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ON THE BORELIAN AND PROJECTIVE TYPES OF LINEAR SUBSPACES

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1. Introduction. In the early years of Studia Mathematica, the problem was raised as to what Borelian and what projective types can be realized by the linear subspaces of Banach spaces. Although some interesting partial results were obtained, the following basic problems remained open:

(Mazur and Sternbach [12]) Are there linear subspaces of arbitrarily high borelian type? [2]

(Banach and Kuratowski [2]) Are there linear subspaces which are analytic but nonborelian?

These questions are answered affirmatively in the present note. We establish a general existence theorem which has the following consequence:

THEOREM A. *If E is an infinite-dimensional separable Banach space, then*

(i) *for each ordinal β with $1 \leqq \beta < \Omega$, there is a dense linear subspace of E which is borelian of additive class β but not of multiplicative class β;*

(ii) *for each ordinal γ with $1 \leqq \gamma < \omega$, there is a dense linear subspace of E which is projective of class γ but not of any lower class.*

The existence theorem appears with related material in § 2 below, while its applications (including the proof of Theorem A) will be found in § 3. In both sections, some unsolved problems are mentioned. Results, terminology, or notation which are employed without specific reference can be found in Day [6] or Kuratowski [11].

2. The existence theorem. Our aim is to find conditions on a subset S of a normed linear space which will ensure that certain topological properties carry over from S to its linear extension le S. The topological properties will be described in terms of the following conditions on a family $\mathscr{F}$ of metric spaces:

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[2] Added in proof: Prof. A. Alexiewics writes that an affirmative solution was anounced by Banach in 1940, and a proof presented by Mazur in 1957 at a conference in Zakopane.

I $\mathscr{F}$ includes the union of each sequence of its members;
II $\mathscr{F}$ includes the cartesian product of each pair of its members;
III $\mathscr{F}$ includes the real number space R;
IV $\mathscr{F}$ includes all closed subsets of its members;
V $\mathscr{F}$ includes all homeomorphic images of its members;
V′ $\mathscr{F}$ includes all biunique continuous images of its members;
V″ $\mathscr{F}$ includes all continuous images of its members.

The above assertions are to be interpreted with respect to the class of all *metric* spaces, and with use of the relative topology where appropriate. Thus (for example) more careful statements of I and V″ would read as follows: I If X_α is a sequence of subsets of a metric space such that each X_i is (under its relative topology) a member of $\mathscr{F}$, then $\bigcup_1^\infty X_i$ is (under its relative topology) a member of $\mathscr{F}$. V″ If X is a member of $\mathscr{F}$ and there is a continuous map of X onto a metric space Y, then Y is a member of $\mathscr{F}$.

2.1 THEOREM. *Suppose $\mathscr{F}$ is a family of separable metric spaces which satisfies conditions* I–IV, *S is a member of $\mathscr{F}$ which lies in a metric linear space, and* le S *is the linear extension of S. Then each of the following supplementary requirements ensures that* le S *shall be a member of $\mathscr{F}$*:

$\mathscr{F}$ satisfies condition V″;

$\mathscr{F}$ satisfies condition V′ *and S is linearly independent*;

$\mathscr{F}$ satisfies condition V, *and the closure of S is compact and linearly independent.*

PROOF. For a linear space X and a positive integer n, let T_n be the function on the product space $R^n \times X^n$ which is defined as follows:

$$T_n\big((r_1, \ldots, r_n), (x_1, \ldots, x_n)\big) = \sum_1^n r_i x_i .$$

The first step in proving 2.1 is to observe

(1) For each n, T_n is a continuous linear transformation of $R^n \times X^n$ onto X. For each $S \subset X$,

$$\operatorname{le} S = \bigcup_1^\infty T_n(R^n \times S^n) .$$

Now suppose the family $\mathscr{F}$ satisfies I–III and V″, and S is a member of $\mathscr{F}$. Then $\mathscr{F}$ includes each set $R^n \times S^n$ by II–III, each set $T_n(R^n \times S^n)$ by V″, and hence includes le S by I. Thus separability and condition IV are unnecessary for the first part of 2.1.

