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ON THE UNIQUENESS OF THE CAUCHY PROBLEM

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1. Introduction. The purpose of this paper is to examine the scope of the method introduced by Carleman [3] for proving uniqueness of the Cauchy problem. The crucial point in this method is the proof of estimates of the form

$$(1) \qquad \tau^{\gamma} \int |Qu|^2 e^{2\tau\varphi} dx \leqq C \int |Pu|^2 e^{2\tau\varphi} dx, \qquad u \in C_0^{\infty}(\Omega), \quad \tau \geqq 1 ,$$

where P and Q are two linear differential operators (or systems), Ω an open set, φ a fixed function, C and γ constants, independent of u and τ. (One has to have $\gamma \geqq 0$ if (1) shall be useful.) Carleman proved such an estimate with $\gamma = 0$ when Q is the identity and P is a first order system in two variables with no multiple characteristics. Douglis [5] proved that it is sufficient to assume that the real characteristics are not multiple. (Strictly speaking, these two authors used L^1 estimates corresponding to the L^2 estimate (1).)

When P is the Laplace operator Δ, Müller [8] proved an inequality of the type (1) with $\gamma \geqq 0$ for $Q = 1$ and Heinz [6] extended it to first order differential operators. Aronszajn [1] and Cordes [4] proved the same result when P is a second order elliptic operator with variable coefficients.

Nirenberg [9] found that when $P = P(D)$, $Q = Q(D)$, $D = -i\partial/\partial x$, are operators with constant coefficients and $\varphi(x) = \langle x, N\rangle$ is a linear form, necessary and sufficient conditions for (1) to hold can be obtained from a result of Hörmander [7]. In particular, (1) then holds with $\gamma \geqq 0$ for all Q of degree smaller then that of a homogeneous operator P if and only if the derivatives $\partial P(\zeta)/\partial \zeta_k$ have no common zero $\neq 0$ of the form $\zeta = \xi + i\tau N$ with ξ and τ real. It then follows that there is unique continuation across a convex surface for the solutions of any differential equation with principal part $P(D)$. This result has been partially extended by Calderon [2] to variable coefficients (and arbitrary non characteristic boundaries).

The starting point of the present investigation was an inequality of the

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form (1) proved by Pederson [10] with $P=\Delta^2$, any Q of order <4, and $\gamma \geqq 0$. According to Nirenberg [9] this would not have been possible with a linear φ since the characteristics are double. Thus it is natural to raise the question of finding general precise conditions on P and Q for the existence of a (non linear) function φ such that (1) holds. We give a complete answer to this question (Theorems 1 and 2). The essential tool in the proof is an inequality given recently by Trêves [11].

For an operator with principal part Δ^2 we get a new proof of the unique continuation by using the arguments of Nirenberg [9] and our new inequalities. However, our results are mostly negative from the point of view of unique continuation. They show, for instance, that the unique continuation of the solutions of an elliptic equation with triple complex characteristics, if true, cannot be proved with the Carleman method in its present form.

2. The main theorems. In order to formulate as general a result as possible on the validity of (1) we have to take into account exactly how non linear the function φ is. Thus consider the linear hull of the vectors

$$\operatorname{grad}\varphi(x) - \operatorname{grad}\varphi(y), \qquad x, y \in \Omega .$$

(We always assume that $\varphi \in C^2$.) This is a subspace of the dual space R_ν, and we assume the coordinates so chosen that the subspace is defined by the equations

$$\xi_{\mu+1} = \ldots = \xi_\nu = 0 . \tag{2}$$

Clearly $\mu = 0$ if and only if φ is a linear function. If $\alpha = (\alpha_1, \ldots, \alpha_j)$ is a multi-index, that is, a set of indices between 1 and ν, we denote as usual by $|\alpha|$ the total number of indices, j. The multi-index obtained by deleting the indices $>\mu$ in α will be denoted by $\alpha^\star$.

THEOREM 1. *If* (1) *is valid and* $\operatorname{grad}\varphi(x) = N$ *at some point in* Ω, *we have with a constant* C_1

$$\tau^\gamma \sum_\alpha |Q^{(\alpha)}(\xi + i\tau N)|^2 \tau^{|\alpha^\star|} \leqq C_1 \sum_\alpha |P^{(\alpha)}(\xi + i\tau N)|^2 \tau^{|\alpha^\star|}, \quad \tau \geqq 1,\ \xi \in R_\nu . \tag{3}$$

The constant C_1 *is bounded when* x *varies in a compact subset of* Ω.

The converse of Theorem 1 does not hold without additional conditions on the function φ (cf. Theorem 4). We shall say that φ is essentially uniformly convex of the quadratic form

$$\sum_1^\nu \sum_1^\nu y^j y^k \, \partial^2\varphi / \partial x^j \partial x^k$$

is positive definite in the plane $y^{\mu+1} = \ldots = y^{\nu} = 0$ (it is independent of these variables in view of the definition of μ).

