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REMARKS ON THE CONNECTION BETWEEN INTUITIONISTIC LOGIC AND A CERTAIN CLASS OF LATTICES

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The purpose of this paper is to give clear and simple proofs of two theorems namely 1) the theorem expressing the correspondence between intuitionistic propositional logic, in the sequel denoted by the letter H, and the theory of certain lattices, and 2) the theorem in H that a formula $A \vee B$ can only be correct, if either A is correct or B is correct.

Let $\wedge$, $\vee$, $\rightarrow \neg$, be the connectives of H, namely conjunction, disjunction, implication and negation respectively. Further, let the two fundamental lattice operations be denoted in the usual way as $\cap$ and $\cup$. We will only consider lattices with a minimal element 0, a maximal element 1 and possessing for arbitrary a and b a maximal solution of the relation

$$a \cap x \leqq b \,. \tag{1}$$

This maximal solution shall be denoted $a \supset b$. Then the relation (1) is equivalent to writing

$$x \leqq a \supset b \,. \tag{2}$$

It is useful to notice that $a \supset b = 1$, if and only if $a \leqq b$. I will also make use of the fact that if

$$b \supset (a \supset c) = 1 \,, \tag{3}$$

then

$$(a \cap b) \supset c = 1 \tag{4}$$

and inversely. Indeed (4) means that $a \cap b \leqq c$, whence $b \leqq (a \supset c)$, whence (3). On the other hand (3) yields $b \leqq (a \supset c)$, whence $a \cap b \leqq c$, whence (4).

In a paper [5], published already in 1919, I proved that every finite distributive lattice admits the operation $\supset$ and that inversely every lattice with the operation $\supset$ is distributive. Of cource I did not use the name "lattice" on these structures. This name has come into use much later.

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Now let F be a formula in H and let generally F' denote the lattice formula obtained from F by replacement of $\wedge$, $\vee$, $\to$, by $\cap$, $\cup$, $\supset$ and replacing $\neg p$ by $p \supset 0$. Then the theorem concerning the correspondence between H and the theory of the lattices with the operation $\supset$ is the following:

THEOREM 1. *As often as F is a correct formula in H, F' takes the maximal value* 1 *for arbitrary values of the variables in any lattice with the operation* $\supset$. *Inversely, if F' always takes the value* 1, *then F is a correct formula in H.*

PROOF. Let us assume that F' always is $=1$. We may distribute the formulas of H in equivalence classes by putting A and B in the same class, if and only if

$$(A \to B) \wedge (B \to A)$$

is a correct formula. Then in an obvious way we may define the operations $\wedge$, $\vee$, $\to$ and $\neg$ for the equivalence classes. It is clear that the relation $\to$ furnishes a partial ordering and that the partial ordered classes constitute a lattice having $\wedge$ and $\vee$ as $\cap$ and $\cup$. The correct formulas of H constitute the class which is the maximal element of this lattice. Further, since

$$a \wedge (a \to b) \to b$$

is correct in H, the operation $\supset$ in this lattice is given by the implication $\to$. Now since F' always takes the value 1, it is evident that F must be a correct formula in H.

By the way this method of distributing the formulas of a logical calculus in equivalence classes furnishes a so called adequate matrix, see [2], for the calculus, the equivalence classes being taken as truth values.

Now let F be correct in H. In order to prove that F' always takes the value 1 we may perform an inductive reasoning. As I did already in an earlier paper [6] we may prove first that our statement is true when F is an axiom. Further we may prove that the statement will remain true for formulas F derived by the rules of inference. Taking for example the system of axioms mentioned in [1], p. 82, we have that the corresponding lattice formulas are

1° $p \supset (q \supset p)$,

2° $\big(p \supset (q \supset r)\big) \supset \big((p \supset q) \supset (p \supset r)\big)$,

3° $p \supset \big(q \supset (p \cap q)\big)$,

4° $(p \cap q) \supset p$,

5° $(p \cap q) \supset q$,

6° $p \supset (p \cup q)$,

7° $q \supset (p \cup q)$,

8° $(p \supset r) \supset \big((q \supset r) \supset ((p \cup q) \supset r)\big)$,

9° $(p \supset q) \supset \big((p \supset (q \supset 0)) \supset (p \supset 0)\big)$,

10° $(p \supset 0) \supset (p \supset q)$.