We denote by L_n the set of all $r = (r_1, \ldots, r_n) \in R^n$ for which no r_i is zero. By a *cross-section* of S^n we shall mean a set $C_n \subset S_n$ such that if

$(x_1, \ldots, x_n) \in C_n$, then the x_i's are all distinct, and such that whenever $y_1, \ldots, y_n$ are n distinct points of S, there is exactly one permutation $z_1, \ldots, z_n$ of the y_i's for which $(z_1, \ldots, z_n) \in C_n$.

(2) Suppose S is a linearly independent subset of X, and for each n, C_n is a cross-section of S^n. Then

$$\operatorname{le} S = \{0\} \cup \bigcup_1^\infty T_n(L_n \times C_n),$$

and for each n the function T_n is biunique on $L_n \times C_n$. We next observe

(3) If S is a separable metric space, then S^n admits a cross-section C_n which is the union of a countable family of closed sets.

Since S is isometric with a subset of the Banach space (m) of all bounded real sequences, it suffices to produce an F_σ cross-section of the space $(m)^n$. For each rational number $\varepsilon > 0$ and each integer $k > 0$, let $Q_{k,\varepsilon}$ be the set of all $x = (x^1, x^2, \ldots) \in (m)$ such that $x^i = 0$ for all $i < k$, and $x^k \geqq \varepsilon$. Let $Q = \bigcup_{k,\varepsilon} Q_{k,\varepsilon}$ and write $x \prec y$ provided $y - x \in Q$. Then (m) is linearly ordered by $\prec$, the usual lexicographic ordering. Let C_n be the set of all $(x_1, \ldots, x_n) \in (m)^n$ such that

$$x_1 \prec x_2 \prec \ldots \prec x_n .$$

Clearly C_n is a cross-section of $(m)^n$. For each $(n-1)$-tuple

$$r = (r_1, \ldots, r_{n-1})$$

of positive rational numbers and each $(n-1)$-tuple

$$j = (j_1, \ldots, j_{n-1})$$

of positive integers, let $V_{r,j}$ be the set of all $(x_1, \ldots, x_n) \in (m)^n$ such that

$$x_{i+1} - x_i \in Q_{j_i, r_i} \quad \text{for} \quad 1 \leqq i \leqq n-1 .$$

Since each set $Q_{k,\varepsilon}$ is closed, so is each set $V_{r,j}$. But of course $C_n = \bigcup_{r,j} V_{r,j}$, so C_n is an F_σ set and (3) has been proved.

Continuing with the proof of 2.1, we observe that L_n is an open subset of R^n, and hence is the union of a sequence $L_n{}^\alpha$ of compact sets. Let C_n be as in (3) and $C_n{}^\alpha$ a sequence of closed subsets of S^n whose union is C_n. Assuming S to be linearly independent, we see from (2) that

$$\operatorname{le} S = \{0\} \cup \bigcup_{i,j,n} T_n(L_n{}^i \times C_n{}^j)$$

and that always T_n is biunique on $L_n{}^i \times C_n{}^j$. Now if $\mathscr{F}$ satisfies I–IV and V′, it follows by II–IV that $\mathscr{F}$ includes $\{0\}$ and each set $L_n{}^i \times C_n{}^j$, then by V′ and I that $\operatorname{le} S$ is a member of $\mathscr{F}$.

For the final assertion of 2.1, assume that the closure K of S is compact and linearly independent, and let D_n be an F_σ cross-section of K^n, D_n^α a sequence of compact sets whose union is D_n, and $C_n^j = D_n^j \cap S^n$. Then T_n is biunique and continuous on each of the compact sets $L_n^i \times D_n^j$, hence a homeomorphism on each, and thus its restriction to the set $L_n^i \times C_n^j$ is also a homeomorphism. Thus in this case condition V is enough to ensure that le S is a member of $\mathscr{F}$. The proof of 2.1 is complete.