THEOREM 2. *If*[1] $\varphi \in C^3(\bar{\Omega})$ *is essentially uniformly convex in* $\bar{\Omega}$, *if* Ω *is bounded and*

$$(4) \qquad \tau^{\gamma} |Q(\xi + i\tau N)|^2 \leqq C_2 \sum_{\alpha} |P^{(\alpha)}(\xi + i\tau N)|^2 \tau^{|\alpha^*|}, \qquad \tau \geqq 1, \quad \xi \in R_\nu ,$$

where $N = \operatorname{grad} \varphi(x)$ *and* C_2 *is independent of* $x \in \Omega$, *then the inequality* (1) *holds true.*

REMARKS. 1) Note that these two theorems mean in particular that (4) implies (3), hence that these conditions are equivalent. This is also easy to prove by a direct algebraic argument.

2) When φ is a constant, the result is identical to Theorem 2.2 in Hörmander [7]; more generally, for linear φ it coincides with the variant of these inequalities given by Nirenberg [9]. In fact, we have $|\alpha^*| = 0$ in that case. The improvements possible by using non linear weight functions φ are thus due to the presence of the factors $\tau^{|\alpha^*|}$ in the right hand side of (4).

An important special case is the following:

COROLLARY. *Let a homogeneous differential operator* $P(D)$ *and a vector* $N \in R_\nu$ *be given. Then, in order that for every* Q *of lower order than* P *there should exist a function* φ *with* $\operatorname{grad} \varphi(x) = N$ *at some point in* Ω *and for which* (1) *holds with* $\gamma \geqq 0$, *it is necessary and sufficient that*

1° *the real characteristics are simple, that is, the polynomials*

$$P^{(\alpha)}(\xi), \qquad |\alpha| \leqq 1 ,$$

have no common real zero $\neq 0$;

2° *the complex characteristics are at most double in the sense that the polynomials*

$$P^{(\alpha)}(\zeta), \qquad |\alpha| \leqq 2 ,$$

have no common zero $\neq 0$ *of the form* $\zeta = \xi + i\tau N$ *with real* ξ *and* τ.

(Note that, as recalled in the introduction, one has to require simple complex characteristics also if one only works with linear functions φ.)

PROOF. The necessity of 1° is proved as follows (see also Theorem 2.3 in Hörmander [7]). Let $P^{(\alpha)}(\xi)$, $|\alpha| \leqq 1$ have a common real zero $\xi_0 \neq 0$. If we take a homogeneous Q with $Q(\xi_0) \neq 0$ of order $m-1$, where m is the order of P, we find that (4) does not hold by taking τ fixed and $\xi = t\xi_0$

[1] The condition $\varphi \in C^3(\bar{\Omega})$ means that φ is the restriction of a function in C^3 in a neighbourhood of $\bar{\Omega}$.

where $t \to \infty$. As shown by Taylor's formula, the right hand side will grow at most as $t^{2(m-2)}$ whereas the left hand side will grow as $t^{2(m-1)}$.

The necessity of 2° is proved similarly, but here the factor $\tau^{|\alpha^*|}$ in (4) is important. Thus assume that

$$P^{(\alpha)}(\zeta) = 0, \ |\alpha| \leqq 2, \quad \text{for} \quad \zeta = \zeta_0 = \xi_0 + i\tau_0 N \neq 0 .$$

Put $\xi = t\xi_0$, $\tau = t\tau_0$ in (4). If m is the degree of P, the right hand side will be of degree at most $2(m-3)+3 = 2m-3$ in t. Thus if Q is homogeneous of degree $m-1$ with $Q(\zeta_0) \neq 0$, we find by letting $t \to \infty$ that (4) does not hold with $\gamma \geqq 0$.

Suppose conversely that 1° and 2° are satisfied. We may also assume that $N \neq 0$ since otherwise we have (1) with $\varphi = 0$ already in virtue of Theorem 2.3 in Hörmander [7]. We shall prove that

$$(5) \quad |\xi + i\tau N|^{2(m-1)} \leqq C \left\{ \sum_{|\alpha|=1} |P^{(\alpha)}(\xi + i\tau N)|^2 + \sum_{|\alpha|=2} |P^{(\alpha)}(\xi + i\tau N)|^2 \, \tau^2 \right\} .$$

In view of the homogeneity it is enough to prove (5) when $|\xi + i\tau N| = 1$. But then the inequality is trivial since the right hand side is non vanishing. Indeed, this follows when $\tau = 0$ from 1° and when $\tau \neq 0$ from 2°. By continuity it follows immediately that, with a different C, (5) is also valid if N is replaced by an arbitrary vector N_1 in a neighbourhood of N. Since some derivative of P is a constant $\neq 0$ we get from (5)

$$(5') \qquad 1 + |\xi + i\tau N_1|^{2(m-1)} \leqq C' \sum_{|\alpha| \neq 0} |P^{(\alpha)}(\xi + i\tau N_1)|^2 \, \tau^{|\alpha|}, \qquad \tau \geqq 1$$

for all N_1 in a neighbourhood of N. Hence the remaining half of the corollary follows immediately from Theorem 2, with $\gamma = 0$, if we take φ uniformly convex such that $\operatorname{grad}\varphi(x)$ is in a sufficiently small neighbourhood of N when $x \in \Omega$. The proof is complete.

