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RELATIONS BETWEEN CONTIGUOUS GENERALIZED LEGENDRE ASSOCIATED FUNCTIONS (RECURRENCE FORMULAS)

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In a previous paper [2] we established two recurrence formulas for the generalized Legendre associated functions $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$ introduced in [3]. In the present paper we amplify this set by a third basic recurrence relation (3) which we find by using intrinsic properties of the Legendres involved, and by proper elimination we derive a complete list of recurrence formulas (Part I), complete meant in the sense in which Erdélyi's list [1, 3.8] is complete. This last set can be found again by putting $m=n$ in the present paper.

As the reader will observe we only consider contiguous functions with consecutive values of k, and (or) where both m and n vary in the same way.

It goes without saying that if we let either m or n be fixed we may find numerous other recurrence formulas. This case is not considered in the present note.

Part II contains a recurrence formula in which a derivative occurs, and Part III proves a relation of an "inhomogeneous" type.

Because of the orthogonality properties of the Legendres the recurrence formulas are of much use when integrals of products of Legendres and other functions must be evaluated. We hope to recur to this subject in another paper.

Part I

The derivative, with respect to t, of

$$\frac{(t-1)^{k-\frac{1}{2}(m-n)+1}(t+1)^{k+\frac{1}{2}(m-n)+1}}{(t-z)^{k+\frac{1}{2}(m+n)+1}}$$

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can be written in the form

$$(1)\quad \begin{cases} \dfrac{2(k+1)(t-1)^{k-\frac12(m-n)}(t+1)^{k+\frac12(m-n)}}{(t-z)^{k+\frac12(m+n)}} + \\ \qquad + \{2(k+1)z-(m-n)\}\dfrac{(t-1)^{k-\frac12(m-n)}(t+1)^{k+\frac12(m-n)}}{(t-z)^{k+\frac12(m+n)+1}} - \\ \qquad - \left(k+\frac12(m+n)+1\right)\dfrac{(t-1)^{k-\frac12(m-n)+1}(t+1)^{k+\frac12(m-n)+1}}{(t-z)^{k+\frac12(m+n)+2}}, \end{cases}$$

as the reader can easily verify. The integral of the above derivative taken along the contour of fig. 1 of [3, p. 438] vanishes. Thus, multiplying the expression (1) by

$$\frac{e^{-\pi i(k-\frac12(m-n))}}{4\pi\sin\left(k-\frac12(m-n)\right)\pi}\cdot\frac{\Gamma\left(k+\frac12(m+n)+1\right)}{\Gamma\left(k-\frac12(m-n)+1\right)}\cdot\frac{(z-1)^{\frac12 m}(z+1)^{\frac12 n}}{2^{k+\frac12(m-n)}} = A ,$$

say, integrating along the path of fig. 1, and using the definition of $P_k^{m,n}(z)$ (see [3, (9)]), we find

$$(2)\quad \begin{cases} A\cdot 2(k+1)\cdot \displaystyle\int^{(z+,1+,z-,1-)} \dfrac{(t-1)^{k-\frac12(m-n)}(t+1)^{k+\frac12(m-n)}}{(t-z)^{k+\frac12(m+n)}}\,dt + \\ \qquad + \{2(k+1)z-(m-n)\}P_k^{m,n}(z) - \\ \qquad - 2\left(k-\frac12(m-n)+1\right)P_{k+1}^{m,n}(z) = 0 . \end{cases}$$

Observing that the integrand of the integral in the first term on the left hand side of (2) is the same as the integrand of the integral occurring in $P_k^{m-1,n-1}(z)$, written in the form [3, (9)], we find that (2) implies

$$(3)\quad \begin{cases} 2(k+1)\left(k+\frac12(m+n)\right)(z^2-1)^{\frac12}P_k^{m-1,n-1}(z) + \\ \qquad + \{2(k+1)z-(m-n)\}P_k^{m,n}(z) - \\ \qquad - 2\left(k-\frac12(m-n)+1\right)P_{k+1}^{m,n}(z) = 0 . \end{cases}$$

Remark 1. For $m=n$ (3) reduces to the well-known relation for the Legendre associated functions:

$$(k+m)(z^2-1)^{\frac12}P_k^{m-1} + zP_k^m - P_{k+1}^m = 0 .$$

REMARK 2. Applying the relation (24) of [4] we easily see that (3) holds for the Q-functions.