The proof of 2.1 can be modified to apply to conv S, the convex hull of S, rather than le S. In place of R^n we employ the set of all $(r_1, \ldots, r_n) \in R^n$ for which each r_i is $\geqq 0$ and $\sum_1^n r_i = 1$. And in this case, the requirements of linear independence may be replaced by those of convex independence. (A set S is *convexly independent* provided no point $s \in S$ is a convex combination of points of $S \sim \{s\}$; equivalently, provided each point of S is an extreme point of conv S).

In conjunction with 2.1, we shall employ a certain embedding theorem, 2.4 below. Its proof depends on the following result, which is implicit in a footnote of Mazur and Sternbach [12] and was first called to my attention by Albert Wilansky.

2.2 PROPOSITION. *If y_α is a linearly independent sequence in a locally convex Hausdorff linear space X, there are biorthogonal sequences x_α in X and f_α in $X^\star$ such that always*

$$\mathrm{le}\{x_1, \ldots, x_n\} = \mathrm{le}\{y_1, \ldots, y_n\}.$$

PROOF. Let $L_n = \mathrm{le}\{y_1, \ldots, y_n\}$. Begin by setting $x_1 = y_1$, $x_2 = y_2$, and letting f_1 and f_2 be continuous linear functionals on X such that

$$f_i x_j = \delta_{ij} \qquad (i, j = 1, 2).$$

(The existence of such f_i is guaranteed by local convexity.) Now having chosen $x_1, \ldots, x_n$ and $f_1, \ldots, f_n$ subject to the desired conditions, let F_n be the set of all $x \in X$ such that $f_i x = 1$ for $1 \leqq i \leqq n$. Since the functions $f_1, \ldots, f_n$ are linearly independent over L_n, the set $F_n \cap L_n$ is one-pointed, while the set $F_n \cap L_{n+1}$ is a flat of deficiency n in L_{n+1}, that is, a line in L_{n+1}. Thus there is a point x_{n+1} in the set $F_n \cap (L_{n+1} \sim L_n)$, and we then choose $f_{n+1} \in X^\star$ such that

$$f_{n+1} x_{n+1} = 1 \qquad \text{and} \qquad f_{n+1} L_n = \{0\}.$$

We proceed by mathematical induction to obtain the desired biorthogonal sequences.

Now a sequence x_α in a Hausdorff linear space X will be called a *quasi-basis* for X provided the following two assertions are true: le$\{x_1, x_2, \ldots\}$ is dense in X; whenever a_α and b_α are sequences in R such that $\sum_1^\infty a_i x_i = \sum_1^\infty b_i x_i$, then $a_j = b_j$ for all j. From 2.2 we have

2.3 COROLLARY. *Each infinite-dimensional separable locally convex Hausdorff linear space admits a quasi-basis.*

PROOF. The hypotheses imply the existence of a linearly independent sequence y_α whose linear extension is dense in the space. Then with x_α and f_α as in 2.2, it is obvious that $\mathrm{le}\{x_1, x_2, \ldots\}$ is dense. And if $\Sigma_1^\infty a_i x_i = \Sigma_1^\infty b_i x_i$, it follows upon application of f_j that $a_j = b_j$.

It appears that existence of a quasi-basis does not imply even the existence of nontrivial continuous linear functionals. In connection with 2.4 and 2.6 below, it would be of interest to know whether every infinite-dimensional separable metric linear space admits a quasi-basis.

2.4 THEOREM. *If X is a complete metric linear space which admits a quasi-basis, there is in X a linearly independent* arc A *such that* $\mathrm{le}\, Z$ *is dense in X for each infinite $Z \subset A$.*

