nu_{-1}}$ satisfying $0 > s_{\nu_{-1}} \geqq -\frac{1}{3}d$. We shall say that a term s_ν is trivial if $\boldsymbol{v}_{\nu'}(s) = \boldsymbol{0}$ when $|s| \geqq |s_\nu|$. Among the non-trivial terms of the finite sequence we choose the one $s_{\nu*}$ with the greatest absolute value.

To fix the ideas, we shall assume that $s_{\nu*} > 0$. If no s_ν satisfies the inequalities $\frac{2}{3}d \leqq |s_\nu| < s_{\nu*}$, we define $r_1 = \frac{2}{3}d$, otherwise $r_1 = s_{\nu*}$. If $s_{\nu*} \geqq \frac{2}{3}d$, we choose $s_{\nu_0} = s_{\nu*}$. Otherwise we choose s_{ν_0} as the first term of the series $s_1, s_3, \ldots$ which is $\geqq \frac{2}{3}d$ (its index may be $> N$). We shall not fix the value of r_2 yet, but we shall require that $0 < r_2 < r_1$ and that no s_ν of the finite sequence satisfies $r_2 < |s_\nu| < r_1$. If we choose

$$\boldsymbol{u}_{\nu'}(s) = \varphi_{r_1 r_2}(s)\, \boldsymbol{v}_{\nu'}(s) ,$$

the vector $\boldsymbol{u}_{\nu'}(s_\nu)$ will be $\boldsymbol{0}$ when $|\nu| \leqq N$ and when $0 < \nu \leqq \nu_0$ except when $\nu = \nu_0$. We choose $\tau = t_{\nu_1} - t_{\nu_0}$, hence $\sigma = s_{\nu_1} - s_{\nu_0}$. We can then be certain that *no segment essential for a* $(\tau, A^m_{m^*})$*-displacement can reach the strip* $|x| < a$. To finish the proof we must determine the interval where the last term in (16) is $\neq \boldsymbol{0}$. The function $\boldsymbol{u}_{\nu'}(s)$ is $\boldsymbol{0}$ outside the interval $r_2 < s < d$. Hence

$$\boldsymbol{u}_{\nu'}(s-\sigma) = \boldsymbol{0} \qquad \text{outside} \qquad r_2+\sigma < s < d+\sigma .$$

From $0 < s_{\nu_1} \leqq \frac{1}{3}d$ and $\frac{2}{3}d \leqq s_{\nu_0} < d$ follows that $-d < \sigma \leqq +\frac{1}{3}d$. We fix the value of r_2 such that $r_2 \geqq \frac{1}{3}d$. Then

$$\boldsymbol{u}_{\nu'}(s-\sigma) = \boldsymbol{0} \qquad \text{outside} \qquad -\tfrac{2}{3}d < s < \tfrac{2}{3}d .$$

The process has thus replaced the vector function by a new function which is $\boldsymbol{0}$ when $s \geqq s_{\nu_0}$. Afterwards we use the symmetric standard operation for the sequence $s_{-1}, \ldots, s_{-N}$. Since the last term in (16) is $\boldsymbol{0}$ when $|s| \geqq \frac{2}{3}d$, this will give us a vector function which is $\boldsymbol{0}$ when $|s| \geqq s_{\nu_0}$.

If $s_{\nu*} < 0$, we take $r_1 = -s_{\nu*}$ or $\frac{2}{3}d$, $s_{\nu_0} = s_{\nu*}$ or $\leqq -\frac{2}{3}d$,

$$\boldsymbol{u}_{\nu'}(s) = \varphi_{r_1 r_2}(-s)\, \boldsymbol{v}_{\nu'}(s) ,$$

and $\tau = t_{\nu_{-1}} - t_{\nu_0}$. The proof will be the same. In the case where $r_1 = \frac{2}{3}d$, the proof is finished with the first step. Otherwise, one more of the terms $s_1, \ldots, s_N$ will be trivial, and after a finite number of steps we get $r_1 = \frac{2}{3}d$, and the proof will then be finished with the following step.