That the expressions 1°, 3°, 4°, 5°, 6°, 7° always take the value 1 is seen so very easily by use of the remarks above, see (1), (2), (3), (4), that this verification can be left to the reader. As to 2° we have that

$$p \cap (p \supset q) \leqq q, \quad p \cap \big(p \supset (q \supset r)\big) \leqq (q \supset r) ,$$

so that

$$p \cap (p \supset q) \cap \big(p \supset (q \supset r)\big) \leqq q \cap (q \supset r) \leqq r .$$

Hence it follows that

$$(p \supset q) \cap \big(p \supset (q \supset r)\big) \supset (p \supset r) = 1$$

and therefore 2° takes the value 1.

Let x be $\leqq (p \supset r) \cap (q \supset r)$. Then

$$(p \cup q) \cap x = (p \cap x) \cup (q \cap x) \leqq \big(p \cap (p \supset r)\big) \cup \big(q \cap (q \supset r)\big) \leqq r ,$$

whence $x \leqq (p \cup q) \supset r$. Thus in particular

$$(p \supset r) \cap (q \supset r) \leqq (p \cup q) \supset r$$

so that 8° has the value 1.

Further

$$p \cap (p \supset q) \cap \big(p \supset (q \supset 0)\big) \leqq q \cap (q \supset 0) \leqq 0 ,$$

whence it follows that 9° is $=1$ (see (3) and (4)).

Since $0 \leqq q$ we have $(p \supset 0) \leqq (p \supset q)$ so that 10° has the value 1.

There are two rules of inference, substitution and modus ponens. It is evident that if a lattice formula F' is always $=1$ for arbitrary values of its variables, then every formula obtained from F' by substitution of formulas instead of its variables will possess the same property. Further, if F' is always $=1$ and $F' \supset G'$ always $=1$, then $G'=1$. This proves our theorem.

Let L be a lattice admitting the operation $\supset$ and let L' be the lattice consisting of L and a new element $1'$ which is greater than all elements of L. It is clear that the values of $a \cap b$ and $a \cup b$, where a and b are $\in L$ will be just the same in L' as in L. Further $a \cap 1' = a$, $a \cup 1' = 1'$. However the value of $a \supset b$ will not always be the same in L' as in L, a and b being elements of L. The value of $a \supset b$ remains unchanged except when $a \leqq b$.

Then the value is 1 in L but $1'$ in L'. I shall now prove a lemma which I shall use later.

LEMMA. *Let F be a lattice formula, its variables having values in L. Then if $F \neq 1$ in L, it will have the same value in L' as in L. If $F = 1$ in L, it will be $= 1$ or $1'$ in L'.*

PROOF. The lemma is of course true when the formula consists of a single letter or in other words when the number of occurring operations $\cap$, $\cup$, $\supset$ is 0. We may therefore use induction on the number of operations. Let first F be of the form $A \cap B$. Here we can according to the hypothesis of induction assume the lemma true for A and B. Then if $A \cap B$ is $\neq 1$ in L, either A or B is $\neq 1$ and remains unchanged by the transition from L to L' with the effect that $A \cap B$ remains unchanged. If $A \cap B = 1$ in L, then $A = 1$, $B = 1$ so that $A = 1$ or $1'$, $B = 1$ or $1'$ in L' and thus $A \cap B = 1$ or $1'$ in L'. Now let F possess the form $A \cup B$. If $A \cup B \neq 1$ in L, then $A \neq 1$, $B \neq 1$ so that they retain their values in L'. Thus $A \cup B$ remains the same in L'. If $A \cup B = 1$ in L, either $A = 1$ or $B = 1$ in L, therefore $= 1$ or $1'$ in L' so that $A \cup B$ is $= 1$ or $1'$ in L'. Finally we must consider the case that F is $A \supset B$. If A is not $\leqq B$ in L, the value of $A \supset B$ in L is $\neq 1$ and remains the same in L', if $A \neq 1$ in L. If $A = 1$ in L, then $A \supset B = B$ in L and remains the same in L' whether A in L' is $= 1$ or $1'$. If $A \leqq B$ in L, $A \supset B = 1$ in L. Now if A and B are $\neq 1$ in L, they remain unchanged in L' so that $A \supset B$ gets the value $1'$. If $A \neq 1$, $B = 1$ in L, we have either $B = 1$ or $= 1'$ in L' with A unchanged which means that $A \supset B = 1'$ in L'. Finally $A = B = 1$ in L yields $A \supset B = 1'$ in L' except when A is changed to $1'$ while B is unchanged. Then $A \supset B = 1$ in L'. Thus our lemma is proved.