PROOF. (This result is a sharpened form of one in [9], and its proof is similar.) With ϱ denoting a complete invariant metric for X, we set

$$|x| = \varrho(0, x) \qquad \text{for each} \quad x \in X .$$

Let q_α be a quasi-basis for X. From the fact that $\lim_{t\to 0}|tx| = 0$ for each $x \in X$, there follows the existence of positive multiples x_i of q_i $(i = 1, 2, \ldots)$ such that always

$$|tx_i| < 2^{-i} \qquad \text{whenever} \qquad |t| \leqq 1 .$$

Of course x_α is also a quasi-basis for X. Observe that if $|t| \leqq 1$ and $m < n$, then

$$\left| \sum_1^m t^i x_i - \sum_1^n t^i x_i \right| \leqq \sum_{m+1}^n |t^i x_i| < \sum_{m+1}^n 2^{-i} ,$$

so (since X is complete) we can define a continuous map φ of $[-1, 1]$ into X by setting

$$\varphi t = \sum_1^\infty t^i x_i \qquad \text{for each} \quad t \in [-1, 1] .$$

Let $J = [1/3, 2/3]$. To show that φ is biunique (hence a homeomorphism) on J and $\varphi\, J$ is linearly independent, it suffices to show that if

$$0 < t_1 < \ldots < t_k \qquad \text{and} \qquad \sum_1^k a_j(\varphi t_j) = 0 ,$$

then $a_k = 0$. Now if

$$\sum_{i=1}^\infty \left(\sum_{j=1}^k a_j t_j^{\,i} \right) x_i = 0 ,$$

then since x_α is a quasi-basis, $\Sigma_{j=1}^k a_j t_j^{\,i} = 0$, whence

$$-a_k = \sum_{j=1}^{k-1} a_j (t_j/t_k)^i \qquad \text{for all} \quad i\ .$$

Of course the right side tends to zero as i increases, and thus $a_k = 0$.

To complete the proof of 2.4 it suffices to show that if Y is an infinite subset of J, $\varepsilon > 0$, and b_α is an eventually-zero sequence of real numbers, then there is a linear combination of members of φY whose distance from the point $\Sigma_1^\infty b_i x_i$ is less than ε. Now for each $t \in Y$, let ξt be the point

$$\xi t = (t, t^2, t^3, \dots)$$

of the Banach space (c_0) of real sequences convergent to zero. We claim that leξY is dense in (c_0). If not, (c_0) admits a nontrivial continuous linear functional f such that $f\xi t = 0$ for all $t \in Y$, and then there are real numbers c_i, not all zero, such that

$$\sum_1^\infty c_i t^i = 0 \qquad \text{for all} \quad t \in Y\ .$$

But of course this power-series in t cannot have infinitely many zeros in J unless the coefficients c_i are all zero, and the contradiction shows that leξY is dense in (c_0). Now let N be such that

$$\sum_N^\infty 2^{-i} < \varepsilon/2\ ,$$

and let $\delta \in]0, 1[$ such that

$$|tx_i| < \varepsilon/2N$$

whenever $1 \leqq i \leqq N$ and $|t| < \delta$. Since leξY is dense in (c_0), there must be points $t_1, \dots, t_k$ of Y and numbers $a_1, \dots, a_k$ such that

$$\left| \sum_{j=1}^k a_j t_j^i - b_i \right| < \delta \qquad \text{for all} \quad i\ ,$$

and it can then be verified that

$$\left| \sum_{j=1}^k a_j(\varphi t_j) - \sum_1^\infty b_i x_i \right| < \varepsilon\ .$$

The proof of 2.4 is complete.

Now the general existence theorem mentioned in the Introduction may be stated as

2.5 COROLLARY. *Suppose* $\mathcal{F}$ *and* $\mathcal{G}$ *are families of metric spaces which satisfy* I–V *and* IV–V *respectively, and* X *is an infinite-dimensional separable Banach space. Then if some subset of* R *is a number of* $\mathcal{F} \sim \mathcal{G}$, *the space* X *must contain a dense linear subspace which is a member of* $\mathcal{F} \sim \mathcal{G}$.

PROOF. (Actually it suffices that X shall be as in 2.4, or, in particular, that X shall be an infinite-dimensional separable locally convex complete metric linear space.) Let A be as in 2.4. Our hypothesis guarantees the existence of an infinite subset S of A such that $S \in \mathscr{F} \sim \mathscr{G}$ and $\operatorname{le} S$ is dense in X. It follows from 2.1 that $\operatorname{le} S$ is a member of $\mathscr{F}$. But of course S is a relatively closed subset of $\operatorname{le} S$, and it follows that $\operatorname{le} S \notin \mathscr{G}$, for $S \notin \mathscr{G}$ and $\mathscr{G}$ satisfies IV.