We shall return later on to the study of operators satisfying 1° and 2°.

3. Proof of Theorem 1. Our argument here is mainly taken over from the proof of Theorem 2.2 in Hörmander [7] but the non linearity of φ makes some modifications necessary. Assuming that $0 \in \Omega$ and writing $N = \operatorname{grad}\varphi(0)$ we have to prove (3). Let V be a function which does not vanish identically such that the functions

$$V_\varepsilon(x) = \varepsilon^{-\frac{1}{2}\mu} V(x^1/\varepsilon, \ldots, x^\mu/\varepsilon, x^{\mu+1}, \ldots, x^\nu)$$

are all in $C_0^\infty(\Omega)$ when $\varepsilon \leqq 1$. With $\xi \in R_\nu$ we shall apply (1) to the functions

$$u(x) = V_\varepsilon(x) e^{i\langle x, \xi\rangle - \tau\langle x, N\rangle} = V_\varepsilon(x) e^{i\langle x, \xi + i\tau N\rangle} .$$

We note that, for instance, $|u|^2 e^{2\tau\varphi} = |V_\varepsilon|^2 e^{2\tau\psi}$ where

$$\psi(x) = \varphi(x) - \langle x, N\rangle . \tag{6}$$

Since by hypothesis $\operatorname{grad}\psi$ is always contained in the subspace (2), ψ is a function of $x^1, \ldots, x^\mu$ only, and assuming (which is no restriction) that $\varphi(0)=0$ we have for small $x^1, \ldots, x^\mu$

$$\psi(x) = 0({x^1}^2 + \ldots + {x^\mu}^2) .$$

We want to choose ε so that the exponent $\tau\psi$ is kept under control in the support of V_ε, which leads us to put $\varepsilon^2 = 1/\tau$.

Application of Leibniz' formula gives

$$\begin{aligned} P(D)u &= e^{i\langle x,\, \xi+i\tau N\rangle} \sum P^{(\alpha)}(\xi+i\tau N)\, D_\alpha V_\varepsilon(x)/|\alpha|! \\ &= e^{i\langle x,\, \xi+i\tau N\rangle} \sum P^{(\alpha)}(\xi+i\tau N)\, \tau^{\frac{1}{2}|\alpha^*|}\, \varepsilon^{|\alpha^*|}\, D_\alpha V_\varepsilon(x)/|\alpha|! , \end{aligned} \tag{7}$$

and similarly for $Q(D)u$. With a constant a we have $\tau|\psi(x)| < a$ when x is in the support of V_ε, in view of our choice of ε. Hence we get from (1)

$$\begin{aligned} &\tau^\gamma \int \Big| \sum_\alpha Q^{(\alpha)}(\xi+i\tau N)\, \tau^{\frac{1}{2}|\alpha^*|}\, D_\alpha V/|\alpha|! \Big|^2 dx \\ &\qquad \leq C e^{4a} \int \Big| \sum_\alpha P^{(\alpha)}(\xi+i\tau N)\, \tau^{\frac{1}{2}|\alpha^*|}\, D_\alpha V/|\alpha|! \Big|^2 dx \end{aligned} \tag{8}$$

after substituting $x^1, \ldots, x^\mu$ for $x^1/\varepsilon, \ldots, x^\mu/\varepsilon$ in the two sides. To complete the proof of Theorem 1 we now only have to recall that, as noted in Hörmander [7], p. 179, if m is the highest of the orders of P and Q, the quadratic form

$$\int \Big| \sum_{|\alpha| \leq m} t_\alpha D_\alpha V \Big|^2 dx$$

of the array t_α, symmetric for permutations of the indices α, is positive definite. Estimating the two sides of (8) by means of this fact, we immediately get (3). Since the last statement of the theorem is an obvious consequence of the above arguments, the proof is now complete.

Remark. It follows with the same proof that the result would remain unchanged if we suppose that an inequality similar to (1) is valid where the square integrals are replaced by the squares of L^p norms. (Carleman used $p=1$.) Hence no improvement in the Carleman method can be expected by operating in such spaces instead of in L^2.

4. A reduction of Theorem 2. The following theorem reduces the proof of Theorem 2 to the study of the special case where $Q = P^{(\alpha)}$.

