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SOME REMARKS ON THE TRIPLE SYSTEMS OF STEINER

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1. A triple system of Steiner over a set of elements is a system of triples such that each pair of elements is contained in one and only one of the triples. In an earlier paper in this journal [2] I mentioned that there is a connection between the problem treated there concerning the distribution of the numbers $1, 2, \ldots, 2n$ in n pairs with differences $1, 2, \ldots, n$ and the problem to construct triple systems of Steiner. In the present paper I shall explain this connection using the results of my previous paper to obtain such triple systems. I shall also give some further methods of construction of such systems.

2. It was proved in [2] that in the cases $n \equiv 0$ or 1 modulo 4 it is possible to distribute the numbers $1, \ldots, 2n$ in n disjoint pairs (a_r, b_r) such that

$$b_r = a_r + r \quad \text{for} \quad r = 1, \ldots, n .$$

Then it is easy to see that the triples

$$(r, a_r + n, b_r + n)$$

yield a distribution of the integers $1, 2, \ldots, 3n$ into n disjoint triples. Indeed, r takes the values $1, 2, \ldots, n$ and $a_r + n$ and $b_r + n$ is a distribution of the integers $n+1, n+2, \ldots, 3n$. This makes it possible to prove the following theorem:

The triples

$$(x, x+r, x+b_r+n)$$

constitute a Steiner triple system built over the set of elements $0, 1, \ldots, 6n$, *when* r *and* x *run through the numbers* $1, 2, \ldots, n$ *and* $0, 1, \ldots, 6n$, *respectively, where all numbers are taken modulo* $6n+1$.

Proof. A pair of integers modulo $6n+1$ can in one and only one way be written as $(z, z+d)$, where $1 \leqq d \leqq 3n$. Now the differences between the numbers in the triple

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(1) $$(x, x+r, x+b_r+n)$$

are

(2) $$(r, a_r+n, b_r+n)\,,$$

and hence there is one and only one r for which d equals one of the differences (2), and with this r one and only one x for which $(z, z+d)$ equals a pair from the triple (1). This proves the theorem.

It was shown in [2] that we can set up a system of pairs (a_r, b_r) for an arbitrary n which is $\equiv 0$ or $1 \pmod 4$. Hence it is possible by the procedure just explained to set up a Steiner triple system for an arbitrary number $6n+1$ of elements, provided that n is $\equiv 0$ or $1 \pmod 4$.

On the other hand I have already shown in my notes added to the second edition of Netto's "Lehrbuch der Combinatorik", how we can construct such triple systems when the number of elements is a product of primes of the form $6n+1$. Indeed, if the number of elements is $6k+1$, my method yields 2^k different Steiner systems [1, p. 330]. It is a regrettable fact that the second edition of Netto's book [1] is very little known so that the work of Viggo Brun and myself printed there is scarcely known.

It is natural to ask if the method shown here could be extended also to the remaining case, $n \equiv 2$ or $3 \pmod 4$. This could be done, if somebody were able to prove the conjecture that the numbers $1, 2, \ldots, 2n-1, 2n+1$ can be distributed into n disjoint pairs (a_r, b_r), $r = 1, \ldots, n$, such that for every r we have $b_r = a_r + r$, provided that $n \equiv 2$ or $3 \pmod 4$. I have investigated a great number of values of n and the conjecture seems to be true. It is easily seen that $n \equiv 2$ or $3 \pmod 4$ is a necessary condition for the existence of such a distribution, but I have not yet been able to set up such a distribution for arbitrary n of this kind. I will not try to pursue this further here in general but only give an example.

For $n = 3$ we have the distribution of 1, 2, 3, 4, 5, 7 in three pairs with differences 1, 2, 3:

$$(1, 2)\ (3, 5)\ (4, 7).$$

Hence we get the Steiner triple system over 19 elements $0, 1, \ldots, 18$ consisting of the following triples

1) All triples $(x, x+1, x+5)$,
2) – – $(x, x+2, x+8)$,
3) – – $(x, x+3, x+10)$,

where x runs through $0, 1, \ldots, 18$ and all numbers are reduced mod 19. The reader will easily verify the correctness of this.

