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ON THE KERNEL OF LINDELÖF'S REPRESENTATION OF ANALYTIC PROLONGATION

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1. For every $\alpha > 0$, the series

$$(1) \qquad L_\alpha(z) = \sum_{n=0}^{\infty} n^{-\alpha n} z^n$$

(where 0^0 denotes $\lim \varepsilon^\varepsilon = 1$) defines an entire function (of order α^{-1}) of z. If $0 < \alpha < 2$, then an application of the residue theorem shows that, for every z outside the infinite sector $|\arg z| \leqq \frac{1}{2}\pi\alpha$,

$$(2) \qquad L_\alpha(z) = 1 + \int_{c-i\infty}^{c+i\infty} \frac{(zw^{-\alpha})^w}{1-e^{2\pi i w}}\, dw ,$$

where c is any constant satisfying $0 < c < 1$ (the first term on the right of (2) is

$$(3) \qquad L_\alpha(0) = 1 ,$$

by the preceding parenthetical proviso). In the limiting case $\alpha = 0$, the integral on the left of (2) is convergent, and the sum on the left of (2) acquires the value $(1-z)^{-1}$, at every z not contained in the half-line $0 \leqq z < \infty$, the limit of the sector excluded in (2) if $0 < \alpha < 2$.

From these two facts, it was concluded by Lindelöf ([4]; reproduced in [5, 122–124] and in [2, 77–79]) that, as $\alpha \to 0$, the entire function $L_\alpha(z)$ tends to $(1-z)^{-1}$ uniformly on every compact z-set which contains no point of the half-line $0 \leqq z < \infty$. Since $(1-z)^{-1}$ is the kernel of Cauchy's integral formula, this enabled Lindelöf to conclude that (2), with $\alpha \to 0$, can serve the same purpose as Mittag-Leffler's E-function, leading to the following result (cf. loc. cit.): If

$$f(z) = c_0 + c_1 z + \ldots$$

is a power series possessing a non-vanishing radius of convergence, then, for every $\alpha > 0$,

$$L_\alpha(z; f) = \sum_{n=0}^{\infty} c_n n^{-\alpha n} z^n$$

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is an entire function having the property that

$$L_\alpha(z;f) \to f(z), \qquad \text{where} \quad \alpha \to 0\,,$$

holds uniformly on every compact subset of the principal star of the analytic function defined by the function element $f(z)=L_0(z;f)$ about $z=0$.

2. In what follows, there will be deduced for the entire function (1), or rather for the entire function $L^*_\alpha(z)$ defined by

$$zL^*_\alpha(z) = L_\alpha(z)-1 \tag{4}$$

(cf. (3)), the *existence* of an integral representation quite different from the explicit formula (2). Whereas (2) assumes $0<\alpha<2$ and excludes the sector $|\arg z| \leqq \frac{1}{2}\pi\alpha$ for every fixed α, the result will assume

$$0 < \alpha < 1 \tag{5}$$

and will be valid on the entire z-plane. Actually, the result to be obtained can also be formulated, *without* any reference to an integral representation, as follows:

(i) *If* α *is an index satisfying* (5) *and if* $L^*_\alpha(z)$ *is the entire function defined by* (4) *and* (1), *that is, by*

$$L^*_\alpha(z) = \sum_{n=1}^{\infty} n^{-\alpha n} z^{n-1}\,, \tag{6}$$

then the function $L^*_\alpha(x)$ *and all of its derivatives are positive at every point of the real axis*

$$-\infty < x < \infty\,. \tag{7}$$

It will be easy to conclude that this holds in the limiting case $\alpha=1$ also:

(i bis) *The entire function*

$$L^*_1(z) = \sum_{n=1}^{\infty} n^{-n} z^{n-1} \tag{8}$$

and all of its derivatives are positive for real z.

The point in (i) or (i bis) is, of course, the inclusion of the lower half, $-\infty<x<0$, of the line (7), since on the upper half of (7) the situation is trivial from the positivity of the coefficients of (6), even if (5) or $\alpha=1$ is replaced by $\alpha>1$.

3. If D denotes d/dx, then (i) is equivalent to the statement that, under the assumption (5),

$$(9) \qquad (-D)^n L^*_\alpha(-x) > 0, \qquad \text{where} \quad n = 0, 1, \ldots ,$$

holds on the line (7) and therefore on the half-line $0 < x < \infty$ (this time $-\infty < x \leqq 0$ is the trivial range). Since this means that $L^*_\alpha(-x)$ is totally monotone for $0 \leqq x < \infty$, it follows from the Hausdorff-Bernstein theorem (cf. [2, 281]) that there exists on the half-line $0 \leqq t < \infty$ a certain monotone function $\mu(t) = \mu_\alpha(t)$ in terms of which a Laplace-Stieltjes representation

$$L^*_\alpha(-x) = \int_0^\infty e^{-xt} d\mu_\alpha(t), \qquad \text{where} \quad d\mu_\alpha(t) \geqq 0 ,$$

holds for $0 < x < \infty$. But this representation must hold on the entire axis (7). This follows from Landau's extension (to Laplace-Stieltjes integrals; cf. [3, pp. 88–89]) of the Vivanti-Pringsheim theorem (for power series), simply because (6), hence $L^*_\alpha(-z)$ as well, is an entire function. Consequently,

$$(10) \qquad L^*_\alpha(z) = \int_0^\infty e^{zt} d\mu_\alpha(t), \qquad \text{where} \quad d\mu_\alpha(t) \geqq 0 ,$$

must hold (in the sense of convergence) on the entire z-plane.

