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ALGEBRAIC RESULTS FOR CERTAIN VALUES OF THE JACOBI THETA-CONSTANT $\theta_3(\tau)$

CARSTEN ELSNER and YOHEI TACHIYA

(Dedicated to Professor Iekata Shiokawa on the occasion of his 75th birthday)

Abstract

In its most elaborate form, the Jacobi theta function is defined for two complex variables z and τ by $\theta(z|\tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i\nu^2\tau+2\pi i\nu z}$, which converges for all complex numbers z, and τ in the upper half-plane. The special case

$$\theta_3(\tau) := \theta(0|\tau) = 1 + 2\sum_{\nu=1}^{\infty} e^{\pi i\nu^2\tau}$$

is called a Jacobi theta-constant or Thetanullwert of the Jacobi theta function $\theta(z|\tau)$. In this paper, we prove the algebraic independence results for the values of the Jacobi theta-constant $\theta_3(\tau)$. For example, the three values $\theta_3(\tau)$, $\theta_3(n\tau)$, and $D\theta_3(\tau)$ are algebraically independent over $\mathbb{Q}$ for any τ such that $q = e^{\pi i\tau}$ is an algebraic number, where $n \geq 2$ is an integer and $D := (\pi i)^{-1}d/d\tau$ is a differential operator. This generalizes a result of the first author, who proved the algebraic independence of the two values $\theta_3(\tau)$ and $\theta_3(2^m\tau)$ for $m \geq 1$. As an application of our main theorem, the algebraic dependence over $\mathbb{Q}$ of the three values $\theta_3(\ell\tau)$, $\theta_3(m\tau)$, and $\theta_3(n\tau)$ for integers $\ell, m, n \geq 1$ is also presented.

1. Introduction and statement of the results

Let τ be a complex variable in the upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. The series

$$\theta_2(\tau) = 2\sum_{\nu=0}^{\infty} q^{(\nu+1/2)^2}, \qquad \theta_3(\tau) = 1 + 2\sum_{\nu=1}^{\infty} q^{\nu^2},$$
$$\theta_4(\tau) = 1 + 2\sum_{\nu=1}^{\infty} (-1)^\nu q^{\nu^2}$$

are known as theta-constants or Thetanullwerte, where $q = e^{\pi i\tau}$. In particular, the function $\theta_3(\tau)$ is called a Jacobi theta-constant or Thetanullwert of the Jacobi theta function $\theta(z|\tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i\nu^2\tau+2\pi i\nu z}$, which is entire in $z \in \mathbb{C}$ and

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holomorphic for $\tau \in \mathbb{H}$. The study of the transcendence and algebraic independence of the values of the theta-constants has made remarkable progress by the celebrated theorem of Yu. V. Nesterenko [7] on the algebraic independence results of the values of the Ramanujan functions

$$P(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)z^n, \qquad Q(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)z^n,$$
$$R(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)z^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$.

THEOREM A ([7, Theorem 1]). *For any complex number s with* $0 < |s| < 1$, *at least three of the numbers*

$$s, \qquad P(s), \qquad Q(s), \qquad R(s)$$

are algebraically independent over $\mathbb{Q}$.

Combining Theorem A and the relations between the Ramanujan functions and the theta-constants (see [6]), D. Bertrand [1] deduced the following Corollaries A.1 and A.2 on algebraic independence of the values of theta-constants. Let $D := (\pi i)^{-1} d/d\tau$ be a differential operator.

COROLLARY A.1. *Let* $j \in \{2, 3, 4\}$ *and* $\tau \in \mathbb{H}$. *Then at least three of the numbers* $e^{\pi i\tau}$, $\theta_j(\tau)$, $D\theta_j(\tau)$, $D^2\theta_j(\tau)$ *are algebraically independent over* $\mathbb{Q}$.

Corollary A.1 was shown independently by D. Duverney, Ke. Nishioka, Ku. Nishioka, and I. Shiokawa [2] in the case where $q = e^{\pi i\tau}$ is algebraic. In particular, the sum $\sum_{n=1}^{\infty} q^{n^2}$ is transcendental for any algebraic number q with $0 < |q| < 1$.

COROLLARY A.2 (see also [9, Corollary 4.7]). *Let* j, k *and* $\ell \in \{2, 3, 4\}$ *with* $j \neq k$. *For any* $\tau \in \mathbb{H}$, *at least three of the numbers* $e^{\pi i\tau}$, $\theta_j(\tau)$, $\theta_k(\tau)$, $D\theta_\ell(\tau)$ *are algebraically independent over* $\mathbb{Q}$.

By Corollary A.2, if $q = e^{\pi i\tau}$ ($\tau \in \mathbb{H}$) is an algebraic number, then any two numbers in the set

$$\{\theta_2(\tau), \theta_3(\tau), \theta_4(\tau)\}$$

are algebraically independent over $\mathbb{Q}$. On the other hand, the three numbers $\theta_2(\tau)$, $\theta_3(\tau)$, and $\theta_4(\tau)$ are algebraically dependent over $\mathbb{Q}$ for any $\tau \in \mathbb{H}$, since the following identity holds:

$$\theta_3^4(\tau) = \theta_2^4(\tau) + \theta_4^4(\tau).$$

We are interested in the algebraic independence (or dependence) problem on the values of the Jacobi theta-constant $\theta_3(\tau)$ at different points. In this direction, recently, the first-named author has proven the following result.

THEOREM B ([3, Theorem 1.1]). *Let $m \geq 1$ be an integer and $\tau \in \mathbb{H}$ such that $q = e^{\pi i \tau}$ is an algebraic number. Then, the two numbers $\theta_3(2^m\tau)$ and $\theta_3(\tau)$ are algebraically independent over $\mathbb{Q}$ as well as the two numbers $\theta_3(n\tau)$ and $\theta_3(\tau)$ for $n = 3, 5, 6, 7, 9, 10, 11, 12$.*

We state the outline of the proof of Theorem B briefly. The first basic tool in proving such algebraic independence results are integer polynomials in two variables X, Y, which vanish identically at certain points $X = X_0$ and $Y = Y_0$ given by rational functions of theta-constants. For instance, in the case of $n = 2^m$ $(m \geq 1)$, there exists a homogeneous polynomial $T_n(t_1, t_2, t_3) \in \mathbb{Z}[t_1, t_2, t_3]$ of total degree λ such that

$$T_n\big(\theta_3^2(n\tau), (\theta_3(\tau) + \theta_4(\tau))^2, \theta_3(\tau)\theta_4(\tau)\big) = 0, \tag{1}$$

with $\lambda = \deg_{t_1} T_n(t_1, t_2, t_3) = 2^{m-2}$ for $m \geq 2$ and $\lambda = 1$ when $m = 1$ (see [3, Lemma 3.1]). The first polynomials T_2, T_4, and T_8 are given by

$$T_2 = 2t_1 - t_2 + 2t_3, \qquad T_4 = 4t_1 - t_2, \qquad T_8 = (8t_1 - t_2)^2 - 8t_3(t_2 - 2t_3).$$

Thus, putting $P_n(X, Y) := T_n(X, (1 + Y)^2, Y)$, we have the following:

THEOREM C ([3, Lemma 3.1]). *For every integer $m \geq 1$, let $n = 2^m$. Then there exists a polynomial $P_n(X, Y) \in \mathbb{Z}[X, Y]$ such that*

$$P_n\left(\frac{\theta_3^2(n\tau)}{\theta_3^2(\tau)}, \frac{\theta_4(\tau)}{\theta_3(\tau)}\right) = 0$$

holds for any $\tau \in \mathbb{H}$, where $\deg_X P_2(X, Y) = 1$, and $\deg_X P_n(X, Y) = 2^{m-2}$ for $m \geq 2$.