THEOREM 3. *Let Ω be a bounded domain, $\varphi \in C^2(\bar{\Omega})$, and assume that* (4) *holds with $N = \operatorname{grad}\varphi(x)$ for all $x \in \Omega$. Then we have*

$$(9) \qquad \tau^\gamma \int |Q(D)u|^2 e^{2\tau\varphi} dx \leqq C_3 \sum_\alpha \tau^{|\alpha^\star|} \int |P^{(\alpha)}(D)u|^2 e^{2\tau\varphi} dx, \qquad u \in C_0^\infty(\Omega) .$$

REMARK. The fact that Theorem 3, which does not involve any convexity assumption on φ, could be extracted from the proof of Theorem 2 given in a preliminary version of this paper was pointed out to the author by B. Malgrange.

PROOF OF THE THEOREM. Let $\chi(x) = \chi(x^1, \ldots, x^\mu)$ be the characteristic function of the unit cube defined by $|x^k| < \frac{1}{2}$, $k = 1, \ldots, \mu$. With a function $\vartheta \geqq 0$ in $C_0^\infty(R^\mu)$ with integral 1, we form the convolution $\Theta = \chi * \vartheta$ in the $x^1, \ldots, x^\mu$ space. This is a function in $C_0^\infty(R^\mu)$; moreover we may assume ϑ so chosen that the support is contained in the cube $|x^k| < 1$, $k = 1, \ldots, \mu$. We obviously have

$$1 = \sum \Theta(x - g)$$

where g runs through all points

$$g = (g^1, \ldots, g^\mu, 0, \ldots, 0)$$

with integral coordinates. This notation will be used in what follows.

In proving inequality (9) we shall use the partition of the unity given by the functions $\Theta\big((x - g\varepsilon)/\varepsilon\big)$ where we put $\varepsilon = \tau^{-\frac{1}{2}}$ as in the proof of Theorem 1. Write

$$(10) \qquad u_g(x) = \Theta\big((x - g\varepsilon)/\varepsilon\big)\, u(x) ,$$

where $u \in C_0^\infty(\Omega)$. Note that at most 2^μ functions u_g can be different from 0 at any point.

Let u_g be one of the functions which does not vanish identically. If x_g is in the support of u_g, we have $x_g \in \Omega$, and the whole support will be contained in the set $|x - x_g|^\star < 2\varepsilon$. (Here and in what follows we denote by $|y|^\star$ the maximum of the numbers $|y^j|$ when $1 \leqq j \leqq \mu$.) Let

$$N_g = \operatorname{grad}\varphi(x_g) .$$

Since $\varphi \in C^2(\bar{\Omega})$ we have

$$|\varphi(x) - \varphi(x_g) - \langle x - x_g, N_g \rangle| \leqq K |x - x_g|^{\star 2}$$

where K is a constant. Hence we have in the whole support of u_g that

$$(11) \qquad |\varphi(x) - \varphi(x_g) - \langle x - x_g, N_g \rangle| \leqq 4K/\tau .$$

Since $x_g \in \Omega$ the inequality (4) is fulfilled with $N = N_g$ by the hypothesis of Theorem 3. Thus we get by Parseval's formula, if $v \in C_0^\infty$,

$$\tau^\gamma \int |Q(D+i\tau N_g)v|^2 \, dx \leqq C_2 \sum_\alpha \int |P^{(\alpha)}(D+i\tau N_g)v|^2 \, \tau^{|\alpha^\star|} \, dx \, . \tag{12}$$

If we set $v = e^{-i\langle x, i\tau N_g\rangle} u_g$, this inequality reduces to

$$\tau^\gamma \int |Q(D)u_g|^2 \, e^{2\tau\langle x, N_g\rangle} \, dx \leqq C_2 \sum_\alpha \int |P^{(\alpha)}(D)u_g|^2 \, e^{2\tau\langle x, N_g\rangle} \, \tau^{|\alpha^\star|} \, dx \, , \tag{13}$$

or, if we multiply both sides by $e^{2\tau(\varphi(x_g) - \langle x_g, N_g\rangle)}$ and use (11),

$$\tau^\gamma \int |Q(D)u_g|^2 \, e^{2\tau\varphi} \, dx \leqq C_2 e^{16K} \sum_\alpha \int |P^{(\alpha)}(D)u_g|^2 \, e^{2\tau\varphi} \, \tau^{|\alpha^\star|} \, dx \, . \tag{14}$$

Obviously (14) is also valid if $u_g = 0$ identically.