I shall now give the promised lattice theoretic proof of the

THEOREM 2. *Let $F_1 \vee F_2$ be a correct formula in H. Then either F_1 is correct or F_2 is correct. The inverse is trivial.*

PROOF. Because of the just proved correspondence between H and the lattices with the operation $\supset$ the theorem can be expressed thus: Let $F_i(x_1, \ldots, x_n)$, $(i = 1, 2)$, be a formula such that a lattice L_i with the operation $\supset$ exists in which we can find elements $a_1^{(i)}, \ldots a_n^{(i)}$ such that $F_i(a_1^{(i)}, \ldots, a_n^{(i)}) \neq 1_i$, 1_i the maximal element of L_i. Then we can find a lattice L and elements $a_1, \ldots, a_n$ of L such that

$$F_1(a_1, \ldots, a_n) \vee F_2(a_1, \ldots, a_n)$$

is $\neq 1$, where 1 is the maximal element of L.

Let L_1 and L_2 be chosen according to the mentioned supposition. Then if $L_3 = L_1 \times L_2$, it is seen that for

$$a_1 = (a_1^{(1)}, a_1^{(2)}), \ldots, a_n = (a_n^{(1)}, a_n^{(2)}),$$

we have

$$F_i(a_1, \ldots, a_n) \neq 1_3, \qquad 1 = 1, 2,$$

where $1_3 = (1_1, 1_2)$ is the maximal element of the product lattice L_3. Indeed

$$F_1(a_1, \ldots, a_n) = \big(F_1(a_1^{(1)}, \ldots, a_n^{(1)}), \quad F_1(a_1^{(2)}, \ldots, a_n^{(2)})\big) \neq 1_3,$$
$$\text{since } F_1(a_1^{(1)}, \ldots, a_n^{(1)}) \neq 1,$$
$$F_2(a_1, \ldots, a_n) = \big(F_2(a_1^{(1)}, \ldots, a_n^{(1)}), \quad F_2(a_1^{(2)}, \ldots, a_n^{(2)})\big) \neq 1_3,$$
$$\text{since } F_2(a_1^{(2)}, \ldots, a_n^{(2)}) \neq 1_2.$$

Now let L be the lattice obtained by adding a still larger element 1 to L_3. According to the lemma above the two formulas F_1 and F_2 will in L retain their values in L_3, that is their values belong to L_3. Then, however, also $F_1 \vee F_2$ has its value in L_3 and is therefore $\neq 1$, q.e.d.

One finds easily that the correct formulas of the classical propositional logic are those corresponding to the lattice formulas which always take the maximal value in the Boolean lattices which constitute a subclass of the lattices having the operation $\supset$. By the way, since the Boolean lattices are powers of the lattice with only the two elements 0 and 1, these formulas are just those which take the value 1 for all choices of the values 0 and 1 for the variables. This yields of course a decision method for the classical formulas. However, the decision problem has also been solved for H. See for example D. Scott [4]. Scott defines an infinite sequence of finite distributive lattices, letting L_1 be the lattice with the two elements 0 and 1 alone and letting L_{n+1} be the lattice obtained by adding a new largest element to L_n^n. Then one can imagine two machines M_1 and M_2 such that M_1 proves the correct formulas of H successively, while M_2 tests the formulas of H by inserting for its variables values from $L_1, L_2, \ldots$. If the formula F is correct, it is proved sooner or later by M_1, whereas, if F is false, this falsehood will be shown some day by M_2. However, a more practicable decision method has been set forth by A. Schmidt [3].

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