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CONVEX IDEALS IN ORDERED GROUP ALGEBRAS AND THE UNIQUENESS OF THE HAAR MEASURE

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Typical ordered rings are given by ordered rings of real-valued functions with pointwise operations and pointwise ordering. When the multiplication is no longer pointwise but a convolution product

$$f*g(x) = \int f(xy^{-1})g(y)\,dy$$

with respect to the Haar measure dy on a locally compact abelian group G, we have (at least) two natural ways to define the order. The order which fits best with the Fourier transformation is, of course, the order which is induced by the pointwise order of the Fourier transforms thus making the Fourier transformation into an order preserving mapping. This is obtained by taking finite sums of functions of the form $f*f^*$ as positive. This ordering is, for instance, well known from spectral theory. On the other hand the family of real-valued integrable functions on G also forms an ordered ring L_R^1 with respect to convolution, pointwise addition and pointwise ordering (almost everywhere). This pointwise ordering of convolution algebras is also pertinent in various contexts. Integration theory is, in fact, based on this ordering; and an invariant Radon measure (i.e. the Haar measure) on G gives rise to an order preserving ring homomorphism of L_R^1 onto the reals which has a closed regular convex maximal ideal of codimension one as kernel. The uniqueness of the Haar measure already gives us some information with respect to the occurrence of such maximal ideals in L_R^1, namely, that there is exactly one of them which contains the set

$$T = \{f-f_a\}_{a\in G,\, f\in L_R^1}$$

where f_a denotes the translate defined by

$$f_a(x) = f(ax)\,.$$

The main purpose of the present note is to show that this result can be

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improved. In fact the uniqueness of the maximal ideal given as the kernel of the Haar measure can still be proved after having removed both the condition that it shall contain T and the condition that it shall be of codimension one. The removal of the former condition is a trivial matter by looking at the expression of the Fourier transform in terms of characters. Corollary 2 of Theorem 1 gives perhaps the most suggestive formulation of what corresponds to the simultaneous removal of the two conditions in question. Looking at the Haar measure from the point of view of its kernel this result may thus be considered as a sort of sharpening supplement to the uniqueness of the Haar measure. We also give a couple of other simple facts which are related to the scarcity of convex ideals in L_R^1, in particular that L_R^1 can never be embedded with preservation of algebraic and order structure in a direct product of linearly ordered rings.

Let us first fix our notations. As already remarked, L_R^1 is the ordered group algebra of all real-valued integrable functions on G under the ordering $f \geqq g$ whenever $f(x) \geqq g(x)$ a.e. on G. L_C^1 shall denote the usual group algebra of all complex-valued integrable functions on G. We recall that an ideal $\mathfrak{a}$ in a (commutative) ring R is called regular whenever $R/\mathfrak{a}$ has an identity. The ideal $\mathfrak{a} \subseteq L_R^1$ is said to be convex if f, $g \in \mathfrak{a}$ and $f \leqq h \leqq g$ implies $h \in \mathfrak{a}$; $\mathfrak{a}$ is absolutely convex if $|g| \leqq |f|$ with $g \in L_R^1$ and $f \in \mathfrak{a}$ implies $g \in \mathfrak{a}$, A convex ideal $\mathfrak{a}$ is absolutely convex if and only if $f \in \mathfrak{a}$ implies $|f| \in \mathfrak{a}$. Or otherwise expressed, $\mathfrak{a}$ is absolutely convex if and only if $f \in \mathfrak{a}$ implies

$$f \circ g = |f| \cap |g| \in \mathfrak{a} \qquad \text{for all} \quad g \in L_R^1 .$$

(Relevant material on convex ideals can be found in [2] Chapter 5[1]). By the term *closure* we shall always mean the topological closure in L_C^1 or L_R^1 induced by the usual L^1-norm. As to Fourier analysis we use the usual notations. $\hat{G}$ is the character group or dual group of G, $\hat{f}$ is the Fourier transform of f and f^* is defined by $f^*(x) = \overline{f(x^{-1})}$.

We give the following two immediate lemmas for ready reference.

Lemma 1. *Any regular maximal ideal $\mathfrak{m}_R$ in L_R^1 is of the form $\mathfrak{m}_C \cap L_R^1$ where $\mathfrak{m}_C$ is a regular maximal ideal in L_C^1; in particular $\mathfrak{m}_R$ is closed.*

Proof. If e is an identity modulo $\mathfrak{m}_R$, i.e.

$$f \equiv e * f (\operatorname{mod} \mathfrak{m}_R) \qquad \text{for all} \quad f \in L_R^1 ,$$

it follows that e is also an indentity in L_C^1 modulo the ideal $\mathfrak{m}_R + i\mathfrak{m}_R$

[1] We wish to thank Professors Gillman and Jerison for allowing us to see the manuscript of their forthcoming book [2].

consisting of all functions $f+ig$ with $f, g \in \mathfrak{m}_R$. Thus $\mathfrak{m}_R + i\mathfrak{m}_R$ being a regular ideal in $L_C{}^1$ is contained in a maximal regular ideal $\mathfrak{m}_C$ in $L_C{}^1$ which clearly satisfies

$$\mathfrak{m}_R = \mathfrak{m}_C \cap L_R{}^1 .$$

The fact that $\mathfrak{m}_R$ is closed in $L_R{}^1$ follows from this equation together with the well-known fact that $\mathfrak{m}_C$ is closed in $L_C{}^1$.