Recalling now that at most 2^μ of the supports of the functions u_g can meet at any point and using Cauchy's inequality, we get since $u = \Sigma u_g$

$$|Q(D)u|^2 \leqq 2^\mu \sum |Q(D)u_g|^2 \, . \tag{15}$$

Adding the inequalities (14) and using (15) we thus get

$$\tau^\gamma \int |Q(D)u|^2 \, e^{2\tau\varphi} \, dx \leqq 2^\mu C_2 e^{16K} \sum_{\alpha, g} \tau^{|\alpha^\star|} \int |P^{(\alpha)}(D)u_g|^2 \, e^{2\tau\varphi} \, dx \, . \tag{16}$$

Writing $\Theta^{(\beta)} = D^\beta \Theta / |\beta|!$ we have

$$P^{(\alpha)}(D)u_g = \sum_\beta P^{(\alpha+\beta)}(D)u \, \varepsilon^{-|\beta^\star|} \, \Theta^{(\beta)}\big((x - g\varepsilon)/\varepsilon\big) \, .$$

Hence Cauchy's inequality gives, if C is an upper bound for $\sum_{|\beta| \leqq m} |\Theta^{(\beta)}|^2$,

$$\begin{aligned} \tau^{|\alpha^\star|} \, |P^{(\alpha)}(D)u_g|^2 &= \Big| \sum_\beta P^{(\alpha+\beta)}(D)u \, \tau^{\frac{1}{2}(|\alpha^\star| + |\beta^\star|)} \, \Theta^{(\beta)}\big((x - g\varepsilon)/\varepsilon\big) \Big|^2 \\ &\leqq C \sum_\beta |P^{(\alpha+\beta)}(D)u|^2 \, \tau^{|\alpha^\star| + |\beta^\star|} \, . \end{aligned}$$

Noting again that no more than 2^μ functions u_g are $\neq 0$ at any point and that the number of multi-indices occurring in the sums is $(\nu + 1)^m$ at most, we get

$$\sum_{\alpha, g} \tau^{|\alpha^\star|} \, |P^{(\alpha)}(D)u_g|^2 \leqq C(\nu + 1)^m \, 2^\mu \sum_\alpha |P^{(\alpha)}(D)u|^2 \, \tau^{|\alpha^\star|} \, , \tag{17}$$

which combined with (16) proves Theorem 3.

REMARK. In the case $\mu = 0$ the proof should be read as follows: The partition of the unity disappears and (13) with $u_g = u$ is already the desired inequality.

5. Proof of Theorem 2. Theorem 3 shows that to complete the proof of Theorem 2 it only remains to prove the inequality

$$\tau^{|\alpha^*|}\int |P^{(\alpha)}(D)u|^2 e^{2\tau\varphi}\,dx \leqq C_4\int |P(D)u|^2 e^{2\tau\varphi}\,dx, \qquad u\in C_0^\infty(\Omega)\,, \tag{18}$$

assuming that φ and Ω fulfill the assumptions in that theorem. When φ is a quadratic polynomial, the inequality (18) is almost identical with the following inequality proved by Trêves [11] (Chap. IV, § 2, Théorème 4.4)

$$\begin{aligned}(t^\alpha)^2\int |P^{(\alpha)}(D)u|^2 e^{t_1{}^2x^{1^2}+\dots+t_\nu{}^2x^{\nu^2}}\,dx \\ \leqq |\alpha|!\,2^{m-|\alpha|}\int |P(D)u|^2 e^{t_1{}^2x^{1^2}+\dots+t_\nu{}^2x^{\nu^2}}\,dx, \qquad u\in C_0^\infty\,.\end{aligned} \tag{19}$$

In fact, let $|x^{\mu+1}|<B, \dots, |x^\nu|<B$ in Ω and set $t_{\mu+1}=\dots=B^{-1}$ in (19). We then obtain

$$(t^{\alpha^*})^2\int |P^{(\alpha)}(D)u|^2 e^{t_1{}^2x^{1^2}+\dots+t_\mu{}^2x^{\mu^2}}\,dx \leqq C\int |P(D)u|^2 e^{t_1{}^2x^{1^2}+\dots+t_\mu{}^2x^{\mu^2}}\,dx\,, \qquad u\in C_0^\infty(\Omega)\,,$$

where $C=|\alpha|!\,2^{m-|\alpha|}B^{2(|\alpha|-|\alpha^*|)}e^{\nu-\mu}$. Replacing u by $ue^{\langle x,\eta\rangle}=ue^{-i\langle x,-i\eta\rangle}$ and $P(D)$ by $P(D+i\eta)$, we obtain

$$\begin{aligned}(t^{\alpha^*})^2\int |P^{(\alpha)}(D)u|^2 e^{t_1{}^2x^{1^2}+\dots+t_\mu{}^2x^{\mu^2}+2\langle x,\eta\rangle}\,dx \\ \leqq C\int |P(D)u|^2 e^{t_1{}^2x^{1^2}+\dots+t_\mu{}^2x^{\mu^2}+2\langle x,\eta\rangle}\,dx\,.\end{aligned} \tag{20}$$