The second tool is an algebraic independence criterion (cf. [4, Lemma 3.1]), from which the algebraic independence of $\theta_3(n\tau)$ and $\theta_3(\tau)$ over $\mathbb{Q}$ can be obtained by proving that the resultant

$$\mathrm{Res}_X\left(P_n(X, Y), \frac{\partial}{\partial Y} P_n(X, Y)\right) \in \mathbb{Z}[Y] \tag{2}$$

does not vanish identically (see [3, Theorem 4.1]). This is true for $n = 2$ and 4, since P_2 and P_4 are given by $P_2 = 2X - Y^2 - 1$ and $P_4 = 4X - (1 + Y)^2$, respectively. Furthermore, for the case of $n = 2^m$ ($m \geq 3$), we can show the identities

$$P_n(X,0) = \left(2^m X - 1\right)^{2^{m-2}},$$
$$\frac{\partial P_n}{\partial Y}(X,0) = -2^{m-1}\left(2^m X - 1\right)^{2^{m-2}-1} + U_n(X),$$

where $U_n(X) \in \mathbb{Z}[X]$ with $U_n(1/2^m) \neq 0$ (see the proof of [3, Theorem 1.1]). Hence, for $Y = 0$ the polynomials $P_n(X,Y)$ and $\partial P_n(X,Y)/\partial Y$ have no common root. Therefore the resultant (2) does not vanish identically, which gives the desired result. This fact will be used again in this paper for proving our Theorem 1.1 below. The algorithm to compute the polynomials P_{2^m} recursively is given by [3, Lemma 3.1]:

$$\begin{aligned}
P_2 &= 2X - Y^2 - 1,\\
P_4 &= 4X - (1+Y)^2,\\
P_8 &= 64X^2 - 16(1+Y)^2 X + (1-Y)^4,\\
P_{16} &= 65536X^4 - 16384(1+Y)^2 X^3\\
&\quad + 512(3Y^4 + 4Y^3 + 18Y^2 + 4Y + 3)X^2\\
&\quad - 64(1+Y)^2(Y^4 + 28Y^3 + 6Y^2 + 28Y + 1)X + (1-Y)^8.
\end{aligned}$$

Similarly, for odd integers $n \geq 3$, the algebraic independence of $\theta_3(n\tau)$ and $\theta_3(\tau)$ over $\mathbb{Q}$ can also be deduced from the nonvanishing of the resultant (2) with the following polynomial $P_n(X,Y)$, which is given by Yu. V. Nesterenko [8].

THEOREM D ([8, Theorem 1, Corollary 4]). *For any odd integer $n \geq 3$ there exists a polynomial $P_n(X,Y) \in \mathbb{Z}[X,Y]$ with* $\deg_X P_n(X,Y) = \psi(n)$, $\deg_Y P_n \leq (n-1)\psi(n)/n$, *such that*

$$P_n\left(n^2\frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, 16\frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}\right) = 0$$

holds for any $\tau \in \mathbb{H}$, where $\psi(n)$ is defined by

$$\psi(n) := n\prod_{p|n}\left(1 + \frac{1}{p}\right). \tag{3}$$

The constitution method for $P_n(X,Y)$ is described in [8], but it is not easy to compute the explicit forms. That is similar to the method for obtaining the modular polynomial $\Phi_n(X, Y) \in \mathbb{Z}[X, Y]$ of order n, which satisfies the identity

$$\Phi_n(j(n\tau), j(\tau)) = 0,$$

for an integer $n \geq 1$, and the modular function

$$j(\tau) := 1728 \frac{Q^3(q^2)}{Q^3(q^2) - R^2(q^2)} = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}, \quad \lambda := \lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$$

(cf. [5, Chapter 5]). The first polynomials P_3 and P_5 are given by

$$\begin{aligned} P_3 &= 9 - (28 - 16Y + Y^2)X + 30X^2 - 12X^3 + X^4, \\ P_5 &= 25 - (126 - 832Y + 308Y^2 - 32Y^3 + Y^4)X \\ &\quad + (255 + 1920Y - 120Y^2)X^2 \\ &\quad + (-260 + 320Y - 20Y^2)X^3 + 135X^4 - 30X^5 + X^6, \end{aligned}$$

and the polynomials P_7, P_9, and P_{11} are listed in the appendix of [3]. The polynomials P_3 and P_5 are already given in [8], P_7, P_9, and P_{11} are the results of computer-assisted computations of the first-named author. The non-vanishing of the resultants (2) for odd integers $n = 3, 5, 7, 9, 11$ is based on these direct computations of $P_n(X, Y)$.

In this paper we give some more precise information of $P_n(X, Y)$ and generalize Theorem B by Theorem 1.2 without computing the explicit form of $P_n(X, Y)$. Our starting point is Theorem 1.1.

THEOREM 1.1. *Let $n \geq 2$ be an integer and $\tau \in \mathbb{H}$. Then the numbers $\theta_2(\tau)$ and $\theta_3(n\tau)$ are algebraic over the fields $\mathbb{Q}(\theta_3(n\tau), \theta_3(\tau))$ and $\mathbb{Q}(\theta_2(\tau), \theta_3(\tau))$, respectively.*

Theorem 1.1 implies that the fields $\mathbb{Q}(\theta_3(n\tau), \theta_3(\tau))$ and $\mathbb{Q}(\theta_2(\tau), \theta_3(\tau))$ have the same algebraic closure in $\mathbb{C}$. Hence by Corollary A.2 we obtain the following Theorem 1.2.

THEOREM 1.2. *Let $n \geq 2$ be an integer and $j \in \{2, 3, 4\}$. Then for any $\tau \in \mathbb{H}$ at least three of the numbers $e^{\pi i \tau}$, $\theta_3(\tau)$, $\theta_3(n\tau)$, $D\theta_j(\tau)$ are algebraically independent over $\mathbb{Q}$.*

COROLLARY 1.3. *Let $n \geq 2$ be an integer and $\tau \in \mathbb{H}$ such that $q = e^{\pi i \tau}$ $(0 < |q| < 1)$ is an algebraic number. Then the three numbers*

$$\theta_3(\tau), \qquad \theta_3(n\tau), \qquad D\theta_3(\tau)$$

are algebraically independent over $\mathbb{Q}$*. This implies that so are the three sums*

$$\sum_{k=1}^{\infty} q^{k^2}, \qquad \sum_{k=1}^{\infty} q^{nk^2}, \qquad \sum_{k=1}^{\infty} k^2 q^{k^2}.$$

This corollary generalizes Theorem B as follows: *Let* $n \geq 2$ *be an integer and* $\tau \in \mathbb{H}$ *such that* $q = e^{\pi i \tau}$ *is an algebraic number. Then the two numbers* $\theta_3(n\tau)$ *and* $\theta_3(\tau)$ *are algebraically independent over* $\mathbb{Q}$*.*

As an application of Theorem 1.1, an algebraic dependence result is also presented. The next theorem asserts that we can not extend the above independence result to many more values.

THEOREM 1.4. *Let* $\ell, m, n \geq 1$ *be integers and* $\tau \in \mathbb{H}$*. Then the three numbers* $\theta_3(\ell\tau), \theta_3(m\tau)$*, and* $\theta_3(n\tau)$ *are algebraically dependent over* $\mathbb{Q}$*.*

EXAMPLE 1.5. Let $\tau \in \mathbb{H}$. From the polynomials P_2 and P_3 above, we obtain the polynomial

$$P(X, Y, Z) := 27X^8 - 18X^4Y^4 - 64X^2Y^4Z^2 + 64X^2Y^2Z^4 - 8X^2Z^6 - Z^8,$$

which vanishes for

$$X = \theta_3(3\tau), \qquad Y = \theta_3(2\tau), \qquad Z = \theta_3(\tau).$$

Theorem 1.1 is founded again on the existence of the integer polynomials in two variables, which vanish identically at certain rational functions of the theta-constants. In order to handle the specific properties of these polynomials, it is necessary to compute $P_n(0, Y)$ for the polynomials $P_n(X, Y)$ from Theorem D. Then it will turn out that $P_n(0, Y)$ is a nonvanishing *constant polynomial* (see Proposition 2.2). This fact plays an important role in the proof of Theorem 1.1 with even integers which are not a power of two. To avoid the complexity, we devote Section 4 to the proof of Proposition 2.2.

It should be noted that our argument is not based on applying the algebraic independence criterion used in [3].

2. Lemmas

In this section, we prepare some lemmas to prove our theorem. In what follows, we distinguish two cases based on the parity of n.

