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AN INTEGRAL FORMULA

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1. Introduction. The present note is concerned with a formula for the mean value of the integrals of a function over all k-dimensional linear varieties passing through the origin in Euclidean n-dimensional space R^n $(k=1, 2, \ldots, n-1)$. This mean value refers to a certain invariant measure μ on the space A^k of all such varieties. The space A^k becomes a metric space (in particular a topological space) when the distance $[\alpha, \beta]$ between two varieties α and β from A^k is defined as the Fréchet distance[1] $\sigma(\Omega_\alpha, \Omega_\beta)$ between the unit spheres Ω_α and Ω_β in α and β:

$$(1) \qquad [\alpha, \beta] = \max\{\varrho(\Omega_\alpha, \Omega_\beta),\ \varrho(\Omega_\beta, \Omega_\alpha)\}\ ,$$

where

$$\varrho(\Omega_\alpha, \Omega_\beta) = \sup_{x\in\Omega_\alpha}\ \inf_{y\in\Omega_\beta} |x-y|\ .$$

It is known from integral geometry that there is a unique invariant normalized measure $\mu \geqq 0$ on this topological space A^k. (The invariance refers to the group $G=O(n)$ of all rotations about the origin $0\in R^n$. Normalization means that the total mass $\mu(A^k)$ equals 1.) This measure μ is known explicitly; it has a known density[2] relative to a suitable parametrization of A^k. For the purpose of the present note, the explicit determination of μ is unnecessary. The mere existence and uniqueness of such a measure will suffice, and they will follow from a known theorem on invariant integration in homogeneous spaces.

The integral formula in question may be stated as follows:

$$(2) \qquad \int_{A^k} F(\alpha)\, d\mu(\alpha) = \frac{\omega_k}{\omega_n} \int_{R^n} |x|^{k-n} f(x)\, dx\ .$$

Here $f=f(x)$ is an arbitrary Baire function on R^n with values

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[1] Cf. e. g. N. Bourbaki [2, Chap. 9, exerc. 7, p. 29]. Actually, $\varrho(\Omega_\alpha, \Omega_\beta) = \varrho(\Omega_\beta, \Omega_\alpha)$ since there always exists an involutory orthogonal transformation in R^n which interchanges α and β. The supremum and infimum may, of course, be replaced by maximum and minimum, respectively, on account of the compactness of Ω_α and Ω_β.

[2] Cf. W. Blaschke [1]. The first determination of μ is due to G. Herglotz [6]. For a general exposition of integral geometry, see L. A. Santaló [10].

$$0 \leqq f(x) \leqq +\infty .$$

The corresponding function $F = F(\alpha)$ is defined on A^k as follows:

$$F(\alpha) = \int_{\alpha} f(x)\, d\sigma(x) , \tag{3}$$

where σ denotes k-dimensional Lebesgue measure in $\alpha \in A^k$. Moreover, dx refers to n-dimensional Lebesgue measure, and $|x|$ denotes the Euclidean length of the vector x (= the distance between the origin 0 and the point x). Finally, ω_m denotes the surface of the unit sphere in R^m, $m = 1, 2, \ldots$:

$$\omega_m = \frac{2\pi^{\frac{1}{2}m}}{\Gamma(\frac{1}{2}m)} .$$

2. Invariant integration in homogeneous spaces. This subject is treated systematically in Weil [11, § 9]. For the sake of convenience we bring here the very simple arguments by which the question of invariant integration in a homogeneous space $\Gamma = G/H$, where G is a compact group and H a closed subgroup of G, is reduced to the well-known special case of invariant integration in a compact group (Haar measure). The homogeneous space $\Gamma = G/H$ is defined as the space of all left cosets $\gamma = gH$ of G modulo H ($g \in G$), the topology on Γ being the finest topology such that the canonical mapping φ of G onto Γ is continuous (cf. N. Bourbaki [2, Chap. 3, p. 18]). The canonical mapping φ is defined by $\varphi(g) = gH$. It is an open mapping, i.e. carries open subsets of G into open subsets of Γ (cf. N. Bourbaki [2, Chap. 3, Prop. 14, p. 18]). The group G acts on the homogeneous space Γ as a transitive transformation group as follows: If $g \in G$ and $\gamma = g_1 H$, then $g\gamma = gg_1 H$. Since G was supposed to be compact, the continuous image $\Gamma = \varphi(G)$ is likewise compact.