Now let φ be a quadratic polynomial whose second order part only involves $x^1, \dots, x^\mu$ and is bounded from below by $c(x^{1^2}+\dots+x^{\mu^2})$ where $c>0$. Then we claim that

$$\tau^{|\alpha^*|}c^{|\alpha^*|}\int |P^{(\alpha)}(D)u|^2 e^{2\tau\varphi}\,dx \leqq C_5\int |P(D)u|^2 e^{2\tau\varphi}\,dx, \qquad u\in C_0^\infty(\Omega)\,, \tag{21}$$

where C_5 is a new constant depending only on B, m and ν. This is an obvious consequence of (20) with $C_5=C$ if the second order part of φ is a sum of squares; the general case is however immediately reduced to this by an orthogonal transformation of the variables $x^1, \dots, x^\mu$. Thus we have now proved that Trêves' inequality really contains (18) when φ is of the second order.

It remains to extend the result to an arbitrary essentially uniformly convex φ. This can be done by means of an argument similar to the proof of Theorem 2. There we used a partition of the unity

$$1 = \sum \Theta\big((x-g\varepsilon)/\varepsilon\big)$$

with $\varepsilon = \tau^{-\frac{1}{2}}$ in order that the exponent $2\tau\varphi$ should be nearly linear in the support of each u_g, which was necessary for the use of Fourier transforms. Here, however, we only have to reduce ourselves to nearly quadratic exponents in order to be able to use Trêves' result (21), so we now choose $\varepsilon = \tau^{-\frac{1}{3}}$. As in the proof of Theorem 2 we again write

$$u = \sum u_g .$$

We denote by φ_g the Taylor expansion of order 2 of φ at the point x_g. Thus we have

$$|\varphi(x)-\varphi_g(x)| = O(|x-x_g|^{\star 3}) .$$

Since by assumption Ω is bounded and $\varphi \in C^3(\overline{\Omega})$, the ordo will be uniform. Since $\tau|x-x_g|^{\star 3} \leqq \varepsilon^3 \tau = 1$ in the support of u_g, we have there

$$\tau|\varphi(x)-\varphi_g(x)| \leqq M , \tag{22}$$

where M is a constant. Now the assumption of essentially uniform convexity for φ means precisely that the assumptions of (21) are satisfied by φ_g with a constant c independent of g and of τ. Thus we get

$$\tau^{|\alpha^\star|} c^{|\alpha^\star|} \int |P^{(\alpha)}(D)u_g|^2 e^{2\tau\varphi_g}\,dx \leqq C_5 \int |P(D)u_g|^2 e^{2\tau\varphi_g}\,dx, \qquad u \in C_0^\infty(\Omega) \tag{23}$$

and using (22) we get if $C_6 \geqq C_5 e^{4M} c^{-|\alpha^\star|}$

$$\tau^{|\alpha^\star|} \int |P^{(\alpha)}(D)u_g|^2 e^{2\tau\varphi}\,dx \leqq C_6 \int |P(D)u_g|^2 e^{2\tau\varphi}\,dx, \qquad u \in C_0^\infty(\Omega) . \tag{24}$$

Using (15) with $Q = P^{(\alpha)}$ we obtain

$$\tau^{|\alpha^\star|} \int |P^{(\alpha)}(D)u|^2 e^{2\tau\varphi}\,dx \leqq C_6 2^\mu \sum_g \int |P(D)u_g|^2 e^{2\tau\varphi}\,dx, \qquad u \in C_0^\infty(\Omega) . \tag{25}$$

With the notations of p. 219 we have

$$P(D)u_g = \sum_{\alpha=\alpha^\star} P^{(\alpha)}(D)u\, \varepsilon^{-|\alpha|} \Theta^{(\alpha)}\big((x-g\varepsilon)/\varepsilon\big)$$

and hence by Cauchy's inequality, noting that $\varepsilon^{-1} = \tau^{\frac{1}{3}}$,

$$|P(D)u_g|^2 \leqq C \sum_{\alpha=\alpha^\star} |P^{(\alpha)}(D)u|^2 \tau^{\frac{2}{3}|\alpha^\star|} . \tag{26}$$

Since no point is in the support of more than 2^μ functions u_g, we get

$$\sum_g |P(D)u_g|^2 \leqq C 2^\mu \sum_{\alpha=\alpha^\star} |P^{(\alpha)}(D)u|^2 \tau^{\frac{2}{3}|\alpha^\star|} . \tag{27}$$

Summing (25) for all multi-indices α with $|\alpha| \leqq m$ and using the inequality (27) in the right hand side, we get with $C_7 = C C_6 4^{\mu} (\nu+1)^m$

$$(28)\quad \sum \tau^{|\alpha^\star|} \int |P^{(\alpha)}(D) u|^2 e^{2\tau\varphi}\, dx \leqq C_7 \sum_{\alpha = \alpha^\star} \tau^{\frac{2}{3}|\alpha^\star|} \int |P^{(\alpha)}(D) u|^2 e^{2\tau\varphi}\, dx .$$