2.1. The case where n is odd

Let $n \geq 3$ be a fixed odd integer and $\tau \in \mathbb{H}$. From Theorem D we know that there exists a nonzero polynomial $P_n(X, Y) \in \mathbb{Z}[X, Y]$ with $\deg_X P_n = \psi(n)$ such that $P_n(X_0, Y_0)$ vanishes for

$$X_0 := h_3(\tau) = n^2 \frac{\theta_3^4(n\tau)}{\theta_3^4(\tau)}, \qquad Y_0 := 16\lambda(\tau) = 16 \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}.$$

LEMMA 2.1. *For any complex number α, the polynomial $P_n(\alpha, Y)$ does not vanish identically.*

PROOF. Suppose on the contrary that there exists an $\alpha \in \mathbb{C}$ such that $P_n(\alpha, Y) = 0$. Then there exists a polynomial $R_n(X, Y) \in \mathbb{C}[X, Y]$ satisfying

$$P_n(X, Y) = (X - \alpha) R_n(X, Y). \tag{4}$$

In accordance with formula (5) in [8], we define

$$x_\nu(\tau) := u^2 \frac{\theta_3^4\left(\frac{u\tau + 2v}{w}\right)}{\theta_3^4(\tau)} \qquad (\nu = 1, 2, \ldots, \psi(n))$$

for any $\tau \in \mathbb{H}$, where the nonnegative integers u, v, w are given by [8, Lemma 1]. These integers depend on n and ν and satisfy the three conditions

$$(u, v, w) = 1, \quad uw = n, \quad 0 \leq v < w. \tag{5}$$

Substituting $Y = 16\lambda(\tau)$ into (4), we have by [8, Corollary 1]

$$\prod_{\nu=1}^{\psi(n)} \bigl(X - x_\nu(\tau)\bigr) = P_n\bigl(X, 16\lambda(\tau)\bigr) = (X - \alpha) R_n\bigl(X, 16\lambda(\tau)\bigr), \tag{6}$$

and hence, by substituting $X = \alpha$ into (6), we see that the holomorphic function

$$\prod_{\nu=1}^{\psi(n)} \bigl(\alpha - x_\nu(\tau)\bigr) = 0$$

is identically zero on $\mathbb{H}$. This implies that there exist integers u, v, and w with (5) such that

$$\frac{\theta_3\left(\frac{u\tau + 2v}{w}\right)}{\theta_3(\tau)} =: \beta \in \mathbb{C} \tag{7}$$

is a constant function on $\mathbb{H}$, where

$$\theta_3(\tau) = 1 + 2 \sum_{\nu=1}^{\infty} \bigl(e^{\pi i \tau / w}\bigr)^{w\nu^2} \tag{8}$$

and

$$\theta_3\left(\frac{u\tau+2v}{w}\right)=1+2\sum_{\nu=1}^{\infty}\xi_w^{v\nu^2}\left(e^{\pi i\tau/w}\right)^{u\nu^2},\qquad \xi_w:=e^{2\pi i/w}. \tag{9}$$

Therefore, by (7), (8), (9) and applying $uw=n$ in (5), we have $\beta=1$, $u=w=\sqrt{n}>1$, $\xi_w^v=1$. Since $\xi_w\neq 1$, we conclude from $\xi_w^v=1$ and $0\leq v<w$ in (5) that $v=0$. Finally, we deduce that $(u,v,w)=(u,0,u)=u>1$, which contradicts the arithmetic condition $(u,v,w)=1$ in (5). This completes the proof of Lemma 2.1.

PROPOSITION 2.2. *Let $n\geq 3$ be an odd integer and $\psi(n)$ be defined as in* (3). *Then we have*

$$\prod\theta_3\left(\frac{u\tau+2v}{w}\right)=\theta_3^{\psi(n)}(\tau), \tag{10}$$

where the product is taken over all $\psi(n)$ triples u,v,w of nonnegative integers satisfying the conditions (5).

We devote Section 4 to a proof of Proposition 2.2.

Throughout this paper, by $\deg Q$ we denote the total degree of an integer polynomial Q in one or two variables. Note that in particular $\deg 0=-\infty$. Moreover for the sake of brevity, for any $j\in\{2,3,4\}$, the notation θ_j is used instead of $\theta_j(\tau)$.

LEMMA 2.3. *Let $m\geq 3$ be an odd integer. Then there is a polynomial $Q_m(X,Y)\in\mathbb{Z}[X,Y]$ such that*

$$Q_m\left(\frac{\theta_3^4(m\tau)}{\theta_3^4},\frac{\theta_2^4}{\theta_3^4}\right)=0 \tag{11}$$

holds for any $\tau\in\mathbb{H}$, where $\deg Q_m(X,Y)=\psi(m)$ *and* $\deg Q_m(0,Y)=0$.

PROOF. The assertion (11) follows immediately from [8, Theorem 1]. For the statement on the total degree of $Q_m(X,Y)$, we have by [8, Corollary 4]

$$Q_m(X,Y)=m^{2\psi(m)}X^{\psi(m)}+\sum_{\nu=1}^{\psi(m)}R_\nu(Y)X^{\psi(m)-\nu}, \tag{12}$$

where for each ν with $1\leq\nu\leq\psi(m)$,

$$\deg\bigl(R_\nu(Y)X^{\psi(m)-\nu}\bigr)\leq\nu\cdot\left(1-\frac{1}{m}\right)+\bigl(\psi(m)-\nu\bigr)<\psi(m).$$

Thus, the total degree of $Q_m(X,Y)$ equals to $\psi(m)$.

Finally we prove that $Q_m(0, Y)$ is a nonzero constant. In Nesterenko's paper [8, pp. 154, lines 21–24], the polynomial $R_{\psi(m)}(Y)$ in (12) is chosen such that the function in τ, defined by

$$\prod u^2 \frac{\theta_3^4\left(\frac{u\tau+2v}{w}\right)}{\theta_3^4(\tau)} - R_{\psi(m)}(16\lambda(\tau)),$$

is identically zero, where the product is taken over all $\psi(m)$ triples u, v, w of nonnegative integers satisfying the conditions in (5). Hence, by Proposition 2.2, the polynomial $R_{\psi(m)}(Y)$ is given by the constant

$$R_{\psi(m)}(Y) = \prod u^2 \frac{\theta_3^4\left(\frac{u\tau+2v}{w}\right)}{\theta_3^4(\tau)} = \prod u^2.$$

Then by (12) we have $Q_m(0, Y) = R_{\psi(m)}(Y)$, and hence the lemma is proved.

EXAMPLE 2.4. We know from Theorem D and the explicit form of P_3 in Section 1 that

$$Q_3(X, Y) = 9 - (252 - 2304Y + 2304Y^2)X + 2430X^2 - 8748X^3 + 6561X^4$$

with $\deg Q_3(X, Y) = 4$ and $Q_3(0, Y) = 9$.

2.2. *The case where n is even*

LEMMA 2.5. *Let $n = 2^\alpha m$ be an integer with $\alpha \geq 1$ and an odd integer $m \geq 3$. Then there exists a polynomial $Q_n(X, Y) \in \mathbb{Z}[X, Y]$ such that*

$$Q_n\left(\frac{\theta_3^4(n\tau)}{\theta_3^4}, \frac{\theta_2^4}{\theta_3^4}\right) = 0 \tag{13}$$

for any $\tau \in \mathbb{H}$. Furthermore, the polynomial $Q_n(X, Y)$ is of the form

$$Q_n(X, Y) = c^{2^\alpha} Y^{2^\alpha \psi(m)} + \sum_{j=0}^{2^\alpha \psi(m)-1} R_{n,j}(X) Y^j \tag{14}$$

with

$$Q_n(0, Y) = c^{2^\alpha} Y^{2^\alpha \psi(m)}, \tag{15}$$

where $\deg R_{n,j}(X) \leq 2^\alpha \psi(m) - j$ $(0 \leq j < 2^\alpha \psi(m))$*, and c equals to the nonzero integer $P_m(0, Y)$, which exists by Lemma 2.3.*

PROOF. Throughout this proof any capital character with subscript(s) defines an integer polynomial. We prove the lemma by induction with respect to α. First we treat the case $\alpha = 1$. Let $m \geq 3$ be an odd integer and

$$Q_m(X, Y) = \sum_{\nu,\mu} a_{\nu,\mu} X^\nu Y^\mu \tag{16}$$

be as in Lemma 2.3, where $\deg Q_m(X, Y) = \psi(m)$. Then we have

$$0 = Q_m\left(\frac{\theta_3^4(2m\tau)}{\theta_3^4(2\tau)}, \frac{\theta_2^4(2\tau)}{\theta_3^4(2\tau)}\right) = \sum_{\nu,\mu} a_{\nu,\mu}\left(\frac{\theta_3^4(2m\tau)}{\theta_3^4(2\tau)}\right)^\nu \left(\frac{\theta_2^4(2\tau)}{\theta_3^4(2\tau)}\right)^\mu \tag{17}$$

for any $\tau \in \mathbb{H}$. Multiplying this identity with $(1 + \theta_4^2/\theta_3^2)^{2\psi(m)}$, we obtain

$$\begin{aligned}
0 &= \left(1 + \frac{\theta_4^2}{\theta_3^2}\right)^{2\psi(m)} \sum_{\nu,\mu} a_{\nu,\mu}\theta_3^{4\nu}(2m\tau) \cdot \theta_2^{4\mu}(2\tau) \cdot \theta_3^{-4(\nu+\mu)}(2\tau) \\
&= \left(1 + \frac{\theta_4^2}{\theta_3^2}\right)^{2\psi(m)} \sum_{\nu,\mu} a_{\nu,\mu}\theta_3^{4\nu}(2m\tau) \cdot \left(\frac{\theta_3^2 - \theta_4^2}{2}\right)^{2\mu} \cdot \left(\frac{\theta_3^2 + \theta_4^2}{2}\right)^{-2(\nu+\mu)} \\
&= \sum_{\nu,\mu} 2^{2\nu} a_{\nu,\mu}\left(\frac{\theta_3(2m\tau)}{\theta_3}\right)^{4\nu} \cdot \left(1 - \frac{\theta_4^2}{\theta_3^2}\right)^{2\mu} \cdot \left(1 + \frac{\theta_4^2}{\theta_3^2}\right)^{2(\psi(m)-\nu-\mu)},
\end{aligned} \tag{18}$$

where we used the identities

$$2\theta_2^2(2\tau) = \theta_3^2(\tau) - \theta_4^2(\tau), \qquad 2\theta_3^2(2\tau) = \theta_3^2(\tau) + \theta_4^2(\tau).$$