REMARK 3. For values of x lying on the crosscut we derive from (3), applying definition (3) of [2],

$$-2(k+1)\big(k+\tfrac{1}{2}(m+n)\big)(1-x^2)^{\frac{1}{2}}P_k^{m-1,n-1}(x)+ \\ +\{2(k+1)x-(m-n)\}P_k^{m,n}(x)- \\ -2\big(k-\tfrac{1}{2}(m-n)+1\big)P_{k+1}^{m,n}(x) = 0 .$$

REMARK 4. In [2] we found a relation between

$$P_{k-1}^{m,n}(z), \qquad P_k^{m,n}(z) \qquad \text{and} \qquad P_{k+1}^{m,n}(z) .$$

Elimination of $P_k^{m,n}(z)$ from this relation and (3) yield

$$(4)\quad \begin{cases} \big(k+\tfrac{1}{2}(m-n)\big)\{2(k+1)z-(m-n)\}P_{k-1}^{m,n}(z)- \\ \qquad -\big(k-\tfrac{1}{2}(m-n)+1\big)(2kz+m-n)P_{k+1}^{m,n}(z) \\ \quad = -\{2k(k+1)(2k+1)z+\tfrac{1}{2}(m^2-n^2)(2k+1)\}(z^2-1)^{\frac{1}{2}}P_k^{m-1,n-1}(z) . \end{cases}$$

For $m=n$, (4) reduces to

$$P_{k-1}^{m}(z)-P_{k+1}^{m}(z) = -(2k+1)(z^2-1)^{\frac{1}{2}}P_k^{m-1}(z) .$$

On the crosscut $-1<x<1$ (4) becomes, as can easily be seen,

$$(5)\quad \begin{cases} \big(k+\tfrac{1}{2}(m-n)\big)\{2(k+1)x-(m-n)\}P_{k-1}^{m,n}(x)- \\ \qquad -\big(k-\tfrac{1}{2}(m-n)+1\big)(2kx+m-n)P_{k+1}^{m,n}(x) \\ \quad = \{2k(k+1)(2k+1)x+\tfrac{1}{2}(m^2-n^2)(2k+1)\}(1-x^2)^{\frac{1}{2}}P_k^{m-1,n-1}(x) . \end{cases}$$

Again, using [4, (24)], we find that (4) holds for the $Q_k^{m,n}(z)$.

Now starting from (4), and [2, (13) and (16)], we are able to derive a whole set of relations between contiguous functions. These relations listed below, (6), ..., (20), are straightforward generalizations of relations between the ordinary contiguous Legendre functions. See for example [1, section 3.8]. The relations (7), (14), ..., (20) hold on the crosscut.

$$P_k^{m+1,n+1}(z) + \{m-n+(m+n)z\}(z^2-1)^{-\frac{1}{2}}P_k^{m,n}(z) - \left(k+\tfrac{1}{2}(m+n)\right)\left(k-\tfrac{1}{2}(m+n)+1\right)P_k^{m-1,n-1}(z) = 0 \tag{6}$$

$$\begin{aligned}&\{2k(k+1)(2k+1)z + \tfrac{1}{2}(m^2-n^2)(2k+1)\}P_k^{m,n}(z)\\ &\quad = 2k\left(k-\tfrac{1}{2}(m-n)+1\right)\left(k-\tfrac{1}{2}(m+n)+1\right)P_{k+1}^{m,n}(z) + 2(k+1)\left(\left(k+\tfrac{1}{2}(m-n)\right)\right)\left(k+\tfrac{1}{2}(m+n)\right)\right)P_{k-1}^{m,n}(z)\end{aligned} \tag{7}$$