LEMMA 2. *The closure of an absolutely convex set is itself absolutely convex.*

PROOF. This is an immediate consequence of the fact that the mapping

$$(f, g) \to f \circ g = |f| \cap |g|$$

of $L_R{}^1 \times L_R{}^1$ into $L_R{}^1$ is continuous in the L^1-norm and that the absolutely convex sets Q are just the sets which have the ideal property that $f \circ g \in Q$ whenever $f \in Q$ and $g \in L_R{}^1$.

THEOREM 1. *The only regular and convex maximal ideal in $L_R{}^1$ is the maximal ideal $\mathfrak{m}_R{}^0$ consisting of all functions in $L_R{}^1$ with zero integral. $L_R{}^1$ does not contain any proper absolutely convex ideal which is either closed or regular.*

PROOF. First, $\mathfrak{m}_R{}^0$ is convex since $f \geqq g$ with $f, g \in \mathfrak{m}_R{}^0$ implies $f=g$. Suppose next that $\mathfrak{m}_R$ is a regular convex maximal ideal in $L_R{}^1$ different from $\mathfrak{m}_R{}^0$. Let by Lemma 1, $\mathfrak{m}_C$ be a regular maximal ideal in $L_C{}^1$ such that

$$\mathfrak{m}_R = \mathfrak{m}_C \cap L_R{}^1 ,$$

and let $\alpha \in \hat{G}$ be the corresponding character. Since α is different from the identity character it is possible to find a compact neighborhood K of the identity in $\hat{G}$ such that

$$\alpha \notin KK^{-1} .$$

Let $\hat{f}$ be a continuous positive non-zero function on $\hat{G}$ with support contained in K. Then $\hat{f} * \hat{f}^*$ will be a non-zero continuous function with support contained in KK^{-1}, i.e. such that

$$\hat{f} * \hat{f}^*(\alpha) = 0 .$$

The function

$$g = f\bar{f} = |f|^2$$

having $\hat{f} * \hat{f}^*$ as Fourier transform will according to the Fourier inversion theorem be a positive non-zero function belonging to $\mathfrak{m}_C$ since $\hat{g}(\alpha) = 0$. Since g is at the same time real-valued we have

$$g \in L_R{}^1 \cap \mathfrak{m}_C \quad \text{and hence} \quad g \in \mathfrak{m}_R .$$

Being translation invariant $\mathfrak{m}_R$ will also contain a positive function h

which is $> \varepsilon > 0$ on a neighborhood of the identity element $e \in G$. Now nh also belongs to $\mathfrak{m}_R$ for any positive integer n and we can choose for any sufficiently small neighborhood V of e a function h_V such that

$$0 < h_V \leqq nh \quad \text{on} \quad V$$

for a suitable n and such that the h_V's constitute an approximate identity for L_R^1, i.e.

$$\lim_V (h_V * f) = f \quad \text{for any} \quad f \in L_R^1 .$$

Since $\mathfrak{m}_R$ is supposed to be convex we have $h_V \in \mathfrak{m}_R$ and because $\mathfrak{m}_R$ is closed by Lemma 1 we get

$$f = \lim_V (h_V * f) \in \mathfrak{m}_R \quad \text{for any} \quad f \in L_R^1$$

contradicting that $\mathfrak{m}_R$ is proper.

For the latter half of the theorem let $\mathfrak{a}$ be an absolutely convex ideal in L_R^1 which is closed in the L^1-norm. If $\mathfrak{a} \neq 0$ we have a function $f \neq 0$ such that $f \in \mathfrak{a}$ and hence also $|f| \in \mathfrak{a}$. By regularization, i.e. by taking a suitable convolution product we can assume that f is continuous and by the translation invariance of $\mathfrak{a}$ we have a function $g \in \mathfrak{a}$ such that $g > \varepsilon > 0$ on a suitable neighborhood of the origin. By the same reasoning as above we get $\mathfrak{a} = L_R^1$, hence L_R^1 contains no proper closed absolutely convex ideal ($\neq 0$). Since by Lemma 2 the closure of an absolutely convex regular ideal is again absolutely convex and regular (counting here L_R^1 as regular) this shows in particular that any absolutely convex regular ideal $\neq 0$ is dense in L_R^1. Such an ideal can therefore not be proper since a proper regular ideal is contained in a maximal regular ideal and such a maximal regular ideal is closed according to Lemma 1. This completes the proof of the theorem.