Let $n = 2m$ and define

$$B_n(X, Y) := \sum_{\nu,\mu} 2^{2\nu} a_{\nu,\mu} X^{4\nu}\left(1 - Y^2\right)^{2\mu}\left(1 + Y^2\right)^{2(\psi(m)-\nu-\mu)}. \tag{19}$$

Then by (18)

$$B_n\left(\frac{\theta_3(n\tau)}{\theta_3}, \frac{\theta_4}{\theta_3}\right) = 0. \tag{20}$$

Furthermore, since $Q_m(0, Y)$ is a nonzero constant by Lemma 2.3, we can apply (16) to get

$$c := Q_m(0, Y) = Q_m(0, 1) = \sum_{\mu \geq 0} a_{0,\mu},$$

and hence by (19) there exists $A_{n,j}(X)$ for each j with $0 \le j < 2\psi(m)$ such that

$$\begin{aligned} B_n(X,Y) &= \Big(\sum_{\mu\ge 0} a_{0,\mu}\Big) Y^{4\psi(m)} + \sum_{j=0}^{2\psi(m)-1} A_{n,j}(X^4)Y^{2j} \\ &= cY^{4\psi(m)} + \sum_{j=0}^{2\psi(m)-1} A_{n,j}(X^4)Y^{2j}, \end{aligned} \tag{21}$$

where $\deg B_n(X,Y) = 4\psi(m)$ follows again from (19). We rewrite $B_n(X,Y)$ by

$$\begin{aligned} B_n(X,Y) &= \sum_{j\ge 0} C_{n,j}(X^4)Y^{2j} \\ &= \sum_{\substack{j\ge 0\\ j\equiv 0 \pmod 2}} C_{n,j}(X^4)Y^{2j} + \sum_{\substack{j\ge 1\\ j\equiv 1 \pmod 2}} C_{n,j}(X^4)Y^{2j} \\ &=: D_n(X^4,Y^4) + Y^2E_n(X^4,Y^4). \end{aligned}$$

Define

$$\begin{aligned} \tilde{Q}_n(X,Y) &:= D_n^2(X,Y) - YE_n^2(X,Y), \\ Q_n(X,Y) &:= \tilde{Q}_n(X, 1-Y). \end{aligned}$$

Note that

$$\begin{aligned} \tilde{Q}_n(X^4,Y^4) &= D_n^2(X^4,Y^4) - Y^4E_n^2(X^4,Y^4) \\ &= B_n(X,Y)B_n(X,iY). \end{aligned} \tag{22}$$

Substituting $X = \theta_3(n\tau)/\theta_3$ and $Y = \theta_4/\theta_3$ into this identity, we have by (20)

$$\tilde{Q}_n\left(\frac{\theta_3^4(n\tau)}{\theta_3^4}, \frac{\theta_4^4}{\theta_3^4}\right) = 0.$$

Since $\theta_4^4/\theta_3^4 = 1 - \theta_2^4/\theta_3^4$, it is clear that

$$Q_n\left(\frac{\theta_3^4(n\tau)}{\theta_3^4}, \frac{\theta_2^4}{\theta_3^4}\right) = 0.$$

Furthermore, by (22) together with (21)

$$\begin{aligned}\tilde{Q}_n(X^4, Y^4) &= \left(cY^{4\psi(m)} + \sum_{j=0}^{2\psi(m)-1} A_{n,j}(X^4)Y^{2j}\right)\\ &\qquad \times \left(cY^{4\psi(m)} + \sum_{j=0}^{2\psi(m)-1} (-1)^j A_{n,j}(X^4)Y^{2j}\right)\\ &= c^2Y^{8\psi(m)} + \sum_{j=0}^{2\psi(m)-1} S_{n,j}(X^4)Y^{4j},\end{aligned}$$

where for each j with $0 \le j < 2\psi(m)$ we have

$$\begin{aligned}\deg \tilde{Q}_n(X^4, Y^4) = 2\deg B_n(X, Y) = 8\psi(m) &\ge \deg\bigl(S_{n,j}(X^4)Y^{4j}\bigr)\\ &= 4\deg S_{n,j}(X) + 4j,\end{aligned}$$

which implies

$$\deg S_{n,j}(X) \le 2\psi(m) - j.$$

Hence we get

$$\tilde{Q}_n(X, Y) = c^2Y^{2\psi(m)} + \sum_{j=0}^{2\psi(m)-1} S_{n,j}(X)Y^j,$$

and so

$$Q_n(X, Y) = \tilde{Q}_n(X, 1-Y) = c^2Y^{2\psi(m)} + \sum_{j=0}^{2\psi(m)-1} R_{n,j}(X)Y^j,$$

where $\deg R_{n,j}(X) \le 2\psi(m) - j$, for $0 \le j < 2\psi(m)$. It remains to prove (15) in the case of $\alpha = 1$. From Lemma 2.3 we get

$$c = P_m(0, Y) = \sum_{\mu \ge 0} a_{0,\mu}Y^\mu,$$

so that $a_{0,0} = c$ and $a_{0,\mu} = 0$ for $\mu \neq 0$. Then, by (16) we obtain

$$B_n(0, Y) = \sum_{\mu\ge 0} a_{0,\mu}(1-Y^2)^{2\mu}(1+Y^2)^{2(\psi(m)-\mu)} = c(1+Y^2)^{2\psi(m)}.$$

By (22), we have

$$\tilde{Q}_n(0, Y^4) = c^2(1+Y^2)^{2\psi(m)}(1-Y^2)^{2\psi(m)} = c^2(1-Y^4)^{2\psi(m)},$$

and therefore

$$\tilde{Q}_n(0, Y) = c^2(1 - Y)^{2\psi(m)},$$

which gives

$$Q_n(0, Y) = c^2 Y^{2\psi(m)}$$

by the above definition of $Q_n(X, Y)$. Hence the proof of Lemma 2.3 with $\alpha = 1$ is completed.

Next, let the lemma be true for some fixed $\alpha \geq 1$ with $n = 2^\alpha m$. In the preceding part of the proof we replace m by n and the polynomial $P_m(X, Y)$ by the polynomial

$$Q_n(X, Y) = \sum_{\nu,\mu} b_{\nu,\mu} X^\nu Y^\mu$$

satisfying (13) to (15). In particular,

$$\deg Q_n(X, Y) = 2^\alpha \psi(m) \tag{23}$$

and

$$b_{0,\mu} = \begin{cases} c^{2^\alpha}, & \text{if } \mu = 2^\alpha \psi(m), \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

Replacing τ by 2τ we thus obtain instead of (17)

$$0 = Q_n\left(\frac{\theta_3^4(2n\tau)}{\theta_3^4(2\tau)}, \frac{\theta_2^4(2\tau)}{\theta_3^4(2\tau)}\right) = \sum_{\nu,\mu} b_{\nu,\mu}\left(\frac{\theta_3^4(2n\tau)}{\theta_3^4(2\tau)}\right)^\nu \left(\frac{\theta_2^4(2\tau)}{\theta_3^4(2\tau)}\right)^\mu.$$

By the method described above for the case $\alpha = 1$, we obtain again polynomials $B_{2n}(X, Y)$, $\tilde{Q}_{2n}(X, Y)$, and $Q_{2n}(X, Y)$ such that

$$\tilde{Q}_{2n}(X^4, Y^4) := B_{2n}(X, Y) B_{2n}(X, iY),$$
$$Q_{2n}(X, Y) := \tilde{Q}_{2n}(X, 1 - Y)$$

and, step by step,

$$0 = \tilde{Q}_{2n}\left(\frac{\theta_3^4(2n\tau)}{\theta_3^4}, \frac{\theta_4^4}{\theta_3^4}\right),$$
$$0 = Q_{2n}\left(\frac{\theta_3^4(2n\tau)}{\theta_3^4}, \frac{\theta_2^4}{\theta_3^4}\right).$$