Now let λ denote Haar measure on the compact group G, normalized so that $\lambda(G) = 1$, and define μ as the image of λ under the continuous mapping φ of G onto Γ (N. Bourbaki [3, § 6, n° 1; in particular Remark 1]). Explicitly,

$$\mu(E) = \lambda\big(\varphi^{-1}(E)\big)$$

for any Borel set $E \subset \Gamma$. Or, on integral form,

$$\int_{\Gamma} f(\gamma)\, d\mu(\gamma) = \int_{G} f(gH)\, d\lambda(g) \tag{4}$$

for any Baire function f on Γ with values $0 \leqq f(\gamma) \leqq +\infty$ (N. Bourbaki [3, Prop. 2, p. 74]). Observe that, since φ is open, a set $E \subset \Gamma$ is a Borel subset of Γ if (and only if) $\varphi^{-1}(E)$ is a Borel subset of G; and similarly a function f on Γ is a Baire function on Γ if (and only if) $f(gH)$ determines a Baire function of g on G.

It is immediately verified that μ is normalized and invariant under G. Conversely, if μ denotes any normalized, G-invariant measure on Γ, then μ satisfies (4) and hence coincides with the above measure μ. In fact, the invariance of μ may be expressed as follows:

$$\int_\Gamma f(\gamma)\, d\mu(\gamma) = \int_\Gamma f(g\gamma)\, d\mu(\gamma)$$

for any $g\in G$ and any function f of the type specified above. Integrating with respect to $d\lambda(g)$, and applying Fubini's theorem, we obtain

$$\int_\Gamma f(\gamma)\, d\mu(\gamma) = \int_\Gamma d\mu(\gamma) \int_G f(g\gamma)\, d\lambda(g)\,.$$

Each $\gamma\in\Gamma$ has the form $\gamma = kH$ for some $k\in G$, and hence

$$\int_G f(g\gamma)\, d\lambda(g) = \int_G f(gkH)\, d\lambda(g) = \int_G f(gH)\, d\lambda(g)$$

in view of the *right* invariance of Haar measure on G (applied to the Baire function $f(gH)$ of $g\in G$). Thus γ does not enter, and since μ is normalized, we are led to the equation (4), q.e.d.

3. Interpretation of A^k as a homogeneous space. From now on we take for G the orthogonal group $O(n)$, the elements of which are all orthogonal matrices of order n, viewed also as rotations about the origin 0 in R^n. In particular, G acts on A^k as a transitive transformation group in the obvious way. The usual topology on $G = O(n)$ can be defined by various well-known metrics, for example by taking for the distance between $g_1\in G$ and $g_2\in G$ the "operator norm" of the matrix difference $g_1 - g_2$:

$$\|g_1 - g_2\| = \max_{\xi\in\Omega} |g_1\xi - g_2\xi|\,. \tag{5}$$

(We denote by Ω the unit sphere in R^n, $\Omega = \{\xi\in R^n \mid |\xi| = 1\}$.) It is well known that G is compact (cf. e.g. C. Chevalley [4, Theorem 1, p. 4]). For fixed $x\in R^n$, the mapping $g\to gx$ of G into R^n is clearly continuous, and hence the scalar product $\langle gx, y\rangle$ is a continuous function of $g\in G$ for fixed x and y in R^n.