When $\tau^{\frac{1}{3}} \geqq C_7 + 1$ we shift the terms on the right hand side with $|\alpha| \neq 0$ to the left hand side using the fact that when $|\alpha^\star| \neq 0$ we have

$$\tau^{|\alpha^\star|} - C_7 \tau^{\frac{2}{3}|\alpha^\star|} \geqq \tau^{|\alpha^\star|}\big(1 - C_7/(C_7+1)\big) = \tau^{|\alpha^\star|}/(C_7+1) .$$

Thus we get

$$\sum \tau^{|\alpha^\star|} \int |P^{(\alpha)}(D) u|^2 e^{2\tau\varphi}\, dx \leqq C_7 (C_7+1) \int |P(D) u|^2 e^{2\tau\varphi}\, dx$$

when $\tau^{\frac{1}{3}} \geqq C_7 + 1$, which clearly proves (18) for all $\tau \geqq 1$. The proof is complete.

6. The role of the convexity of φ. Most of the inequalities of type (1) referred to in the introduction involve a non convex function φ such as $-\log|x|$. As we shall see in the next section, the applications to unique continuation theorems here get too special since we only have results for convex φ. However, the following theorem shows that our convexity assumption cannot be relaxed without the loss of part of our results.

THEOREM 4. *Let $\varphi \in C^2$ be a function such that* (18) *holds true for every first order operator P, with a constant C_4 independent of P. Then φ is essentially uniformly convex.*

PROOF. To get simpler notations we place the origin at a general point in Ω. We write $\operatorname{grad}\varphi(0) = N$ and

$$\varphi(x) = \varphi(0) + \langle x, N\rangle + A(x) + o(|x|^2) .$$

Let y be a vector with $|y|^\star \neq 0$; we have to prove that $A(y) > 0$. To do so we apply (1) to $P(D) = \langle D - i\tau N, y\rangle$ and replace u by $u e^{-\langle x, N\tau\rangle}$. Noting that $P^{(\alpha)} = y^j$, $1 \leqq j \leqq \mu$, for an α with $|\alpha^\star| = 1$, we get

$$\tau |y|^{\star 2} \int |u|^2 e^{2\tau(A(x) + o(|x|^2))}\, dx \leqq C_4 \int |\langle D, y\rangle u|^2 e^{2\tau(A(x)+o(|x|^2))}\, dx .$$

We now set $u(x) = \varepsilon^{-\frac{1}{2}\nu} V(x/\varepsilon)$, where $\varepsilon^2 = \tau^{-1}$ and $V \in C_0^\infty(R^\nu)$. When ε is small enough, $u \in C_0^\infty(\Omega)$, and passing to the limit when $\varepsilon \to 0$ after a change of variables we get

$$(29)\qquad |y|^{\star 2} \int |V|^2 e^{2A(x)}\, dx \leqq C_4 \int |\langle D, y\rangle V|^2 e^{2A(x)}\, dx .$$

As a limiting case (29) also holds if $A(x)$ is the second order part of the Taylor expansion of φ at a point on the boundary of Ω.

Now (29) implies that $A(y) \geqq 0$ if $|y|^\star \neq 0$. In fact, if $A(y) < 0$ for some such y we set

$$V(x) = V_0(x) f(\varepsilon x)$$

where $V_0 \not\equiv 0$ satisfies the equation $\langle D, y\rangle V_0 = 0$ and has compact support modulo $\{ty\}$; $f \in C_0^\infty$, $f(0) = 1$. Writing $\langle D, y\rangle f = g$, we get

$$\int |V_0(x) f(\varepsilon x)|^2 e^{2A(x)}\, dx \leqq C_4 \varepsilon^2 \int |V_0(x) g(\varepsilon x)|^2 e^{2A(x)}\, dx \tag{30}$$

and since

$$\int |V_0(x)|^2 e^{2A(x)}\, dx < \infty$$

we get a contradiction when $\varepsilon \to 0$. Hence $A(y)$ is semi-definite.

Now if $A(y) = 0$ for some y satisfying $|y|^\star \neq 0$ we can repeat the same argument. Indeed, since we already know that A is semi-definite, we have $A(x + ty) \equiv A(x)$, hence the left hand side of (30) grows like ε^{-1} and the right hand side tends to 0 as ε. Thus we get again a contradiction. The proof is complete.

Remark. Even if it is only required that (18) holds for every fixed P of order 1, with C_4 depending on P, it is still easily proved that the level surfaces of φ are convex.

7. Results on unique continuation. We do not formulate the most general result which can be deduced from Theorem 2, but give an important special case which already illustrates what can be achieved by our inequalities and the Carleman method.

