MATH. SCAND. 6 (1958), 226–230

NOTE ON A PAPER OF SPARRE ANDERSEN

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1. Several years ago Erik Sparre Andersen [6] gave proofs by induction for the following relations (1) and (2) involving binomial coefficients:

$$(1)\qquad \sum_{k=0}^{\alpha}\binom{x}{k}\binom{-x}{n-k} = \frac{n-\alpha}{n}\binom{x-1}{\alpha}\binom{-x}{n-\alpha}$$
$$= -\frac{\alpha+1}{n}\binom{x}{\alpha+1}\binom{-x-1}{n-\alpha-1}$$
$$= -\frac{x-\alpha}{n}\binom{x}{\alpha}\binom{-x-1}{n-\alpha-1},$$

valid for all integral n and α such that $n \geqq 1$ and $0 \leqq \alpha \leqq n$ and for all real values of x; and

$$(2)\qquad \sum_{k=0}^{\alpha}\binom{x}{k}\binom{1-x}{n-k} = \frac{(n-1)(1-x)-\alpha}{n(n-1)}\binom{x-1}{\alpha}\binom{-x}{n-\alpha-1},$$

valid for all integral n and α such that $n \geqq 2$ and $0 < \alpha \leqq n-1$ and for all real values of x.

It is the purpose of this note to indicate how one is led naturally to the indicated values of the sums in (1) and (2). This will be done by establishing a slightly more general relation.

2. Use will be made of several formulas whose proofs are given in most standard books on finite differences. For example, they may be found in [1], [3], [4], or [5]. These relations are as follows:

$$(3)\qquad \sum_{k=0}^{m}(-1)^k\binom{x}{k} = (-1)^m\binom{x-1}{m} = \prod_{k=1}^{m}\left(1-\frac{x}{k}\right)$$

$$(4)\qquad \sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{x+j}{n} = (-1)^k\binom{x}{n-k}$$

Received May 20, 1958.

$$\binom{x}{k}\binom{k}{j} = \binom{x}{j}\binom{x-j}{k-j} \tag{5}$$

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{1}{z-j} = \frac{(-1)^k}{(z-k)\binom{z}{k}}. \tag{6}$$

3. After these preliminary remarks, the main theorem may be stated.

THEOREM 1. *Let*

$$F(k) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}f(j).$$

Then

$$\sum_{k=0}^{\alpha}(-1)^k\binom{x}{k}F(k) = (-1)^\alpha\binom{x-1}{\alpha}\sum_{j=0}^{\alpha}(-1)^j\binom{\alpha}{j}\frac{x}{x-j}f(j).$$

PROOF.

$$\begin{aligned}\sum_{k=0}^{\alpha}(-1)^k\binom{x}{k}F(k) &= \sum_{k=0}^{\alpha}(-1)^k\binom{x}{k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}f(j)\\ &= \sum_{j=0}^{\alpha}(-1)^jf(j)\sum_{k=j}^{\alpha}(-1)^k\binom{x}{k}\binom{k}{j}\\ &= \sum_{j=0}^{\alpha}(-1)^j\binom{x}{j}f(j)\sum_{k=j}^{\alpha}(-1)^k\binom{x-j}{k-j}\\ &= \sum_{j=0}^{\alpha}\binom{x}{j}f(j)\sum_{k=0}^{\alpha-j}(-1)^k\binom{x-j}{k}\\ &= \sum_{j=0}^{\alpha}\binom{x}{j}f(j)(-1)^{\alpha-j}\binom{x-j-1}{\alpha-j}\\ &= (-1)^\alpha\binom{x-1}{\alpha}\sum_{j=0}^{\alpha}(-1)^j\binom{\alpha}{j}\frac{x}{x-j}f(j),\end{aligned}$$

where successive use was made of (5), (3), and (5). Thus Theorem 1 depends on nothing more complicated than (3).

The value of Theorem 1 lies in this: If one can determine a function f such that F may be evaluated and in fact taken equal to some desirable function, then it may be possible to evaluate the sum occuring in Theorem 1 by means of the transformation there indicated. This will be demonstrated now by proving relation (1).