This proves (13) for n replaced by $2n = 2^{\alpha+1} m$. Next, we consider (14). We have by (23) and (24) instead of (21)

$$
\begin{aligned}
B_{2n}(X,Y) &= \sum_{\nu,\mu} 2^{2\nu} b_{\nu,\mu} X^{4\nu} (1-Y^2)^{2\mu} (1+Y^2)^{2(2^\alpha\psi(m)-\nu-\mu)} \\
&= \Big(\sum_{\mu\geq 0} b_{0,\mu}\Big) Y^{2^{\alpha+2}\psi(m)} + \sum_{j=0}^{2^{\alpha+2}\psi(m)-1} A_{2n,j}(X^4) Y^{2j} \\
&= c^{2^\alpha} Y^{2^{\alpha+2}\psi(m)} + \sum_{j=0}^{2^{\alpha+2}\psi(m)-1} A_{2n,j}(X^4) Y^{2j},
\end{aligned}
$$

where $\deg B_{2n}(X,Y) = 2^{\alpha+2}\psi(m)$. Then, by the same arguments as in the case of $\alpha = 1$, we obtain (14) with α replaced by $\alpha+1$. Finally, we show (15) for $\alpha+1$. With the above formula for $B_{2n}(X,Y)$ we obtain

$$
\begin{aligned}
B_{2n}(0,Y) &= \sum_{\mu\geq 0} b_{0,\mu} (1-Y^2)^{2\mu} (1+Y^2)^{2(2^\alpha\psi(m)-\mu)} \\
&= b_{0,2^\alpha\psi(m)} (1-Y^2)^{2^{\alpha+1}\psi(m)} \\
&= c^{2^\alpha} (1-Y^2)^{2^{\alpha+1}\psi(m)},
\end{aligned}
$$

where we used (24). Thus, using the same arguments as in the case $\alpha = 1$, we obtain

$$
\tilde{Q}_{2n}(0,Y^4) = c^{2^{\alpha+1}} (1-Y^4)^{2^{\alpha+1}\psi(m)},
$$

hence

$$
Q_{2n}(0,Y) = c^{2^{\alpha+1}} Y^{2^{\alpha+1}\psi(m)}.
$$

This completes the proof of the lemma.

EXAMPLE 2.6. With the polynomial from Example 2.4, we obtain for $n = 6$

$$
\begin{aligned}
B_6(X,Y) = {}& 9Y^{16} + 72Y^{14} + (-1008X^4 + 252)Y^{12} + (30816X^4 + 504)Y^{10} \\
&+ (38880X^8 - 15120X^4 + 630)Y^8 \\
&+ (155520X^8 - 93888X^4 + 504)Y^6 \\
&+ (-559872X^{12} + 233280X^8 - 15120X^4 + 252)Y^4 \\
&+ (-1119744X^{12} + 155520X^8 + 30816X^4 + 72)Y^2 \\
&+ 1679616X^{16} - 559872X^{12} + 38880X^8 - 1008X^4 + 9
\end{aligned}
$$

and

$$
\begin{aligned}
Q_6(X,Y) = 81Y^8 &+ 18144XY^7 + (1715904X^2 - 5344704X)Y^6 \\
&+ (88459776X^3 + 907448832X^2 + 58392576X)Y^5 \\
&+ (2670589440X^4 - 11804341248X^3 + 1470721536X^2 \\
&\quad - 180486144X)Y^4 \\
&+ (46921752576X^5 - 92553560064X^4 \\
&\quad + 34882265088X^3 - 4756340736X^2 + 212336640X)Y^3 \\
&+ (444063596544X^6 - 148021198848X^5 + 96423395328X^4 \\
&\quad - 23254843392X^3 + 2378170368X^2 - 84934656X)Y^2 \\
&+ (1880739938304X^7 - 1044855521280X^6 + 162533081088X^5 \\
&\quad - 7739670528X^4)Y \\
&+ 2821109907456X^8 - 3761479876608X^7 + 1044855521280X^6 \\
&\quad - 108355387392X^5 + 3869835264X^4.
\end{aligned}
$$

3. Proofs of Theorems 1.1, 1.2, and 1.4

PROOF OF THEOREM 1.1. Let $n \geq 2$ be an integer and $\tau \in \mathbb{H}$. We first show the algebraicity of θ_2 over the field

$$F := \mathbb{Q}(\theta_3, \theta_3(n\tau)).$$

Let $n \geq 3$ be an odd integer. Applying Lemma 2.1 with $\alpha = h_3(\tau) = n^2\theta_3^4(n\tau)/\theta_3^4$, we obtain the nonzero polynomial

$$f_n(Y) := P_n\left(n^2\frac{\theta_3^4(n\tau)}{\theta_3^4}, 16\frac{Y}{\theta_3^4}\right) \in F[Y]$$

satisfying $f_n(\theta_2^4) = P_n(h_3(\tau), 16\lambda(\tau)) = 0$. This implies that the number θ_2 is algebraic over F. Next we put $n = 2^\alpha m$ ($\alpha \geq 1$) with an odd number $m \geq 3$. Then by Lemma 2.5 there exists a polynomial $Q_n(X,Y) \in \mathbb{Z}[X,Y]$ satisfying (13) and (14), namely, the number $Y = \theta_2^4$ is a zero of a nonzero polynomial $g_n(Y) \in F(Y)$ defined by

$$
\begin{aligned}
g_n(Y) &:= Q_n\left(\frac{\theta_3^4(n\tau)}{\theta_3^4}, \frac{Y}{\theta_3^4}\right) \\
&= c^{2^\alpha}\left(\frac{Y}{\theta_3^4}\right)^{2^\alpha\psi(m)} + \sum_{j=0}^{2^\alpha\psi(m)-1} R_{n,j}\left(\frac{\theta_3^4(n\tau)}{\theta_3^4}\right)\left(\frac{Y}{\theta_3^4}\right)^j,
\end{aligned}
$$

which yields the algebraicity of θ_2 over F. Finally we treat the case of $n = 2^\alpha$ ($\alpha \geq 1$). Let $P_n(X,Y) \in \mathbb{Z}[X,Y]$ be the polynomial given in Theorem C.

Then $Y = \theta_4/\theta_3$ is a zero of the polynomial

$$h_n(Y) := P_n\left(\frac{\theta_3^2(n\tau)}{\theta_3^2}, Y\right) \in F[X].$$

We check that $h_n(Y)$ is nonzero. This holds for the cases $\alpha = 1$ and 2 from the explicit forms of P_2 and P_4 given in Section 1. For $\alpha \geq 3$ we see this by considering the Taylor expansion

$$h_n(Y) = P_n\left(\frac{\theta_3^2(n\tau)}{\theta_3^2}, 0\right) + \frac{\partial}{\partial Y} P_n\left(\frac{\theta_3^2(n\tau)}{\theta_3^2}, 0\right) Y + O(Y^2)$$

and the fact that the polynomials $P_n(X, 0)$ and $\partial P_n(X, 0)/\partial Y$ have no common root as pointed out in Section 1. Hence the number θ_4/θ_3 is algebraic over F, and so is θ_2 because of the relation $\theta_3^4 = \theta_2^4 + \theta_4^4$. Thus, in any case, the number θ_2 is algebraic over F.

Next we prove that $\theta_3(n\tau)$ is algebraic over the field $E := \mathbb{Q}(\theta_2, \theta_3)$. Let $n = 2^\alpha m$ be an integer with $\alpha \geq 0$ and an odd integer $m \geq 3$. Then, by Lemma 2.3 and Lemma 2.5, there exists a nonzero integer polynomial $Q_n(X, Y)$ such that

$$Q_n\left(\frac{\theta_3^4(n\tau)}{\theta_3^4}, \frac{\theta_2^4}{\theta_3^4}\right) = 0$$

and $Q_n(0, Y) = c^{2^\alpha} Y^{2^\alpha \psi(m)}$, where $c := Q_m(0, Y)$ is a nonzero integer. Define

$$f_n(X) := Q_n\left(\frac{X^4}{\theta_3^4}, \frac{\theta_2^4}{\theta_3^4}\right) \in E[X].$$

It is clear that $f_n\big(\theta_3(n\tau)\big) = 0$. Furthermore the polynomial $f_n(X)$ is nonzero, since

$$f_n(0) = Q_n\big(0, \theta_2^4/\theta_3^4\big) = c^{2^\alpha} (\theta_2^4/\theta_3^4)^{2^\alpha \psi(m)} \neq 0.$$

Hence the assertion holds. For the case of $n = 2^m$ $(m \geq 1)$, we consider the polynomial

$$g_n(X) := T_n\big(X^2, (\theta_3 + \theta_4)^2, \theta_3\theta_4\big) \in E(\theta_4)[X],$$

where the polynomial $T_n(t_1, t_2, t_3) \in \mathbb{Z}[t_1, t_2, t_3]$ is given in (1) and it has the form

$$T_n(t_1, t_2, t_3) = a_n t_1^\lambda + \sum_{\substack{\nu=(\nu_1,\nu_2,\nu_3)\in\mathbb{N}^3 \\ \nu_1+\nu_2+\nu_3=\lambda \\ \nu_1<\lambda}} a_\nu t_1^{\nu_1} t_2^{\nu_2} t_3^{\nu_3}, \qquad a_n \neq 0,$$

with $\lambda = 2^{m-2}$ for $m \geq 2$ and $\lambda = 1$ when $m = 1$. Hence $g_n(X)$ is nonzero and $g_n\big(\theta_3(n\tau)\big) = 0$. This implies that the number $\theta_3(n\tau)$ is algebraic over $E(\theta_4)$. Since θ_4 is algebraic over E, we have the conclusion. The proof of Theorem 1.1 is completed.