Theorem 5. *Let u be a classical solution of an equation $P(x, D)u = 0$ with continuous coefficients, which is defined in a neighbourhood of* 0, *and let $u = 0$ outside of a sphere passing through* 0 *with normal N at* 0. *Suppose that the principal part of $P(x, D)$ has constant coefficients and satisfies the conditions* 1° *and* 2° *of the Corollary to Theorems* 1 *and* 2. *Then u vanishes in a neighbourhood of* 0.

Proof. Let $A(x) \leqq 0$ define the interior of a sphere through 0 containing the sphere mentioned in the hypothesis, except for the origin; A is a polynomial of the second order. Let $P(D) = P(0, D)$. From (5′) it follows immediately, since $\alpha \neq 0$ in the terms on the right hand side, that

$$\sum_0^{m-1} \tau^{2(m-1-k)} |\xi + i\tau N_1|^{2k} \leqq C'' \sum |P^{(\alpha)}(\xi + i\tau N_1)|^2 \tau^{|\alpha|}, \qquad \tau \geqq 1,$$

for all N_1 in a neighbourhood of N. Let Ω be a spherical neighbourhood of 0 such that $\operatorname{grad} A(x)$ is in the permissible neighbourhood of N for all $x \in \Omega$. Then we get from Theorem 2

$$(31)\quad \sum_{|\alpha| \leqq m-1} \tau^{2(m-1-|\alpha|)} \int |D^\alpha v|^2 e^{2\tau A(x)}\, dx \leqq C_2 \int |P(D)v|^2 e^{2\tau A(x)}\, dx, \qquad \tau \geqq 1\,,$$

provided that $v \in C_0^\infty(\Omega)$; obviously (31) extends by continuity to all $v \in C_0^m$. Now the operator $P(x, D) - P(D)$ is of order $m-1$ and its coefficients tend to 0 when $x \to 0$. Let Ω_0 be the smallest of the spheres Ω and $\{x;\, |x| < \delta\}$. Then we have

$$(32)\qquad |P(D)v|^2 \leqq 2|P(x, D)v|^2 + \omega(\delta) \sum_{|\alpha| \leqq m-1} |D^\alpha v|^2\,,$$

where the function $\omega(\delta) \to 0$ when $\delta \to 0$. Combining (31) with (32) and choosing δ so small that $C_2\omega(\delta) < 1$, we obtain

$$(33)\quad \sum_{|\alpha| \leqq m-1} \tau^{2(m-1-|\alpha|)} \int |D^\alpha v|^2 e^{2\tau A(x)}\, dx \leqq C_3 \int |P(x, D)v|^2 e^{2\tau A(x)}\, dx, \quad \tau \geqq 1\,.$$

Now let ψ be a function in $C_0^\infty(\Omega_\delta)$ such that $\psi = 1$ in a neighbourhood of 0. Put $v = \psi u$ in (33), which is permitted since $v \in C_0^m(\Omega)$. The function

$$f = P(x, D)v$$

vanishes when $A(x) > -\varkappa$, for some $\varkappa > 0$. In fact, $f = P(x, D)u = 0$ in a neighbourhood of 0, and since u vanishes outside a sphere contained in the interior of the sphere $A(x) \leqq 0$ except for the point 0 the assertion is obvious. Now (33) gives if we only keep one term on the left and restrict the range of integration there

$$e^{2\tau(-\frac{1}{2}\varkappa)} \int_{A(x) > -\frac{1}{2}\varkappa} |v|^2 dx = O(e^{2\tau(-\varkappa)})\,,$$

and hence

$$\int_{A(x) > -\frac{1}{2}\varkappa} |v|^2 dx = O(e^{-\tau\varkappa}) \to 0 \quad \text{when} \quad \tau \to \infty\,.$$

Thus v and consequently u also vanishes in a neighbourhood of 0. The proof is complete.

This theorem is weak in two respects: it assumes that the principal part has constant coefficients and only gives unique continuation across convex surfaces. The reason for the latter defect is that Theorem 2 is restricted to convex functions φ, a fact which was discussed in the previous section. To extend the results to variable coefficients is probably considerably more difficult than in the case of simple characteristics studied by Calderon [2]. Our reason for believing that the situation is much more delicate is that when there are double characteristics the constant γ in (1) cannot be taken >0 in estimating the derivatives of order $m-1$, in the constant coefficient case, whereas we can take $\gamma = 1$ if all characteristics are simple.

Theorem 5 applies in particular to differential operators with principal part Δ^2. Thus Theorem 5 combined with an argument based on reflection in a spherical surface used for Δ in Nirenberg [9] proves the following

THEOREM 6. *A solution of a differential equation with principal part* Δ^2 *and continuous coefficients vanishes identically if it vanishes in an open set.*

This result is due to Pederson [10].

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