COROLLARY 1. *Any regular maximal ideal in L_R^1 which is convex has codimension* 1.

In fact $\mathfrak{m}_R^0$ is according to Theorem 1 the only regular maximal ideal in L_R^1 which is convex and it has codimension one being the kernel of the Haar measure.

A more striking formulation of the same fact is the following

COROLLARY 2. *Let μ be an order-preserving ring homomorphism of L_R^1 onto an ordered field F. Then the field F is isomorphic to the field of real numbers and μ is the Haar measure of G.*

The last part of the theorem can be interpreted similarly. That there are no proper regular and absolutely convex ideals in L_R^1 gives for instance the following

COROLLARY 3. *No non-trivial lattice ordered ring with an identity can occur as the homomorphic image of L_R^1 if the homomorphism also preserves the lattice operations.*

We shall now derive a stronger result than Theorem 1 which will determine the necessary and sufficient condition that an intersection of regular maximal ideals in L_R^1 be convex. The expected condition is of course that the intersection in question is contained in $\mathfrak{m}_R^0$. But for the convenience of the proof of this we shall first give a lemma which expresses this condition in terms of the Stone topology in the space $\mathfrak{M}_C$ of maximal regular ideals of the complex algebra L_C^1. The Stone topology is what is called the hull-kernel topology in [3], p. 56. The Gelfand topology is the topology in $\mathfrak{M}_C$ which transferred to the characters gives the usual topology in $\hat{G}$. We note further that by Lemma 1 we can associate to each $\mathfrak{m}_R$ in the family $\mathfrak{M}_R$ of maximal regular ideals in L_R^1 an

$$\mathfrak{m}_C \in \mathfrak{M}_C \qquad \text{such that} \qquad \mathfrak{m}_R = \mathfrak{m}_C \cap L_R^1 .$$

In this connection we need not care about the uniqueness of this mapping $\mathfrak{m}_R \to \mathfrak{m}_C$. If we associate to each $\mathfrak{m}_R^{(i)}$ in the family

$$\{\mathfrak{m}_R^{(i)}\}_{i \in I} = \mathfrak{M}_R^{(I)} \subseteq \mathfrak{M}_R$$

just one $\mathfrak{m}_C^{(i)}$ such that

$$\mathfrak{m}_R^{(i)} = \mathfrak{m}_C^{(i)} \cap L_R^1 \qquad \text{for each} \quad i \in I$$

we get what we shall call a corresponding family of complex regular maximal ideals and which we shall denote $\mathfrak{M}_C^{(I)}$. We note, however, that for the ideal $\mathfrak{m}_R^0$ we have trivially a unique choice; namely, m_C^0 since $\mathfrak{m}_C^0 = \mathfrak{m}_R^0 + i\mathfrak{m}_R^0$.

LEMMA 3. *Let $\mathfrak{M}_R^{(I)}$ be a non-void subfamily of $\mathfrak{M}_R$. Then the necessary and sufficient condition that the ideal*

$$\mathfrak{a} = \bigcap_{i \in I} \mathfrak{m}_R^{(i)}$$

be convex is that we can find to $\mathfrak{M}_R^{(I)}$ a corresponding family $\mathfrak{M}_C^{(I)}$ such that $\mathfrak{m}_C^0$ belongs to the closure of $\mathfrak{M}_C^{(I)}$ in the Stone topology of $\mathfrak{M}_C$.

PROOF. The sufficiency is obvious since the condition implies

$$\bigcap_{i \in I} \mathfrak{m}_C^{(i)} \subseteq \mathfrak{m}_C^0$$

and by taking the intersection with L_R^1 on both sides of this inclusion we get $\mathfrak{a} \subseteq \mathfrak{m}_R^0$ which shows the convexity of $\mathfrak{a}$ since any subset of $\mathfrak{m}_R^0$

is convex. Assume conversely that the condition of the theorem is not satisfied. Thus since the closure of $\mathfrak{M}_C^{(I)}$ in the Stone topology does not contain $\mathfrak{m}_C^0$ and the Stone topology is coarser than the Gelfand topology, it follows that $\mathfrak{m}_C^0$ is also not contained in the closure of $\mathfrak{M}_C^{(I)}$ in the Gelfand topology. Denoting by α_i the character corresponding to $\mathfrak{m}_C^{(i)}$ we can therefore find a neighborhood K of the origin in $\hat{G}$ such that

$$\alpha_i \notin KK^{-1} \qquad \text{for all} \quad i \in I .$$

We can further pick a positive continuous non-zero function $\hat{g}$ on $\hat{G}$ with support in K, i.e. such that

$$\hat{g}(\alpha_i) = 0 \qquad \text{for all} \quad i \in I .$$

Considering the non-zero positive function f having $\hat{g} * \hat{g}^*$ as Fourier transform we get in exactly the same way as before — using an approximate identity — that $\mathfrak{a}$ must be non-convex since it is proper.