§ 5. The limit process.

We consider a skeleton with the corresponding function $F_0(z)$ which is regular in each of the strips $|x| < a_0$ and $|y| < b_0$. We choose a positive number $\varrho < \min(a_0, b_0)$ and a convergent series

$$\varepsilon_1+\varepsilon_2+\ldots = \varepsilon$$

where each ε_ν is positive and ε is so small that the inequality

$$|F_0(z_2)-F_0(z_1)| > 2\varepsilon \tag{17}$$

is satisfied for some values of z_1 and z_2 in the union of the strips $|x|<a_0-\varrho$ and $|y|<b_0-\varrho$. Finally, we choose two increasing sequences

$$a_0 < a_1 < \ldots \to \infty \qquad \text{and} \qquad b_0 < b_1 < \ldots \to \infty .$$

According to theorem 2 and theorem 3 we can find a sequence of skeletons with the same basis as the first skeleton such that the sequence $F_0(z)$, $F_1(z)$, ... of corresponding functions satisfies the following conditions:

1) $F_n(z)$ is regular in each of the strips $|x|<a_n$ and $|y|<b_n$.
2) $|F_n(z)-F_{n-1}(z)| \leqq \varepsilon_n$ when $|x| \leqq a_{n-1}-\varrho$ and when $|y| \leqq b_{n-1}-\varrho$.

The last condition implies that the sequence $F_N(z)$, $F_{N+1}(z)$, ... converges uniformly in each of the strips $|x| \leqq a_{N-1}-\varrho$ and $|y| \leqq b_{N-1}-\varrho$. The limit function $f(z)$ is regular in each of these strips, hence, in the whole z-plane since $a_{N-1}-\varrho \to \infty$ and $b_{N-1}-\varrho \to \infty$. It is not constant since the condition (17) implies that $|f(z_2)-f(z_1)|>0$. If the skeleton is of the first kind, the function $f(z)$ will be limit periodic in the imaginary direction with the α from the skeleton as basis number, and it will be almost periodic in the real direction with the numbers β_1 and β_2 from the skeleton as an integral basis. If the skeleton is of the second kind, the function $f(z)$ will be almost periodic in both directions and have the basis numbers of the skeleton as entire bases. This completes the construction.

§ 6. Some variants of the method.

The preceding construction has a number of variants and we shall mention a few of them.

If we use the simple skeleton $\big(it_\nu,\, v(s_\nu)\big)$ and displacements in the direction of the y-axis, we obtain an entire function almost periodic with entire basis in every strip $a<x<b$ where $ab>0$, but with different Fourier-Dirichlet series when $x<0$ and when $x>0$. If we start from the skeleton $\big(a+it_\nu,\, v(s_\nu)\big)$, we can use displacements in the direction of the y-axis alternating with small displacements in the direction of the x-axis such that $a \to 0$. If we choose the standard operations such that we always get a fine approximation on the y-axis, we can get an entire function almost periodic on every vertical line but not in every finite vertical strip.

These examples are analogous to examples given by R. Petersen in the limit periodic case. R. Petersen's results [3] concerning the existence of entire functions with a given everywhere dense set of vertical strips as maximal strips for limit periodicity can then be parallelled for functions with integral basis.

If we return to the main construction, but use displacements in one direction alternating with small displacements in the other direction, we can obtain functions almost periodic in every finite strip in one direction and almost periodic on every line in the other direction, but with an infinity of maximal strips of almost periodicity in this direction. We can also use a skeleton where the pairs (p, q) run through all pairs of integers. We can then obtain an entire function almost periodic in every horizontal strip and with two maximal strips of almost periodicity in the imaginary direction.

R. Petersens original example can be varied such that the resulting function becomes almost periodic in 3 directions, but it is not known if almost periodicity in more than 3 directions can occur for entire functions.

It is an unsolved problem whether or not the class $\mathscr{D}$ contains functions of finite order. It is also an open question whether or not functions of $\mathscr{D}$ with a finite basis may have an analytic spatial extension (for the definition of this notion, see e.g. [2]).

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