PROOF OF THEOREM 1.2. Theorem 1.2 follows from Theorem 1.1 applied twice and from Corollary A.2 by establishing the inequality

$$\begin{aligned} 3 &\leq \text{trans. deg}\,\mathbb{Q}\big(e^{\pi i\tau}, \theta_2(\tau), \theta_3(\tau), D\theta_j(\tau)\big)/\mathbb{Q} \\ &= \text{trans. deg}\,\mathbb{Q}\big(e^{\pi i\tau}, \theta_2(\tau), \theta_3(\tau), \theta_3(n\tau), D\theta_j(\tau)\big)/\mathbb{Q} \\ &= \text{trans. deg}\,\mathbb{Q}\big(e^{\pi i\tau}, \theta_3(\tau), \theta_3(n\tau), D\theta_j(\tau)\big)/\mathbb{Q}. \end{aligned}$$

Thus Theorem 1.2 is proved.

PROOF OF THEOREM 1.4. Let $\ell, m, n \geq 1$ be integers and $\tau \in \mathbb{H}$. Then by Theorem 1.1 the field

$$K := E\big(\theta_3(\ell\tau), \theta_3(m\tau), \theta_3(n\tau)\big)$$

is an algebraic extension of $E := \mathbb{Q}(\theta_2, \theta_3)$, and hence

$$\text{trans. deg}\,K/\mathbb{Q} = \text{trans. deg}\,E/\mathbb{Q} \leq 2.$$

Therefore the three numbers $\theta_3(\ell\tau)$, $\theta_3(m\tau)$, and $\theta_3(n\tau)$ are algebraically dependent over $\mathbb{Q}$.

4. Proof of Proposition 2.2

For the sake of brevity, we put $\theta_3(\tau) = \theta_3(q)$ ($q = e^{\pi i\tau}, \tau \in \mathbb{H}$). Then equation (10) is equivalent to

$$\prod_{u,w} \prod_{\substack{v=0 \\ (u,v,w)=1}}^{w-1} \theta_3(\zeta_w^v q^{u/w}) = \theta_3^{\psi(n)}(q), \tag{25}$$

where $\zeta_w := e^{2\pi i/w}$ is a primitive wth root of unity and the left product is taken over u and w with $uw = n$. In what follows, we prepare some lemmas for the proof of (25).

LEMMA 4.1. *Let p be an odd prime. Then*

$$\prod_{k=0}^{p-1} \theta_3(\zeta_p^k q) = \frac{\theta_3^{p+1}(q^p)}{\theta_3(q^{p^2})}. \tag{26}$$

PROOF. Define

$$F(q) := \prod_{\ell=1}^{\infty}(1-q^{\ell})$$

for $q \in \mathbb{C}$ with $|q| < 1$. Then we have

$$\begin{aligned}
\prod_{k=0}^{p-1} F(\zeta_p^k q) &= \prod_{\ell=1}^{\infty}\prod_{k=0}^{p-1}\bigl(1-(\zeta_p^k q)^{\ell}\bigr) \\
&= \Biggl(\prod_{\substack{\ell=1 \\ p\mid\ell}}^{\infty}\prod_{k=0}^{p-1}\bigl(1-(\zeta_p^k q)^{\ell}\bigr)\Biggr)\Biggl(\prod_{\substack{\ell=1 \\ p\nmid\ell}}^{\infty}\prod_{k=0}^{p-1}\bigl(1-(\zeta_p^k q)^{\ell}\bigr)\Biggr) \\
&= \prod_{\substack{\ell=1 \\ p\mid\ell}}^{\infty}(1-q^{\ell})^{p}\prod_{\substack{\ell=1 \\ p\nmid\ell}}^{\infty}(1-q^{p\ell}) \\
&= \prod_{\ell=1}^{\infty}(1-q^{p\ell})^{p}\prod_{\substack{\ell=1 \\ p\nmid\ell}}^{\infty}(1-q^{p\ell}),
\end{aligned}$$

and hence

$$\begin{aligned}
F\bigl(q^{p^2}\bigr)\prod_{k=0}^{p-1} F(\zeta_p^k q) &= \prod_{\ell=1}^{\infty}(1-q^{p^2\ell})\prod_{\ell=1}^{\infty}(1-q^{p\ell})^{p}\prod_{\substack{\ell=1 \\ p\nmid\ell}}^{\infty}(1-q^{p\ell}) \\
&= \prod_{\ell=1}^{\infty}(1-q^{p\ell})^{p+1} \qquad (27) \\
&= F(q^p)^{p+1}. \qquad (28)
\end{aligned}$$

On the other hand, using Jacobi's triple product expression for $\theta_3(q)$, we have

$$\theta_3(q) = \prod_{\ell=1}^{\infty}(1-q^{2\ell})(1+q^{2\ell-1})^2 = \frac{F(q^2)^5}{F(q)^2F(q^4)^2}, \qquad (29)$$

where we used the equalities

$$\prod_{\ell=1}^{\infty}(1+q^{2\ell-1}) = \prod_{\ell=1}^{\infty}\frac{1+q^{\ell}}{1+q^{2\ell}} = \prod_{\ell=1}^{\infty}\frac{(1-q^{2\ell})^2}{(1-q^{\ell})(1-q^{4\ell})}.$$

Therefore we obtain by (27) and (29)

$$\begin{aligned}\theta_3\big(q^{p^2}\big)\prod_{k=0}^{p-1}\theta_3(\zeta_p^k q) &= \frac{F(q^{2p^2})^5}{F(q^{p^2})^2F(q^{4p^2})^2}\prod_{k=0}^{p-1}\frac{F(\zeta_p^{2k}q^2)^5}{F(\zeta_p^k q)^2F(\zeta_p^{4k}q^4)^2}\\ &= \frac{F(q^{2p^2})^5}{F(q^{p^2})^2F(q^{4p^2})^2}\prod_{k=0}^{p-1}\frac{F(\zeta_p^{k}q^2)^5}{F(\zeta_p^k q)^2F(\zeta_p^{k}q^4)^2}\\ &= \left(\frac{F(q^{2p})^5}{F(q^p)^2F(q^{4p})^2}\right)^{p+1} = \theta_3^{p+1}(q^p),\end{aligned}$$

which proves the lemma.