4. To prove (1), let

$$f(j) = \binom{-x+j}{n}. \tag{7}$$

Then by application of (4) it follows that

$$F(k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{-x+j}{n} = (-1)^k \binom{-x}{n-k}.$$

Therefore Theorem 1 gives

$$\sum_{k=0}^{\alpha} (-1)^k \binom{x}{k} F(k) = \sum_{k=0}^{\alpha} \binom{x}{k} \binom{-x}{n-k}$$

$$= (-1)^\alpha \binom{x-1}{\alpha} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \binom{-x+j}{n} \frac{x}{x-j}$$

$$= -x(-1)^\alpha \binom{x-1}{\alpha} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \binom{-x+j}{n} \frac{1}{-x+j}$$

$$= -\frac{x}{n} (-1)^\alpha \binom{x-1}{\alpha} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \binom{-x-1+j}{n-1}, \quad n \geqq 1$$

$$= -\frac{x}{n} (-1)^\alpha \binom{x-1}{\alpha} (-1)^\alpha \binom{-x-1}{n-1-\alpha}, \qquad \text{by (4)},$$

$$= -\frac{x-\alpha}{n} \binom{x}{\alpha} \binom{-x-1}{n-\alpha-1},$$

so that relation (1) of Sparre Andersen is seen to be valid for $0 \leqq \alpha \leqq n$, $n \geqq 1$, and all real values of x.

5. With only a little more work involved in the calculations one may now obtain relation (2) by taking

$$f(j) = \binom{1-x+j}{n}. \tag{8}$$

By (4) one has

$$F(k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{1-x+j}{n} = (-1)^k \binom{1-x}{n-k},$$

and then by Theorem 1 it follows that

$$\sum_{k=0}^{\alpha} (-1)^k \binom{x}{k} F(k) = \sum_{k=0}^{\alpha} \binom{x}{k} \binom{1-x}{n-k}$$

$$= (-1)^\alpha \binom{x-1}{\alpha} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} \frac{x}{x-j} \binom{1-x+j}{n}.$$

At this step, however, one must break the sum into two parts to proceed readily. This is done by observing that

$$\frac{1}{x-j}\binom{1-x+j}{n} = -\frac{1-x}{n(n-1)}\binom{-1-x+j}{n-2} - \frac{j}{n(n-1)}\binom{-1-x+j}{n-2}.$$

Making this substitution, and simplifying, one finds the expression

$$(-1)^\alpha\binom{x-1}{\alpha}\left\{-\frac{x(1-x)}{n(n-1)}(-1)^\alpha\binom{-x-1}{n-2-\alpha}- \right.$$

$$\left. -\frac{x\alpha}{n(n-1)}\sum_{j=1}^{\alpha}{}'(-1)^j\binom{\alpha-1}{j-1}\binom{-x-1+j}{n-2}\right\}$$

$$= (-1)^\alpha\binom{x-1}{\alpha}\left\{-\frac{x(1-x)}{n(n-1)}(-1)^\alpha\binom{-x-1}{n-2-\alpha}+ \right.$$

$$\left. +\frac{x\alpha}{n(n-1)}(-1)^{\alpha-1}\binom{-x}{n-1-\alpha}\right\}$$

$$= \binom{x-1}{\alpha}\binom{-x}{n-\alpha-1}\frac{x}{n(n-1)}\left\{\frac{(1-x)(n-\alpha-1)}{x}-\alpha\right\}$$

$$= \binom{x-1}{\alpha}\binom{-x}{n-\alpha-1}\frac{(n-1)(1-x)-\alpha}{n(n-1)}.$$

A few of the steps in the above deduction have been omitted for brevity, but the essential steps have been indicated.

6. It is observed that the summing of the series

$$\sum_{k=0}^{\alpha}\binom{x}{k}\binom{z}{n-k}, \tag{9}$$

depends on the possibility of summing

$$(-1)^\alpha\binom{x-1}{\alpha}\sum_{j=0}^{\alpha}(-1)^j\binom{\alpha}{j}\binom{z+j}{n}\frac{x}{x-j}, \tag{10}$$

which by Theorem 1 is the same number. For certain values of the parameters involved, the latter sum may be evaluated, and in any case the transformation provides an alternative form of the sum from which bounds may be obtained if desired.

7. As a final example, let

$$f(j) = \frac{1}{z-j}. \tag{11}$$

It follows in this case that

$$(12)\qquad \sum_{k=0}^{\alpha}\binom{x}{k}\frac{1}{\binom{z}{k}(z-k)} = (-1)^{\alpha}x\binom{x-1}{\alpha}\sum_{j=0}^{\alpha}(-1)^{j}\binom{\alpha}{j}\frac{1}{(x-j)(z-j)}.$$

This last series may be readily evaluated. For example one may apply Theorem III in [2], taking $\varphi \equiv 1$. More general results using formulas given in [5] are possible.

8. The writer would like to pose the problem of finding something about the sum

$$(13)\qquad \sum_{k=0}^{\alpha} A_k(x,\beta)A_{n-k}(y,\beta)\,,$$

where

$$A_k(x,\beta) = \frac{x}{x+\beta k}\binom{x+\beta k}{k}.$$

For some discussion of this type of series, see [7].

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