Theorem 2. *An ideal $\mathfrak{a}$ in L_R^1 which is equal to the intersection of a non-void family of regular maximal ideals in L_R^1 is convex if and only if it is contained in the ideal $\mathfrak{m}_R^0$ consisting of functions with zero integral.*

Proof. This is an immediate consequence of Lemma 3. If $\mathfrak{a} \subseteq m_R^0$, $\mathfrak{a}$ is convex. Conversely if $\mathfrak{a} \nsubseteq \mathfrak{m}_R^0$ we have

$$\bigcap_{i \in I} \mathfrak{m}_C^{(i)} \nsubseteq \mathfrak{m}_C^0$$

by the beginning of the proof of Lemma 3. This means that $\mathfrak{m}_C^0$ does not belong to the closure of the set $\mathfrak{M}_C^{(I)}$ in the Stone topology and $\mathfrak{a}$ is not convex according to Lemma 3.

Comparing Lemma 3 and Theorem 2 we see that the condition that $\mathfrak{m}_C^0$ shall belong to the Stone closure of $\mathfrak{M}_C^{(I)}$ is equivalent to the corresponding real assertion that $\mathfrak{m}_R^0$ shall belong to the Stone closure of $\mathfrak{M}_R^{(I)}$ in $\mathfrak{M}_R$.

Remarks. The above results are in striking contrast to analogous results concerning rings of functions where the multiplication is pointwise. Let $C(X)$ (resp. $C^*(X)$) denote the ordered ring of continuous (resp. continuous and bounded) real-valued functions on the topological space X with pointwise operations and pointwise ordering. Here convex maximal ideals occur in abundance. In fact any prime ideal in these rings is absolutely convex (Theorem 5.5 in [2]). But we do not have

$$C(X)/\mathfrak{m} \cong R$$

for all convex maximal ideals $\mathfrak{m} \subseteq C(X)$. In fact every maximal (convex) ideal in $C(X)$ is of codimension 1 if and only if $C(X)$ does not contain any unbounded function ([2, Theorem 5.8]).

The following result is also directly related to the scarcity of convex ideals in L_R^1.

THEOREM 3. *The ordered group algebra L_R^1 of a locally compact abelian group G can never (i.e. for any $G \neq \{e\}$) be embedded as an ordered ring in a direct product of totally ordered rings.*

PROOF. This follows from a general embedding theorem proved in [1]. But since the argument needed is very simple we can repeat it here. Suppose we had an embedding

$$L_R^1 \subseteq \prod_{i \in I} R_i \tag{1}$$

where each ring R_i is totally ordered. Then the set $\mathrm{pr}_i^{-1}(0) = \mathfrak{p}_i$ consisting of the elements in L_R^1 which has i'th coordinate equal to zero in the given embedding is a convex ideal in L_R^1; and since the embedding by definition is one-one we have the "semi-simplicity"

$$\bigcap_{i \in I} \mathfrak{p}_i = \{0\} \tag{2}$$

Since $L_R^1/\mathfrak{p}_i$ is order-isomorphic with a subring of R_i and hence totally ordered it is obvious that $\mathfrak{p}_i$ must have the following primeness property with respect to the operation $f \circ g = |f| \cap |g|$:

$$\text{If} \quad f \circ g \in \mathfrak{p}_i \quad \text{then} \quad \text{either } f \in \mathfrak{p}_i \text{ or } g \in \mathfrak{p}_i .$$

Using this together with the multiplicative ideal property of $\mathfrak{p}_i$ (with respect to convolution) it further follows that for positive f, g, $h \in L_R^1$ such that $f \cap g \in \mathfrak{p}_i$ we also have

$$f \cap (g * h) \in \mathfrak{p}_i .$$

By (2) this carries over to the ideal $\{0\}$ such that the implication

$$f \cap g = 0 \;\Rightarrow\; f \cap (g * h) = 0 \tag{3}$$

occurs as a consequence of the embeddability assumption (1). It is, however, easy to see that (3) is not satisfied in L_R^1. In fact let K_1 and K_2 be two disjoint compact neighborhoods of two distinct points $a_1, a_2 \in G$ and let χ_1 and χ_2 be the characteristic functions of K_1 and K_2. Then $\chi_1 \cap \chi_2 = 0$, but

$$\chi_1 \cap (\chi_2 * h) \neq 0$$

if h is for instance chosen to be a positive continuous function with compact support and which is non-zero at the point $a_1 a_2^{-1}$. This completes the proof of the theorem.

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