Next, let $\alpha_0\in A^k$ denote a fixed k-dimensional subspace of R^n, and denote by H the subgroup of G consisting of all rotations $g\in G$ which leave α_0 invariant (not necessarily point-wise):

$$g\alpha_0 = \alpha_0 \quad \text{for} \quad g \in H .$$

This amounts to the requirement that $\langle gx, y\rangle = 0$ for every pair (x, y) with $x \in \alpha_0$ and $y \perp \alpha_0$. Hence H is closed. Consider now the homogeneous space $\Gamma = G/H$, and define a correspondence ψ between Γ and A^k as follows. All representatives $g \in \gamma$ of a given left coset $\gamma \in \Gamma$ carry the given subspace α_0 into one and the same subspace $\alpha = g\alpha_0 \in A^k$. Writing $\alpha = \psi(\gamma)$, we have obtained a one-to-one mapping of Γ onto A^k. Composing this mapping ψ with the canonical mapping φ of G onto Γ, we obtain the following mapping $\chi = \psi \circ \varphi$ of G onto A^k:

$$\chi(g) = g\alpha_0 .$$

This mapping χ is continuous because it is a contraction with respect to the metrics (1) and (5) on A^k and G:

$$[\alpha_1, \alpha_2] \leqq \|g_1 - g_2\| . \tag{6}$$

In fact, let $x \in \Omega_{\alpha_1}$, and put $\xi = g_1^{-1}x$, $y = g_2\xi$. Then $\xi \in \Omega$, $y \in \Omega_{\alpha_2}$, and

$$|x - y| = |g_1\xi - g_2\xi| \leqq \|g_1 - g_2\| .$$

Hence $\varrho(\Omega_{\alpha_1}, \Omega_{\alpha_2}) \leqq \|g_1 - g_2\|$, and (6) follows by symmetry. The continuity of χ implies that of ψ (cf. N. Bourbaki [2, Chap. I, Théorème 1, p. 53]). Being a one-to-one continuous mapping of the compact space Γ onto A^k, ψ is a homeomorphism of Γ onto A^k. Moreover, the actions of G on Γ and on A^k correspond to each other by the mapping ψ: If $g\gamma_1 = \gamma_2$, then $g\alpha_1 = \alpha_2$ (with the notation $\alpha_1 = \psi(\gamma_1)$, $\alpha_2 = \psi(\gamma_2)$). For the purpose of the present note we may, consequently, identify Γ with A^k.

4. Proof of the integral formula (2). By application of the results of § 2 to the homogeneous space $\Gamma = G/H$ described in § 3 (and identified with A^k), we obtain

$$\int_{A^k} F(\alpha)\, d\mu(\alpha) = \int_G F(g\alpha_0)\, d\lambda(g) \tag{7}$$

for any Baire function F on A^k with values $0 \leqq F(\alpha) \leqq +\infty$. Now let F be defined as in (4) in terms of a given Baire function f on R^n (with $0 \leqq f(x) \leqq +\infty$). Then we have

$$F(g\alpha_0) = \int_{g\alpha_0} f(y)\, d\sigma(y) = \int_{\alpha_0} f(gx)\, d\sigma(x) .$$

According to Fubini's theorem, this represents a Baire function of $g \in G$, and hence F is itself a Baire function on A^k (cf. § 2). Moreover,

$$(8) \qquad \int_G F(g\alpha_0)\, d\lambda(g) = \int_{\alpha_0} \left(\int_G f(gx)\, d\lambda(g) \right) d\sigma(x) .$$

For $x \neq 0$, the inner integral on the right is simply the usual mean value of f over the sphere $\Sigma_x \subset R^n$ of centre 0 and passing through the point x. It is, in fact, well known (and may be verified by arguments similar to those employed in § 3, cf. e.g. C. Chevalley [4, p. 32–33]) that Σ_x may be identified with the homogeneous space G/G_x, where $G = O(n)$ as before, and G_x consists of all $g \in G$ which leave the point x fixed: $gx = x$. The invariant integral on Σ_x is, however, the usual mean value, and hence we obtain from (4):

$$\int_G f(gx)\, d\lambda(g) = \frac{1}{\omega_n} \int_\Omega f(|x|\,\xi)\, d\omega(\xi) ,$$