Lemma 4.2. *Let p be an odd prime. For any integer $j \geq 0$, we have*

$$\prod_{k=0}^{p^j-1}\theta_3(\zeta_{p^j}^k q) = \frac{\theta_3^{b_{j+1}}(q^{p^j})}{\theta_3^{b_j}(q^{p^{j+1}})}, \tag{30}$$

where b_n $(n \geq 0)$ are nonnegative integers defined by

$$b_n := \frac{p^n-1}{p-1}.$$

Proof. The assertion is trivial for $j = 0$, and Lemma 4.1 is the case of $j = 1$. Suppose that (30) holds for $j \geq 1$. Then, for each $v = 0, 1, \dots, p-1$, we replace q by $\zeta_{p^{j+1}}^v q$ in (30):

$$\prod_{k=0}^{p^j-1}\theta_3(\zeta_{p^j}^k\zeta_{p^{j+1}}^v q) = \frac{\theta_3^{b_{j+1}}\big(\zeta_p^v q^{p^j}\big)}{\theta_3^{b_j}\big(q^{p^{j+1}}\big)},$$

where $\zeta_{p^j}^k\zeta_{p^{j+1}}^v = \zeta_{p^{j+1}}^{pk+v}$. Taking the product of the both sides above for v from $v = 0$ to $p-1$, we get

$$\prod_{v=0}^{p-1}\prod_{k=0}^{p^j-1}\theta_3\big(\zeta_{p^{j+1}}^{pk+v}q\big) = \frac{\prod_{v=0}^{p-1}\theta_3^{b_{j+1}}\big(\zeta_p^v q^{p^j}\big)}{\theta_3^{pb_j}\big(q^{p^{j+1}}\big)}. \tag{31}$$

Since the integers $pk+v$ $(0 \leq k < p^j, 0 \leq v \leq p-1)$ are distinct,

$$\prod_{v=0}^{p-1}\prod_{k=0}^{p^j-1}\theta_3\big(\zeta_{p^{j+1}}^{pk+v}q\big) = \prod_{k=0}^{p^{j+1}-1}\theta_3\big(\zeta_{p^{j+1}}^k q\big). \tag{32}$$

Furthermore, replacing q by q^{p^j} in (26), we have

$$\prod_{v=0}^{p-1} \theta_3\big(\zeta_p^v q^{p^j}\big) = \frac{\theta_3^{p+1}\big(q^{p^{j+1}}\big)}{\theta_3\big(q^{p^{j+2}}\big)}. \tag{33}$$

Applying the results (32) and (33) to (31), we obtain

$$\begin{aligned}\prod_{k=0}^{p^{j+1}-1} \theta_3(\zeta_{p^{j+1}}^k q) &= \left(\frac{\theta_3^{p+1}\big(q^{p^{j+1}}\big)}{\theta_3\big(q^{p^{j+2}}\big)}\right)^{b_{j+1}} \frac{1}{\theta_3^{pb_j}\big(q^{p^{j+1}}\big)} \\ &= \frac{\theta_3^{(p+1)b_{j+1}-pb_j}\big(q^{p^{j+1}}\big)}{\theta_3^{b_{j+1}}\big(q^{p^{j+2}}\big)} = \frac{\theta_3^{b_{j+2}}\big(q^{p^{j+1}}\big)}{\theta_3^{b_{j+1}}\big(q^{p^{j+2}}\big)}.\end{aligned}$$

This implies that the equality (30) also holds for $j+1$, and hence Lemma 4.2 is proved.

LEMMA 4.3. *The equality* (25) *holds for any power of an odd prime* $n = p^\ell$ $(\ell \geq 1)$*, namely,*

$$\prod_{j=0}^{\ell} \prod_{\substack{v=0 \\ (p^{\ell-j},v,p^j)=1}}^{p^j-1} \theta_3\big(\zeta_{p^j}^v q^{p^{\ell-2j}}\big) = \theta_3^{\psi(p^\ell)}(q). \tag{34}$$

PROOF. If $n = p^\ell$ $(\ell \geq 1)$ is a power of an odd prime, then the product on the left hand side in (25) is taken over all triples $u = p^{\ell-j}$, v, $w = p^j$ satisfying the conditions

$$(p^{\ell-j}, v, p^j) = 1, \qquad 0 \leq v < p^j, \qquad 0 \leq j \leq \ell,$$

and hence the product is given by the left hand side in (34).

In what follows, we show the equality in (34). Let $j \geq 0$ be a fixed integer. We first replace q by $q^{p^{\ell-2j}}$ in (30):

$$\prod_{v=0}^{p^j-1} \theta_3\big(\zeta_{p^j}^v q^{p^{\ell-2j}}\big) = \frac{\theta_3^{b_{j+1}}\Big(\big(q^{p^{\ell-2j}}\big)^{p^j}\Big)}{\theta_3^{b_j}\Big(\big(q^{p^{\ell-2j}}\big)^{p^{j+1}}\Big)} = \frac{\theta_3^{b_{j+1}}\big(q^{p^{\ell-j}}\big)}{\theta_3^{b_j}\big(q^{p^{\ell-j+1}}\big)}, \tag{35}$$

and then we replace j by $j-1$ and q by $q^{p^{\ell-2j}}$ in (30):

$$\prod_{v=0}^{p^{j-1}-1} \theta_3\big(\zeta_{p^{j-1}}^v q^{p^{\ell-2j}}\big) = \frac{\theta_3^{b_j}\Big(\big(q^{p^{\ell-2j}}\big)^{p^{j-1}}\Big)}{\theta_3^{b_{j-1}}\Big(\big(q^{p^{\ell-2j}}\big)^{p^j}\Big)} = \frac{\theta_3^{b_j}\big(q^{p^{\ell-j-1}}\big)}{\theta_3^{b_{j-1}}\big(q^{p^{\ell-j}}\big)}. \tag{36}$$

Since the condition $(p^{\ell-j}, v, p^j) = 1$ is equivalent to either $j = 0, 1$ or $p \nmid v$ when $j \neq 0, 1$, we have by (35) and (36)

$$\begin{aligned}
\prod_{j=0}^{\ell} \prod_{\substack{v=0 \\ (p^{\ell-j},v,p^j)=1}}^{p^j-1} & \theta_3\big(\zeta_{p^j}^v q^{p^{\ell-2j}}\big) \\
&= \theta_3\big(q^{p^\ell}\big)\Bigg(\prod_{j=1}^{\ell-1} \prod_{\substack{v=0 \\ p\nmid v}}^{p^j-1} \theta_3\big(\zeta_{p^j}^v q^{p^{\ell-2j}}\big)\Bigg) \prod_{v=0}^{p^\ell-1} \theta_3\big(\zeta_{p^\ell}^v q^{p^{-\ell}}\big) \\
&= \theta_3\big(q^{p^\ell}\big)\Bigg(\prod_{j=1}^{\ell-1} \frac{\prod_{v=0}^{p^j-1} \theta_3\big(\zeta_{p^j}^v q^{p^{\ell-2j}}\big)}{\prod_{v=0}^{p^{j-1}-1} \theta_3\big(\zeta_{p^{j-1}}^v q^{p^{\ell-2j}}\big)}\Bigg) \prod_{v=0}^{p^\ell-1} \theta_3\big(\zeta_{p^\ell}^v q^{p^{-\ell}}\big) \\
&= \frac{\prod_{j=0}^{\ell} \prod_{v=0}^{p^j-1} \theta_3\big(\zeta_{p^j}^v q^{p^{\ell-2j}}\big)}{\prod_{j=1}^{\ell-1} \prod_{v=0}^{p^{j-1}-1} \theta_3\big(\zeta_{p^{j-1}}^v q^{p^{\ell-2j}}\big)} \\
&= \Bigg(\prod_{j=0}^{\ell} \frac{\theta_3^{b_{j+1}}\big(q^{p^{\ell-j}}\big)}{\theta_3^{b_j}\big(q^{p^{\ell-j+1}}\big)}\Bigg)\Bigg(\prod_{j=1}^{\ell-1} \frac{\theta_3^{b_{j-1}}\big(q^{p^{\ell-j}}\big)}{\theta_3^{b_j}\big(q^{p^{\ell-j-1}}\big)}\Bigg) \\
&= \theta_3^{b_{\ell+1}-b_{\ell-1}}(q) = \theta_3^{\psi(p^\ell)}(q),
\end{aligned}$$

where we used the following equalities for the second equality above:

$$\prod_{\substack{v=0 \\ p\nmid v}}^{p^j-1} \theta_3\big(\zeta_{p^j}^v q^{\ell-2j}\big) = \frac{\prod_{v=0}^{p^j-1} \theta_3\big(\zeta_{p^j}^v q^{\ell-2j}\big)}{\prod_{v=0}^{p^{j-1}-1} \theta_3\big(\zeta_{p^j}^{pv} q^{\ell-2j}\big)} = \frac{\prod_{v=0}^{p^j-1} \theta_3\big(\zeta_{p^j}^v q^{\ell-2j}\big)}{\prod_{v=0}^{p^{j-1}-1} \theta_3\big(\zeta_{p^{j-1}}^v q^{\ell-2j}\big)}. \tag{37}$$

Thus the proof of Lemma 4.3 is completed.

Proof of Proposition 2.2. We prove (25) by induction on the number of distinct prime factors of an odd integer n. If the integer n has only one prime factor, namely, n is a power of some odd prime, then the assertion follows immediately from Lemma 4.3. Suppose that (25) holds for an odd integer $m \geq 3$ having $s(\geq 1)$ distinct prime factors:

$$\prod_{u,w} \prod_{\substack{v=0 \\ (u,v,w)=1}}^{w-1} \theta_3(\zeta_w^v q^{u/w}) = \theta_3^{\psi(m)}(q), \tag{38}$$

where the product is taken over u and w with $uw = m$.