Although 2.5 suffices for the applications in § 3, it seems of interest to prove a stronger result. For that, we need a different embedding theorem.

2.6 THEOREM. *Suppose Y is an infinite subset of a separable metric space M, and X is a complete metric linear space which admits a quasi-basis. Then there is a homemorphism h of M into X such that* $\operatorname{le} hY$ *is dense in X and the closure of hM is compact and linearly independent.*

PROOF. Let P denote the Hilbert parallelotope, here most conveniently represented as the product space $[1/3, 2/3]^{\aleph_0}$. We show first how the proof of 2.6 can be reduced to the following

(1) There are in X a linearly independent homeomorph Q of P and a convergent sequence u_α of points of Q such that $\operatorname{le}\{u_1, u_2, \ldots\}$ is dense in X.

Suppose (1) holds, let $v = \lim u_\alpha$, and let v_α be the sequence of those u_i's which differ from v. By a well-known embedding theorem, there is a homeomorphism g_1 of M into Q. Since Q is compact and $g_1 Y$ is infinite, there must exist in $g_1 Y$ a sequence w_α convergent to a point $w \in Q$, with always $w_i \neq w$. Now it is known [10] that P (and hence Q) is homogeneous with respect to its countable closed subsets, and hence there is a homeomorphism g_2 of Q onto Q such that always $g_2 w_i = v_i$. The map $g_2 g_1$ is the desired homeomorphism of M into X. It remains, then, to establish (1).

The points τ of P will be represented in the form $\tau = (\tau 1, \tau 2, \ldots)$ with always $\tau i \in [1/3, 2/3]$. Now for each j and k (positive integers) let τ_{jk} denote the point of P defined as follows:

$$\tau_{jk} i = 2/3 \quad \text{for} \quad i \neq k; \qquad \tau_{jk} k = \frac{2}{3} \frac{j}{j+1}.$$

Let W denote the set of all points of the form τ_{jk}; then the points of W can be arranged in a sequence converging to the point $(2/3, 2/3, 2/3, \ldots)$, so to prove 2.6 it will suffice to define a homeomorphism h of P onto a linearly independent subset of X such that $\operatorname{le} hW$ is dense in X.

Let the quasi-basis x_α for X be as in 2.4, and let z_{ij} be an enumeration of the x_i's in a double sequence. For each point $\tau \in P$, let

$$h\tau = \sum_{i,j} (\tau i)^j z_{ij} .$$

Then by an elaboration of the arguments used for 2.4, it can be proved that h has the desired properties. We leave the details to the reader.

As a consequence of 2.1 and 2.6 we have the following extension of 2.5, stated in a somewhat different way.

2.7 THEOREM. *Suppose $\mathscr{F}$ and $\mathscr{G}$ are families of separable metric spaces which satisfy* I–V *and* IV–V *respectively, and X is an infinite-dimensional separable Banach space. Then to show that $\mathscr{F} \subset \mathscr{G}$ it suffices to show that every dense linear subspace of X which is a member of $\mathscr{F}$ is also a member of $\mathscr{G}$.*

In particular, $\mathscr{F} \subset \mathscr{G}$ provided every inner-product space which is a member of $\mathscr{F}$ is also a member of G.

Some of the preceding results can be extended to certain families of nonseparable metric spaces with the aid of the following embedding theorem.

2.8 THEOREM. *Each metric space is homeomorphic with a closed linearly independent subset of a* (*suitably chosen*) *normed linear space.*

The result 2.8 is due independently to Blumenthal and Klee [4] and Arens and Eells [1], the latter paper having the stronger result in that it obtains not merely a homeomorphism but even an isometry. Actually, closedness is not mentioned in [4] nor linear independence in [1], but it is easily seen that the embeddings described have these properties.