(also on the crosscut)

$$\begin{aligned}&\left(k+\tfrac{1}{2}(m-n)\right)\{2(k+1)z-(m-n)\}P_{k-1}^{m,n}(z) - \left(k-\tfrac{1}{2}(m-n)+1\right)(2kz+m-n)P_{k+1}^{m,n}(z)\\ &\quad = -\{2k(k+1)(2k+1)z + \tfrac{1}{2}(m^2-n^2)(2k+1)\}(z^2-1)^{\frac{1}{2}}P_k^{m-1,n-1}(z)\end{aligned} \tag{8}$$

$$\begin{aligned}&\left(k-\tfrac{1}{2}(m+n)\right)\left(k-\tfrac{1}{2}(m+n)+1\right)\left(k-\tfrac{1}{2}(m-n)+1\right)(m-n-2kz)P_{k+1}^{m,n}(z)+\\ &\qquad + \left(k+\tfrac{1}{2}(m+n)\right)\left(k+\tfrac{1}{2}(m-n)\right)\left(k+\tfrac{1}{2}(m+n)+1\right)\{m-n+2(k+1)z\}P_{k-1}^{m,n}(z)\\ &\quad = \{2k(k+1)(2k+1)z + \tfrac{1}{2}(m^2-n^2)(2k+1)\}(z^2-1)^{\frac{1}{2}}P_k^{m+1,n+1}(z)\end{aligned} \tag{9}$$

$$(2k+m-n)P_{k-1}^{m,n}(z) - (2kz+m-n)P_k^{m,n}(z) = -2k\left(k-\tfrac{1}{2}(m+n)+1\right)(z^2-1)^{\frac{1}{2}}P_k^{m-1,n-1}(z) \tag{10}$$

$$\{2(k+1)z-(m-n)\}P_k^{m,n}(z) - 2\left(k-\tfrac{1}{2}(m-n)+1\right)P_{k+1}^{m,n}(z) = -2(k+1)\left(k+\tfrac{1}{2}(m+n)\right)(z^2-1)^{\frac{1}{2}}P_k^{m-1,n-1}(z) \tag{11}$$

$$\left(k-\tfrac{1}{2}(m+n)\right)(m-n-2kz)P_k^{m,n}(z) + 2\left(k+\tfrac{1}{2}(m+n)\right)\left(k+\tfrac{1}{2}(m-n)\right)P_{k-1}^{m,n}(z) = -2k(z^2-1)^{\frac{1}{2}}P_k^{m+1,n+1}(z) \tag{12}$$

$$2\left(k-\tfrac{1}{2}(m+n)+1\right)\left(k-\tfrac{1}{2}(m-n)+1\right)P_{k+1}^{m,n}(z) - \left(k+\tfrac{1}{2}(m+n)+1\right)\{m-n+2(k+1)z\}P_k^{m,n}(z) = 2(k+1)(z^2-1)^{\frac{1}{2}}\,P_k^{m+1,n+1}(z) \tag{13}$$

$$P_k^{m+1,n+1}(x) + \{m-n+(m+n)x\}(1-x^2)^{-\frac{1}{2}}\,P_k^{m,n}(x) + \left(k+\tfrac{1}{2}(m+n)\right)\left(k-\tfrac{1}{2}(m+n)+1\right)P_k^{m-1,n-1}(x) = 0 \tag{14}$$

$$\left(k+\tfrac{1}{2}(m-n)\right)\{2(k+1)x-(m-n)\}P_{k-1}^{m,n}(x) - \left(k-\tfrac{1}{2}(m-n)+1\right)(2kx+m-n)P_{k+1}^{m,n}(x) = \{2k(k+1)(2k+1)x+\tfrac{1}{2}(m^2-n^2)(2k+1)\}(1-x^2)^{\frac{1}{2}}\,P_k^{m-1,n-1}(x) \tag{15}$$

$$\left(k-\tfrac{1}{2}(m+n)\right)\left(k-\tfrac{1}{2}(m+n)+1\right)\left(k-\tfrac{1}{2}(m-n)+1\right)(m-n-2kx)P_{k+1}^{m,n}(x) + \left(k+\tfrac{1}{2}(m+n)\right)\left(k+\tfrac{1}{2}(m-n)\right)\left(k+\tfrac{1}{2}(m+n)+1\right)\{m-n+2(k+1)x\}P_{k-1}^{m,n}(x) = \{2k(k+1)(2k+1)z+\tfrac{1}{2}(m^2-n^2)(2k+1)\}(1-x^2)^{\frac{1}{2}}\,P_k^{m+1,n+1}(x) \tag{16}$$