3. I will now show some other simple ways of constructing Steiner triple systems for arbitrary element numbers of the forms $6n+1$ and $6n+3$. I take the number $6n+3$ first, because the construction is simpler in this case. By the way, I already gave a construction in [1, pp. 332–334] but that below is simpler.

Let the set E of $6n+3$ given elements be $0, 1, \ldots, 6n+2$. We may arrange them in three rows, namely

$$\begin{array}{lll} 0, & 1, \ldots, & 2n \\ 2n+1, & 2n+2, \ldots, & 4n+1 \\ 4n+2, & 4n+3, \ldots, & 6n+2\,. \end{array} \tag{3}$$

and we may define a cyclical order of the three rows by saying that the second, third and first row is respectively the next row to the first, second and third one. We say that (x, y) is a vertical pair, if x and y are written in the same vertical column in (3), further that (x, y) is a horizontal pair if x and y occur in te same horizontal row in (3). The remaining pairs are called oblique pairs.

S shall denote the system of triples obtained by taking first all vertical triples

$$(x,\, 2n+1+x,\, 4n+2+x),\quad x = 0, 1, \ldots, 2n\,, \tag{4}$$

and secondly all triples

$$(x, y, z) \quad \text{with} \quad x+y \equiv 2z \pmod{2n+1}\,, \tag{5}$$

where x and y belong to the same horizontal row and z to the next one.

I assert that *S is a Steiner system.*

PROOF. It is immediately clear that every vertical pair is contained in just one of the vertical triples (4). If (x, y) is a horizontal pair, then in the next row there is a unique z such that $x+y \equiv 2z \pmod{2n+1}$, because each row is a complete set of residues modulo $2n+1$. Further $x \not\equiv z$, $y \not\equiv z$ modulo $2n+1$ since $x \not\equiv y$. Hence each horizontal pair occurs in just one of the triples (5). Any oblique pair has the form (x, z), where z is in the next row to that where x occurs and $x \not\equiv z \pmod{2n+1}$. Then y can be uniquely determined in the same row as x such that $x+y \equiv 2z \pmod{2n+1}$ and again $x \not\equiv y$ and $y \not\equiv z$, because $x \not\equiv z \pmod{2n+1}$. Hence also each oblique pair occurs in just one of the triples (5). Thus it is proved that S is a Steiner system.

As an example, let $n=2$. Then, denoting the elements of E by $0, 1, \ldots, 14$, T turns out to be the following system:

1,4,5	0,2,6	0,4,7	0,1,8	0,3,9	0,5,10
2,3,5	3,4,6	1,3,7	2,4,8	1,2,9	1,6,11
6,9,10	5,7,11	5,9,12	5,6,13	5,8,14	2,7,12
7,8,10	8,9,11	6,8,12	7,9,13	6,7,14	3,8,13
0,11,14	1,10,12	2,10,14	3,10,11	4,10,13	4,9,14
0,12,13	1,13,14	2,11,13	3,12,14	4,11,12	

4. After this I shall show how to construct a Steiner system when the set E of given elements contains $6n+1$ elements. Let these be $0, 1, \ldots, 6n$, and let us write them in three rows

$$\begin{array}{c} 0,\ 1,\ \ldots,\ 2n-1 \\ 2n,\ \ldots,\ 4n-1 \\ 4n,\ \ldots,\ 6n-1 \end{array} \tag{6}$$

followed by the last element $6n$. I will use the same cyclical order of the three rows as before and the denotations vertical, horizontal and oblique pair shall retain their meaning. In each row in (6) the numbers $\equiv 0, 1, \ldots, n-1$ modulo $2n$ constitute the left half of the row, while the numbers $\equiv n, n+1, \ldots, 2n-1$ constitue the right half of it.