For extending our results to nonseparable spaces, it would be useful to have more information about cross-sections in such spaces. In that connection, the following observation may be of interest.

2.9 PROPOSITION. *For a normed linear space X and an infinite cardinal number w, the assertions* (i) *and* (ii) *below are equivalent and imply* (iii):

(i) *the product space $X \times X$ admits a cross-section which is the union of w closed sets;*

(ii) *in the unit sphere S of X there are w closed sets whose union includes exactly one point from each antipodal pair;*

(iii) *S is the union of w closed sets, none of which includes two antipodal points.*

PROOF. (By "unit sphere" we mean the set $\{x \in X: \|x\| = 1\}$.) Let L be the linear subspace $\{(x, -x): x \in X\}$ and for each $x \in X$, let

$$gx = (x, -x) \in L .$$

Then g is a linear homeomorphism of X onto L. Now if C is a cross-section of $X \times X$ then the set $C \cap L$ includes (for each $x \in X \sim \{0\}$) exactly one of the pairs $(x, -x)$ and $(-x, x)$, so of course the set $g^{-1}(C \cap L) \cap S$ includes exactly one point from each antipodal pair in S. Thus i implies ii.

Now suppose conversely that ii holds, and let $\{F_a : a \in A\}$ be the special family of w closed subsets of S. Let D be the diagonal in $X \times X$ and for each positive integer n let $F_{a,n}$ be the linear sum

$$[1/n, n]gF_a + D .$$

Note that $X \times X$ is topologically and algebraically the direct sum of D and L, the relevant decomposition being

$$(x, y) = \left(\tfrac{1}{2}(x+y), \tfrac{1}{2}(x+y)\right) + \left(\tfrac{1}{2}(x-y), \tfrac{1}{2}(y-x)\right);$$

it follows readily that each set $F_{a,n}$ is closed and that $\bigcup_{a,n} F_{a,n}$ is a cross-section of $X \times X$. Since $\aleph_0 w = w$, it follows that ii implies i.

With $2w = w$, it is obvious that ii implies iii, so the proof of 2.9 is complete.

A theorem of Lyusternik and Schnirelmann, and of Borsuk [5], asserts that if X is of finite dimension n, then iii fails for $w = n$. From this it follows that if X is infinite-dimensional, then iii fails whenever w is finite. But in proving 2.1 we have shown that iii holds with $w = \aleph_0$ even for the nonseparable Banach space (m). It would be of interest to study the behavior in this respect of other important nonseparable Banach spaces.

3. Applications. Recall that the class P_0X of *borelian* sets relative to a metric space X is defined as the smallest class of sets which includes all closed subsets and all open subsets of X, and includes the union and the intersection of each of its countable subclasses. The class P_0X is the union of its *additive* subclasses $A_\beta X$ for $1 \leqq \beta < \Omega$, A_1X being the class of all F_σ subsets of X, A_2X the class of $G_{\delta\sigma}$ sets, etc. Similarly, P_0X is the union of its *multiplicative* subclasses $M_\beta X$ for $1 \leqq \beta < \Omega$, where M_1X is the class of G_δ sets, M_2X the class of $F_{\sigma\delta}$ sets, etc. The entire class P_0X consitutes the *projective* class of order 0 relative to X, and the further projective classes $P_\gamma X$ (for $1 \leqq \gamma < \omega$) are defined as follows: P_{2n+1} is the class of all subsets of X which are continuous images of members of $P_{2n}X$; $P_{2n}X$ is the class of all subsets of X which are complements in X of members of $P_{2n-1}X$. The members of P_1X are also called *analytic* subsets of X.