$$(2k+m-n)P_{k-1}^{m,n}(x) - (2kx+m-n)P_k^{m,n}(x) = 2k\left(k-\tfrac{1}{2}(m+n)+1\right)(1-x^2)^{\frac{1}{2}}\,P_k^{m-1,n-1}(x) \tag{17}$$

$$\{2(k+1)x-(m-n)\}P_k^{m,n}(x) - 2\left(k-\tfrac{1}{2}(m-n)+1\right)P_{k+1}^{m,n}(x) = 2(k+1)\left(k+\tfrac{1}{2}(m+n)\right)(1-x^2)^{\frac{1}{2}}\,P_k^{m-1,n-1}(x) \tag{18}$$

$$\left(k-\tfrac{1}{2}(m+n)\right)(m-n-2kx)P_k^{m,n}(x) + 2\left(k+\tfrac{1}{2}(m+n)\right)\left(k+\tfrac{1}{2}(m-n)\right)P_{k-1}^{m,n}(x) = -2k(1-x^2)^{\frac{1}{2}}\,P_k^{m+1,n+1}(x) \tag{19}$$

$$2\left(k-\tfrac{1}{2}(m+n)+1\right)\left(k-\tfrac{1}{2}(m-n)+1\right)P_{k+1}^{m,n}(x) - \left(k+\tfrac{1}{2}(m+n)+1\right)\{m-n+2(k+1)x\}P_k^{m,n}(x) = 2(k+1)(1-x^2)^{\frac{1}{2}}\,P_k^{m+1,n+1}(x)\,. \tag{20}$$

Part II

From

$$(21)\quad P_k^{m,n}(z) = \frac{1}{\Gamma(1-m)} \cdot \frac{(z+1)^{\frac{1}{2}n}}{(z-1)^{\frac{1}{2}m}} F\big(k-\tfrac{1}{2}(m-n)+1,\ -k-\tfrac{1}{2}(m-n);\ 1-m;\ \tfrac{1}{2}(1-z)\big)$$

and

$$F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

follows

$$(22)\quad P_k^{m,n}(z) = \frac{2^n}{\Gamma(1-m)(z+1)^{\frac{1}{2}n}(z-1)^{\frac{1}{2}m}} \cdot F\big(-k-\tfrac{1}{2}(m+n),\ k-\tfrac{1}{2}(m+n)+1;\ 1-m;\ \tfrac{1}{2}(1-z)\big).$$

According to

$$\frac{d}{dz} F(a, b; c; z) = \frac{ab}{c} F(a+1, b+1; c+1; z)$$

we have by (21)

$$(23)\quad \begin{cases} \dfrac{dP_k^{m,n}(z)}{dz} = \dfrac{\big(k+\frac{1}{2}(m+n)\big)\big(k-\frac{1}{2}(m+n)+1\big)}{(z^2-1)^{\frac{1}{2}}} P_k^{m-1,n-1}(z) - \\ \qquad - \dfrac{(m+n)z+m-n}{2(z^2-1)} P_k^{m,n}(z). \end{cases}$$

(Calculations are omitted.)

On the crosscut (23) reduces to

$$(24)\quad \begin{cases} \dfrac{dP_k^{m,n}(x)}{dx} = \dfrac{\big(k+\frac{1}{2}(m+n)\big)\big(k-\frac{1}{2}(m+n)+1\big)}{(1-x^2)^{\frac{1}{2}}} P_k^{m-1,n-1}(x) + \\ \qquad + \dfrac{(m+n)x+m-n}{2(1-x^2)} P_k^{m,n}(x). \end{cases}$$

Elimination of $P_k^{m-1,n-1}(x)$ from (24) and (18) yields

$$(25)\quad 2(k+1)(1-x^2)\frac{dP_k^{m,n}(x)}{dx}$$
$$= \{2(k+1)^2 x + \tfrac{1}{2}(m^2-n^2)\} P_k^{m,n}(x) - 2\big(k-\tfrac{1}{2}(m+n)+1\big)\big((k-\tfrac{1}{2}(m-n)+1\big) P_{k+1}^{m,n}(x).$$