Now let S here be the system of triples obtained by taking first the vertical triples from the left halves of the rows

$$(x,\ 2n+x,\ 4n+x),\quad x = 0, 1, \ldots, n-1\ , \tag{7}$$

thereafter all triples

$$\left.\begin{array}{l} (x+n,\ \ x+2n,\ 6n) \\ (x+3n,\ x+4n,\ 6n) \\ (x+5n,\ \quad x\quad,\ 6n) \end{array}\right\}\ x = 0, 1, \ldots, n-1\ , \tag{8}$$

and finally the triples (x, y, z), where z belongs to the next row to that, where x and y occur, while

$$\begin{array}{l} x+y \equiv 2z \pmod{2n},\ z \text{ in the left half of its row, if } x+y \text{ is even,} \\ x+y \equiv 2z+1 \pmod{2n},\ z \text{ in the right half of its row, if } x+y \text{ is odd.} \end{array} \tag{9}$$

I assert that *S is a Steiner system.*

PROOF. It is seen at once that every pair $(u, 6n)$ is contained in one and only one of the triples (8) and elsewhere it does not occur. It remains to prove that every pair of the numbers $0, 1, \ldots, 6n-1$ occurs in one and only one of the triples in S.

Each vertical pair from the left halfs of the rows in (6) is obviously contained in just one of the vertical triples (7). Any other vertical pair has the form (x, z), where x is in the right half of its row, while $z \equiv x \pmod{2n}$ is in the next row. This pair occurs in the triple

$$(x, y, z) \text{ with } y \equiv x+1 \pmod{2n} \text{ in the same row as } x\,,$$

which is one of the triples (9).

To any horizontal pair (x, y) a unique z can be found in the next row to that containing x and y so that modulo $2n$ we have $x+y \equiv 2z$, z in the left half of its row, resp. $x+y \equiv 2z+1$, z in the right half, according as $x+y$ is even or odd. Therefore (x, y) is contained in just one of the triples (9). It does not occur anywhere else.

Every oblique pair is of the form (x, z), where x belongs to a certain row and z to the next one, while $x \not\equiv z$ modulo $2n$. If in particular $z \equiv x \pmod{n}$, the pair occurs in one and only one of the triples (8). If $z \not\equiv x \pmod{n}$, we can, if $z < n$, determine y in a unique way so that $x+y$ is even and $\equiv 2z \pmod{2n}$, and if $z \geqq n$, determine y so that $x+y$ becomes odd and $\equiv 2z+1 \pmod{2n}$. Further $y \not\equiv x \pmod{2n}$ since $z \not\equiv x \pmod{n}$. Thus (x, z) is contained in just one of the triples (9). This completes the proof that S is a Steiner system.

As an example, let $n=4$. The elements are 0, 1, . . ., 24. The system S is:

0,8,16	1,9,17	2,10,18	3,11,19
1,7,8	0,2,9	0,4,10	0,6,11
2,6,8	3,7,9	1,3,10	1,5,11
3,5,8	4,6,9	5,7,10	2,4,11
9,15,16	8,10,17	8,12,18	8,14,19
10,14,16	11,15,17	9,11,18	9,13,19
11,13,16	12,14,17	13,15,18	10,12,19
0,1,12	0,3,13	0,5,14	0,7,15
2,7,12	1,2,13	1,4,14	1,6,15
3,6,12	4,7,13	2,3,14	2,5,15
4,5,12	5,6,13	6,7,14	3,4,15
8,9,20	8,11,21	8,13,22	8,15,23
10,15,20	9,10,21	9,12,22	9,14,23
11,14,20	12,15,21	10,11,22	10,13,23
12,13,20	13,14,21	14,15,22	11,12,23
4,16,17	5,16,19	6,16,21	7,16,23
4,18,23	5,17,18	6,17,20	7,17,22
4,19,22	5,20,23	6,18,19	7,18,21
4,20,21	5,21,22	6,22,23	7,19,20

0,17,23	1,16,18	2,16,20	3,16,22
0,18,22	1,19,23	2,17,19	3,17,21
0,19,21	1,20,22	2,21,23	3,18,20
4,8,24	5,9,24	6,10,24	7,11,24
12,16,24	13,17,24	14,18,24	15,19,24
0,20,24	1,21,24	2,22,24	3,23,24