We shall deal here only with separable metric spaces. In order to facilitate application of the results of § 2, and to avoid confusion between the relative and absolute borelian classes, it is convenient to employ the following definition: A separable metric space will be said to be of a certain borelian or projective class if and only if it is homeomorphic with a subset of the Hilbert parallelotope P which is of that class relative to P. The classes A_β, M_β, and P_γ so obtained are exactly the absolute classes relative to complete separable metric spaces. Specifically, consider a separable metric space S and a homeomorph S' of S which lies in a complete metric space M. Then if $S \in A_\beta$, $S' \in A_\beta M$ (and similarly for M_β, P_γ); conversely, if $S' \in A_\beta M$ with $\beta \geqq 2$, then $S \in A_\beta$ (and similarly for M_β and P_γ with arbitrary $\beta \geqq 1$, $\gamma \geqq 0$). From the existing theory of borelian and projective classes (as expounded, for example, by Kuratowski in [11]), it follows that each multiplicative class M_β satisfies conditions II–V of § 2, each additive class A_β satisfies I–V, each projective class $P_{2\gamma}$ satisfies I–V′, and each projective class $P_{2\gamma+1}$ satisfies I–V″. It is known further that the real number space R contains for each β a member of $A_\beta \sim M_\beta$ and a member of $M_\beta \sim A_\beta$, and for each $\gamma > 0$ a member of P_γ which is not a member of P_δ for any $\delta < \gamma$.

Theorem A of the Introduction follows at once from the above-mentioned facts in conjunction with 2.5. It seems, though, that our method sheds no light on the existence of linear subspaces which are members of $M_\beta \sim A_\beta$. It has been proved by Mazur and Sternbach [12] that a G_δ linear subspace of a Banach space must be closed, and by Banach and Mazur [3] that there are linear subspaces which are $F_{\sigma\delta}$'s without being $G_{\delta\sigma}$'s. These papers and [2] contain further interesting examples.

The following is a consequence of 2.1.

3.1 PROPOSITION. *Suppose S is a subset of a separable Banach space. Then if S is a member of a projective class $P_{2\gamma+1}$, so are* $\operatorname{le} S$ *and* $\operatorname{conv} S$. *If S is linearly independent (resp. convexly independent) and $S \in P_{2\gamma}$, then* $\operatorname{le} S \in P_{2\gamma}$ *(resp.* $\operatorname{conv} S \in P_{2\gamma}$*).*

In particular, the linear extension of an analytic set is analytic, and of a linearly independent borelian set is borelian. It would be of interest to determine whether similar assertians are valid for the individual borelian classes, and to avoid if possible the requirements of independence.

For another application of 2.5, we turn to the $\aleph_0$-*dimensional* spaces of Hurewicz [7] [8], these being the infinite-dimensional separable metric spaces which are the union of countably many finite-dimensional subsets. (To avoid confusion, we shall refer to these spaces as *topologically* $\aleph_0$-*dimensional.*) Hurewicz showed [8] that Hilbert space is not topologically $\aleph_0$-dimensional, whence it follows that no Banach space is topologically $\aleph_0$-dimensional. On the other hand, a normed linear space of algebraic dimension $\aleph_0$ is obviously topologically $\aleph_0$-dimensional.

3.2 PROPOSITION. *There exist separable inner-product spaces (even of arbitrarily high borelian or projective type) which are topologically $\aleph_0$-dimensional but algebraically of uncountable dimension.*

PROOF. Let $\mathscr{F}_1$ be an arbitrary class A_β for $\beta \geqq 2$ (or P_γ for $\gamma \geqq 1$) and let $\mathscr{G}$ be the class M_β (or $P_{\gamma-1}$). Let $\mathscr{F}$ be the class of all topologically finite- or $\aleph_0$-dimensional members of $\mathscr{F}_1$. Then from 2.5 we conclude the existence of a separable inner-product space which is a member of $\mathscr{F} \sim \mathscr{G}$. Surely such a space is not algebraically of countable dimension, for it is not a member of the class A_1.

It follows from 3.2 that there are uncountably many distinct topological types even among the separable inner-product spaces which are topologically $\aleph_0$-dimensional. Thus if m denotes the number of topological types represented by the separable normed linear spaces, we know (granting the generalized continuum hypothesis) that $m = c$ or $m = 2^c$. It would be interesting to have more information on this question, and on the corresponding problem regarding dimension types in the sense of Frechet.

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