Part III

Many of the preceding recurrence formulas can be proved by using other representations of the Legendres. A well-known device is to replace the hypergeometric function by a Barnes integral. We illustrate this method by deriving a new generalization of [1, 3.8(3)], namely

$$
(26)\begin{cases}
\left(k+\tfrac{1}{2}(m-n)\right)\left(k+\tfrac{1}{2}(m-n)+1\right)P_{k-1}^{m,n}(z)- \\
\qquad -\left(k-\tfrac{1}{2}(m-n)+1\right)\left(k-\tfrac{1}{2}(m-n)\right)P_{k+1}^{m,n}(z) \\
\qquad = -(2k+1)\left(k-\tfrac{1}{2}(m-n)\right)\left(k+\tfrac{1}{2}(m-n)+1\right)(z^2-1)^{\frac{1}{2}}\,P_k^{m-1,n-1}(z)+ \\
\qquad +\dfrac{(2k+1)(m-n)}{\Gamma(1-m)}\,\dfrac{(z+1)^{\frac{1}{2}n}}{(z-1)^{\frac{1}{2}m}}\times \\
\qquad \times F\{k-\tfrac{1}{2}(m-n),\ -k-\tfrac{1}{2}(m-n)-1;\ 1-m;\ \tfrac{1}{2}(1-z)\}\,.
\end{cases}
$$

The starting-point is

$$
(27)\qquad \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}F(a,b;c;z)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\,\Gamma(b+s)\,\Gamma(-s)}{\Gamma(c+s)}(-z)^s\,ds\,,
$$

where $|\arg(-z)|<\pi$. The path of integration is such that the poles at $s=0, 1, 2, \ldots$ are separated from the poles at $s=-a-n$, $s=-b-n$ $(n=0, 1, 2, \ldots)$ of the integrand. To that end we suppose that a and b are different from $0, -1, -2, \ldots$.

It is easily seen that

$$
\int_{-i\infty}^{i\infty}\frac{\Gamma\left(k-\tfrac{1}{2}(m-n)+s\right)\Gamma\left(-k-\tfrac{1}{2}(m-n)+1+s\right)\Gamma(-s)}{\Gamma(1-m+s)}\left(-\frac{1-z}{2}\right)^s ds-
$$
$$
-\int_{-i\infty}^{i\infty}\frac{\Gamma\left(k-\tfrac{1}{2}(m-n)+2+s\right)\Gamma\left(-k-\tfrac{1}{2}(m-n)-1+s\right)\Gamma(-s)}{\Gamma(1-m+s)}\left(-\frac{1-z}{2}\right)^s ds
$$
$$
=(2k+1)(m-n)\int_{-i\infty}^{i\infty}\frac{\Gamma\left(k-\tfrac{1}{2}(m-n)+s\right)\Gamma\left(-k-\tfrac{1}{2}(m-n)-1+s\right)\Gamma(-s)}{\Gamma(1-m+s)}\left(-\frac{1-z}{2}\right)^s ds
$$
$$
+2(2k+1)\int_{-i\infty}^{i\infty}\frac{\Gamma\left(k-\tfrac{1}{2}(m-n)+s\right)\Gamma\left(-k-\tfrac{1}{2}(m-n)-1+s\right)\Gamma(-s)\,(-s)}{\Gamma(1-m+s)}\left(-\frac{1-z}{2}\right)^s ds\,.
$$

Set in the last integral $s=s'+1$, apply definition (21) in connection with (27), then it follows that (26) is true.

REFERENCES

1. A. Erdélyi a. o., *Higher transcendental functions* I, New York · Toronto · London, 1953.
2. L. Kuipers, *Generalized Legendre's associated functions* (*Integral Theorems, Recurrence formulas*), Monatsh. Math. 62 (1958), 24–31.
3. L. Kuipers and B. Meulenbeld, *On a generalization of Legendre's associated differential equation* I, Nederl. Akad. Wetensch. Proc. Ser. A 60 (1957), 436–443.
4. L. Kuipers and B. Meulenbeld, *Related generalized Legendre's associated functions*, Archiv Math. 9 (1958), 394–400,

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