5. In the case of an odd n, I have found also another procedure leading to a different triple system.

Let S_r, $r=0, 1, \ldots, 2n-1$, denote the set of all pairs (x, y) (where $x \not\equiv y \pmod{2n}$) such that $x+y \equiv r \pmod{2n}$, but omitting for $r=4s+1$ the pair $(2s, 2s+1)$ and for $r=4s+3$ the pair $(n+2s+1.\ n+2s+2)$. Then every S_r contains just $n-1$ pairs. The omitted pairs are $(2x, 2x+1)$ for $x=0, 1, \ldots, n-1$. Let us call them o-pairs. Obviously they are disjoint. The numbers occurring in the pairs of S_r for even r are all numbers $0, 1, \ldots, 2n-1$ with the exception of the two a and b which satisfy the congruence $2x \equiv r \pmod{2n}$. Since n is odd, a is $\not\equiv b \pmod 2$.

Now, for even r let S_r' be the result of performing the cyclic permutation

$$(n, 1, n+2, 3, \ldots, n-2, 2n-1)$$

in S_r. Since only odd numbers are permuted, the union of the S_r' with even r is just the same as the union of all S_r with even r. Further it is clear that the S_r' are mutually disjoint. Letting S_r', for odd r, be $=S_r$ we now have a sequence of disjoint systems of pairs

$$S_0', S_1', \ldots, S_{2n-1}',$$

all of them containing just $n-1$ pairs of the numbers $0, 1, \ldots, 2n-1$. Further it is seen that in the pairs of S_{2r}' and S_{2r+1}' all the $2n$ numbers occur with exception of the same o-pair. Now we change the enumeration of the systems by putting $\Sigma_s = S_r'$ when s is the odd number such that

$$2s \equiv r+1 \pmod{2n}$$

for odd r, but s the even number such that

$$2s \equiv r \pmod{2n}$$

for even r. Then taking again the $6n+1$ elements of E to be the element e together with the pairs (x, r) for $x=0, 1, \ldots, 2n-1$ and $r=0, 1, 2$ we build the following system T of triples:

T shall contain all triples

$$(10) \quad \big(e, (2x, r), (2x+1, r)\big), \qquad x = 0, 1, \ldots, n-1, \quad r = 0, 1, 2 .$$

Further it shall contain all triples

$$\big((x, r), (y, r), (z, r+1)\big), \tag{11}$$

where $(x, y) \in \Sigma_z$. Finally T shall contain for $x = 0, 1, \ldots, n-1$ the triples

$$\begin{matrix} \big((2x, 0), (2x, 1), (2x, 2)\big) & \big((2x, 0), (2x+1, 1), (2x+1, 2)\big) \\ \big((2x+1, 0), (2x, 1), (2x+1, 2)\big) & \big((2x+1, 0), (2x+1, 1), (2x, 2)\big) \end{matrix} \tag{12}$$

The proof of the statement that *T is a Steiner system* is fairly easy when one first notices that the numbers which do not occur in the pairs of Σ_{2r} as well as in the pairs of Σ_{2r+1} are just $2r$ and $2r+1$. For brevity I omit the proof and confine myself to give an example.

For $n = 5$ the elements are $0, 1, \ldots, 30$, putting $(x, r) = x + 2nr$ and $e = 6n$, and we get the following system T:

(13)

0,1,30	2,3,30	4,5,30	6,7,30	8,9,30
10,11,30	12,13,30	14,15,30	16,17,30	18,19,30
20,21,30	22,23,30	24,25,30	26,27,30	28,29,30

(14)

5,7,10	2,9,11	0,4,12	0,5,13	0,8,14	0,9,15	0,2,16	0,3,17
2,8,10	3,8,11	7,9,12	1,4,13	3,7,14	1,8,15	5,9,16	1,2,17
3,9,10	4,7,11	1,5,12	6,9,13	2,6,14	2,7,15	4,8,16	4,9,17
4,6,10	5,6,11	6,8,12	7,8,13	1,9,14	3,6,15	1,3,16	5,8,17