Let $\ell \geq 1$ be an integer and p be an odd prime number not dividing m. In what follows, under the induction hypothesis (38), we prove that (25) also holds for an integer $n = mp^\ell$ having $s+1$ distinct prime factors. If $n = mp^\ell$ ($\ell \geq 1$), then the product on the left hand side in (25) is taken over all triples $u' = up^{\ell-j}$, $v' = v$, $w' = wp^j$ satisfying the conditions

$$(u', v', w') = (up^{\ell-j}, v, wp^j) = 1,$$
$$uw = m, \qquad 0 \leq v < wp^j, \qquad 0 \leq j \leq \ell.$$

Hence the product in (25) is given by

$$\prod_{j=0}^{\ell} \prod_{u,w} \prod_{\substack{v=0 \\ (up^{\ell-j}, v, wp^j)=1}}^{wp^j-1} \theta_3\big(\zeta_{wp^j}^v q^{\frac{u}{w} p^{\ell-2j}}\big), \tag{39}$$

where the product is taken over u and w with $uw = m$. Since $m = uw$ is not divided by p, we have the equivalent conditions

$$(up^{\ell-j}, v, wp^j) = 1 \iff \begin{cases} (u, v, w) = 1, & \text{if } j = 0, \ell, \\ (u, v, w) = 1 \text{ and } p \nmid v, & \text{otherwise}, \end{cases}$$

and hence the product (39) is divided into three parts as follows:

$$\Bigg(\prod_{u,w} \prod_{\substack{v=0 \\ (u,v,w)=1}}^{w-1} \theta_3\big(\zeta_w^v q^{\frac{u}{w} p^\ell}\big)\Bigg)\Bigg(\prod_{j=1}^{\ell-1} \prod_{u,w} \prod_{\substack{v=0 \\ (u,v,w)=1 \\ p\nmid v}}^{p^j w-1} \theta_3\big(\zeta_{p^j w}^v q^{\frac{u}{w} p^{\ell-2j}}\big)\Bigg)$$
$$\times \Bigg(\prod_{u,w} \prod_{\substack{v=0 \\ (u,v,w)=1}}^{p^\ell w-1} \theta_3\big(\zeta_{wp^\ell}^v q^{\frac{u}{w} p^{-\ell}}\big)\Bigg),$$

where, similarly as in (37), we have

$$\prod_{\substack{v=0 \\ (u,v,w)=1 \\ p\nmid v}}^{p^j w-1} \theta_3\big(\zeta_{p^j w}^v q^{\frac{u}{w} p^{\ell-2j}}\big) = \frac{\displaystyle\prod_{\substack{v=0 \\ (u,v,w)=1}}^{p^j w-1} \theta_3\big(\zeta_{p^j w}^v q^{\frac{u}{w} p^{\ell-2j}}\big)}{\displaystyle\prod_{\substack{v=0 \\ (u,v,w)=1}}^{p^{j-1} w-1} \theta_3\big(\zeta_{p^{j-1} w}^v q^{\frac{u}{w} p^{\ell-2j}}\big)}.$$

Therefore the product (39) is rewritten again as

$$\prod_{j=0}^{\ell}\prod_{u,w}\prod_{\substack{v=0\\(up^{\ell-j},v,wp^j)=1}}^{wp^j-1}\theta_3\big(\zeta_{wp^j}^{v}q^{\frac{u}{w}p^{\ell-2j}}\big)=\frac{\prod_{j=0}^{\ell}\prod_{u,w}\prod_{\substack{v=0\\(u,v,w)=1}}^{p^jw-1}\theta_3\big(\zeta_{p^jw}^{v}q^{\frac{u}{w}p^{\ell-2j}}\big)}{\prod_{j=1}^{\ell-1}\prod_{u,w}\prod_{\substack{v=0\\(u,v,w)=1}}^{p^{j-1}w-1}\theta_3\big(\zeta_{p^{j-1}w}^{v}q^{\frac{u}{w}p^{\ell-2j}}\big)}. \tag{40}$$

Now we simplify the products on the right hand side above. Let $j\geq 0$ be a fixed integer. For each $k=0,1,\ldots,p^j-1$, we replace q by $\zeta_{p^j}^{k}q^{p^{\ell-2j}}$ in (38):

$$\prod_{u,w}\prod_{\substack{v=0\\(u,v,w)=1}}^{w-1}\theta_3\big(\zeta_{p^jw}^{p^jv+uk}q^{\frac{u}{w}p^{\ell-2j}}\big)=\theta_3^{\psi(m)}\big(\zeta_{p^j}^{k}q^{p^{\ell-2j}}\big).$$

We take the product on both sides above for k from 0 to p^j-1 and obtain

$$\prod_{k=0}^{p^j-1}\prod_{u,w}\prod_{\substack{v=0\\(u,v,w)=1}}^{w-1}\theta_3\big(\zeta_{p^jw}^{p^jv+uk}q^{\frac{u}{w}p^{\ell-2j}}\big)=\prod_{k=0}^{p^j-1}\theta_3^{\psi(m)}\big(\zeta_{p^j}^{k}q^{p^{\ell-2j}}\big). \tag{41}$$

Then the left hand side in (41) is expressed by

$$\prod_{k=0}^{p^j-1}\prod_{u,w}\prod_{\substack{v=0\\(u,v,w)=1}}^{w-1}\theta_3\big(\zeta_{p^jw}^{p^jv+uk}q^{\frac{u}{w}p^{\ell-2j}}\big)=\prod_{u,w}\prod_{\substack{v=0\\(u,v,w)=1}}^{p^jw-1}\theta_3\big(\zeta_{p^jw}^{v}q^{\frac{u}{w}p^{\ell-2j}}\big). \tag{42}$$

On the other hand, by (35), the right hand side in (41) is expressed by

$$\prod_{k=0}^{p^j-1}\theta_3^{\psi(m)}\big(\zeta_{p^j}^{k}q^{p^{\ell-2j}}\big)=\left(\frac{\theta_3^{b_{j+1}}\big(q^{p^{\ell-j}}\big)}{\theta_3^{b_j}\big(q^{p^{\ell-j+1}}\big)}\right)^{\psi(m)}. \tag{43}$$

Thus, by (42) and (43), we rewrite (41) as follows:

$$\prod_{u,w}\prod_{\substack{v=0\\(u,v,w)=1}}^{p^jw-1}\theta_3\big(\zeta_{p^jw}^{v}q^{\frac{u}{w}p^{\ell-2j}}\big)=\left(\frac{\theta_3^{b_{j+1}}\big(q^{p^{\ell-j}}\big)}{\theta_3^{b_j}\big(q^{p^{\ell-j+1}}\big)}\right)^{\psi(m)}.$$

Taking the product on both sides for j from 0 to ℓ, we have

$$\prod_{j=0}^{\ell}\Bigg(\prod_{u,w}\prod_{\substack{v=0\\(u,v,w)=1}}^{p^jw-1}\theta_3\big(\zeta_{p^jw}^{v}q^{\frac{u}{w}p^{\ell-2j}}\big)\Bigg)=\theta_3^{b_{\ell+1}\psi(m)}(q). \tag{44}$$

Similarly, by replacing q by $\zeta_{p^{j-1}}^{k} q^{p^{\ell-2j}}$ in (38) and proceeding by the same arguments as above, we get

$$\prod_{j=1}^{\ell-1}\Bigg(\prod_{u,w}\prod_{\substack{v=0\\(u,v,w)=1}}^{p^j w-1} \theta_3\big(\zeta_{p^j w}^{v} q^{\frac{u}{w}p^{\ell-2j}}\big)\Bigg) = \theta_3^{b_{\ell-1}\psi(m)}(q). \tag{45}$$

Therefore, applying the consequences (44) and (45) to (40), we have

$$\prod_{u,w}\prod_{j=0}^{\ell}\prod_{\substack{v=0\\(up^{\ell-j},v,wp^j)=1}}^{wp^j-1} \theta_3\big(\zeta_{wp^j}^{v} q^{\frac{u}{w}p^{\ell-2j}}\big) = \theta_3^{(b_{\ell+1}-b_{\ell-1})\psi(m)}(q) = \theta_3^{\psi(n)}(q).$$

The proof of Proposition 2.2 is completed.

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FACHHOCHSCHULE FÜR DIE WIRTSCHAFT
UNIVERSITY OF APPLIED SCIENCES
FREUNDALLEE 15
D-30173 HANNOVER
GERMANY
E-mail: carsten.elsner@fhdw.de

HIROSAKI UNIVERSITY
GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY
HIROSAKI 036-8561
JAPAN
E-mail: tachiya@hirosaki-u.ac.jp