where ω is the usual surface measure on the unit sphere $\Omega \subset R^n$. Inserting this result on the right of (8), and performing the integration over α_0 by use of polar coordinates (observing that only $|x| = r$ enters), we obtain in view of (7) the desired identity (2):

$$\begin{aligned}
\int_{A^k} F(\alpha)\, d\mu(\alpha) &= \frac{\omega_k}{\omega_n} \int_0^\infty r^{k-1} dr \int_\Omega f(r\xi)\, d\omega(\xi) \\
&= \frac{\omega_k}{\omega_n} \int_0^\infty \int_\Omega r^{k-n} f(r\xi)\, r^{n-1}\, dr\, d\omega(\xi) \\
&= \frac{\omega_k}{\omega_n} \int_{R^n} |x|^{k-n} f(x)\, dx .
\end{aligned}$$

5. Applications. Replacing the origin 0 by an arbitrary point $x \in R^n$, we obtain for the mean value of the integrals of the function f over all k-dimensional linear varieties passing through x the expression

$$\frac{\omega_k}{\omega_n} \int_{R^n} |x-y|^{k-n} f(y)\, dy = \frac{\omega_k}{\omega_n} U_k^f(x) ,$$

where U_k^f is the potential of order k of the "mass distribution" with density f. In the sequel we assume that $(1+|x|)^{k-n} f(x)$ has a finite integral over R^n. If the integrals of f over (almost) all k-dim. linear varieties in R^n are known, then so is, therefore, the potential U_k^f of f of order k. For even k, one obtains f itself (apart from a constant factor) from U_k^f by applying the Laplace operator Δ successively $k/2$ times. This follows from the well-known identity due to M. Riesz [9, Chap. I, § 2]:

$$\Delta U_k^f = (2-k)(n-k)\,U_{k-2}^f \qquad (2 < k < n)\,,$$

together with Poisson's formula

$$\Delta U_2^f = (n-2)\,\omega_n \cdot f \qquad (n > 2)\,,$$

valid in the classical sense provided f is sufficiently smooth. In the case where k is odd, one may determine U_1^f in the above way. Next, U_2^f is the potential of order 1 of U_1^f (multiplied by a certain constant, cf. M. Riesz [9, loc. cit.]; we assume here $n > 2$). Finally f is obtained by application of Poisson's formula.

Thus the integral formula (2) gives rise to a solution of the problem of determining a function on R^n when its integrals over all k-dim. linear varieties in R^n are given. The first solution of this problem was given by J. Radon [8]. The role of this and related problems in the theory of partial differential equations is described in the book of F. John [7].

A different type of applications of formula (2) arose in the author's study of "exceptional systems" of surfaces (cf. [5, Chap. 2, § 3]).

REFERENCES

1. W. Blaschke, *Integralgeometrie* (Actual. sci. et ind. 252), Paris, 1935.
2. N. Bourbaki, *Topologie générale* (Les structures fondamentales de l'analyse, Livre III), Paris, 1940–1953.
3. N. Bourbaki, *Intégration des mesures* (Les structures fondamentales de l'analyse, VI, Chap.5), Paris, 1956.
4. C. Chevalley, *Theory of Lie groups*. Princeton, 1946.
5. B. Fuglede, *Extremal length and functional completion*, Acta math. 98 (1957), 171–219.
6. G. Herglotz, *Geometrische Wahrscheinlichkeiten*, Unpublished lectures in Göttingen, 1933.
7. F. John, *Plane waves and spherical means*, New York, 1955.
8. J. Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*, Ber. Verh. Sächs. Akad. 69 (1917), 262–277.
9. M. Riesz, *Intégrales de Riemann-Liouville et potentiels*, Acta Szeged. 9 (1938), 1–42.
10. L. A. Santaló, *Introduction to integral geometry* (Actual. sci. et ind. 1198), Paris, 1953.
11. A. Weil, *L'intégration dans les groupes topologiques et ces applications*, (Actual. sci. et ind. 869) Paris, 1940.

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