0,6,18	0,7,19
1,7,18	1,6,19
2,4,18	2,5,19
3,5,18	3,4,19

15,17,20	12,19,21	10,14,22	10,15,23	10,18,24	10,19,25	10,12,26	10,13,27
12,18,20	13,18,21	17,19,22	11,14,23	13,17,24	11,18,25	15,19,26	11,12,27
13,19,20	14,17,21	11,15,22	16,19,23	12,16,24	12,17,25	14,18,26	14,19,27
14,16,20	15,16,21	16,18,22	17,18,23	11,19,24	13,16,25	11,13,26	15,18,27

10,16,28	10,17,29
11,17,28	11,16,29
12,14,28	12,15,29
13,15,28	13,14,29

0,25,27	1,22,29	2,20,24	3,20,25	4,20,28	5,20,29	6,20,22	7,20,23	8,20,26	9,20,27
0,22,28	1,23,28	2,27,29	3,21,24	4,23,27	5,21,28	6,25,29	7,21,22	8,21,27	9,21,26
0,23,29	1,24,27	2,21,25	3,26,29	4,22,26	5,22,27	6,24,28	7,24,29	8,22,24	9,22,25
0,24,26	1,25,26	2,26,28	3,27,28	4,21,29	5,23,26	6,21,23	7,25,28	8,23,25	9,23,24

(15)

0,10,20	0,11,21	5,14,25	5,15,24
1,10,21	1,11,20	6,16,26	6,17,27
2,12,22	2,13,23	7,16,27	7,17,26
3,12,23	3,13,22	8,18,28	8,19,29
4,14,24	4,15,25	9,18,29	9,19,28

I have known these constructions for some time, but I did not publish them before, because I thought they were probably not new, since they are rather simple. However, quite recently I read in the "Abstracts of short communications etc." from the International Congress of Mathematicians in Edinburgh, 1958, that H. Hanani had found a construction of Steiner triple systems for arbitrary numbers $6n+1$ and $6n+3$. I do not know Hanani's method but it appeared to me that I ought to publish my procedure.

6. I would like to add the remark that it is possible in a simple way to build a Steiner triple system S_{mn} over mn elements if we have such a system S_m over a set E_m of m elements and a system S_n over a set E_n of n elements. Let generally $\bar{S}_l$ denote the system of triples obtained by adding to a Steiner system S_l all triples (x, x, x), where x runs through the l elements. Further let $\bar{S}_m \times \bar{S}_n$ denote the set of triples of the form

$$\big((a, \alpha), (b, \beta), (c, \gamma)\big),$$

where $(a, b, c) \in \bar{S}_m$ and $(\alpha, \beta, \gamma) \in \bar{S}_n$. Then it is easily seen that if we again remove from $\bar{S}_m \times \bar{S}_n$ all triples in which $a=b=c$ and $\alpha=\beta=\gamma$, we get a Steiner system over the mn elements (x, ξ), where $x \in E_m$ and $\xi \in E_n$. I proved already a certain generalization of this in [1]. I also proved a generalization of the procedure to build a Steiner triplesystem over $2n+1$ objects having such a system over n objects. Letting an m-n system of N objects mean a set of n-tuples of these objects such that every m-tuple is contained in just one of the n-tuples, I showed how to construct a 2-ν system over $(\nu-1)n+1$ elements if we have such a system over n elements. I also gave in [1] some examples of m-n systems with either $m \neq 2$ or $n \neq 3$.

A more difficult problem, however, will be to find out how many essentially different triple systems (and more generally m-n systems) exist over a given number of elements. Two systems are called essentially different if they cannot be transformed into each other by a permutation of the elements. It was already mentioned in the first edition of Netto's book that 2 essentially different 2-3 systems exist over 13 objects and more than one over 15 objects as well, but on the whole very little is known in this direction.

REFERENCES

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