Conversely, if (10) is granted, then it is clear that $L^*_\alpha(x)$ and all of its derivatives are positive at every point of the line (7), simply because $\mu_\alpha(t)$ cannot be a constant (since (8) is not). Accordingly, (i) is equivalent to the following assertion:

(ii) *Corresponding to every index* α *satisfying* (5), *there exists on the half-line* $0 \leqq t < \infty$ *a monotone function* $\mu_\alpha(t)$ *in terms of which the Laplace representation* (10) *of the entire function* (6) *is valid for all* z.

Similarly, (i bis) can be rephrased as follows:

(ii bis) *The assertion of* (ii) *remains true in the limiting case* $\alpha = 1$.

I was unable to deduce from (4) and from Lindelöf's integral (2) the assertion (10) of (ii). If the line

$$w = c + it$$

iwhich, with $-\infty < t < \infty$ and with any positive constant $c < 1$, is the (ntegration path of (2)) is chosen to be

$$w = \tfrac{1}{2} + it ,$$

then (2) and (4) lead to a Fourier integral in $\log z$. But it is hard to see how what thus results (for $0<\alpha<2$) can be transformed into a Laplace representation (10) if $z=-x$ (and $0<\alpha<1$). The emphasis lies, of course, on the assertion $d\mu_\alpha(t)\geqq 0$ of (10).

4. For these reasons, the proof will be made to depend, not on a residue formula like (2), but on a fact which lies entirely within the real field. It is the following fact, derived in [7, § 10] (from another result): For every index satisfying (5), the Stieltjes moment problem

$$\int_0^\infty u^n d\varphi_\alpha(u) = \Gamma(n+1)/\Gamma(\alpha n+1), \qquad \text{where} \quad n = 0, 1, \ldots, \tag{11}$$

has a solution $d\varphi_\alpha(u)\geqq 0$ (the same is true, but trivial, in the limiting case $\alpha=1$, since (11) is then satisfied by the function $\varphi_1(u)$ which is 0 or 1 according as $0\leqq u<1$ or $1\leqq u<\infty$).

Let this fact be combined with the relation which results if β is chosen to be $\alpha n+1$, and s is replaced by nv, in

$$\Gamma(\beta) = \int_0^\infty e^{-s}s^{\beta-1}ds$$

(the substitution $s=nv$ *is not allowed if* $n=0$). Since this leads to

$$\Gamma(\alpha n+1) = n^{1+\alpha n}\int_0^\infty (e^{-v}v^\alpha)^n dv\ ,$$

it follows from (11) that, if $n=1, 2, \ldots$,

$$(n-1)!\,n^{-\alpha n} = \int_0^\infty\int_0^\infty (ue^{-v}v^\alpha)^n d\varphi_\alpha(u)dv\ . \tag{12}$$

5. Since $d\varphi_\alpha(u)\geqq 0$, it follows from Fubini's theorem on product measures that the integral (12) can be rearranged into

$$\int_0^\infty t^n d\lambda_\alpha(t)\ ,$$

if $d\lambda_\alpha(t)$ is defined for $0\leqq t<\infty$ by

$$d\lambda_\alpha(t) = \int_{C_t} d\varphi_\alpha(u)dv, \qquad \text{where} \qquad C_t\colon\ ue^{-v}v^\alpha = t \tag{13}$$

(the integration extends in (13) over those points (u, v) of the quadrant $(0 \leqq u < \infty, 0 \leqq v < \infty)$ which are situated on the analytic curve or curves $ue^{-v}v^{\alpha} = t$, defined by a fixed t). It follows from (12) that there exists on the half-line $0 \leqq t < \infty$ a monotone function $\lambda_{\alpha}(t)$ satisfying

$$(14) \quad n^{-n\alpha} = \int_0^{\infty} t^n d\lambda_{\alpha}(t)/(n-1)!, \qquad \text{where } d\lambda_{\alpha}(t) \geqq 0 \quad (n = 1, 2, \ldots).$$

If the (monotone) function $\mu_{\alpha}(t)$ is defined for $0 \leqq t < \infty$ by

$$(15) \qquad \mu_{\alpha}(t) = \int_0^t s\, d\lambda_{\alpha}(s), \qquad d\mu_{\alpha}(t) \geqq 0,$$

then the integral which is the numerator on the right of (14) is the $(n-1)$-st Stieltjes moment of $\mu_{\alpha}(t)$. Hence the definition (6) can be written in the form

$$(16) \qquad L^*_{\alpha}(z) = \sum_{m=0}^{\infty} z^m \int_0^{\infty} t^m d\mu_{\alpha}(t)/m!$$

($m = n-1$ runs here through all non-negative integers, that is, $m = 0$ *is included*).

Clearly, (10) follows from (16) by term-by-term integration. The latter is readily justified from $d\mu_{\alpha}(t) \geqq 0$ and from the circumstance that z can be assumed to be positive.

This proves (ii), hence (i) as well.

6. In order to prove (ii bis), it is sufficient to ascertain that (i bis) follows from (i) or (ii). To this end, let $\alpha \to 1$. Then it is clear from (6) that

$$L^*_{\alpha}(z) \to L^*_1(z)$$

holds uniformly on every fixed z-circle, which implies that

$$D^n L^*_{\alpha}(z) \to D^n L^*_1(z)$$

holds for every fixed n. It follows therefore from (i) that

$$D^n L^*_1(x) \geqq 0 \qquad \text{on} \quad (7).$$

But this means that $L^*_1(-x)$ is totally monotone and so, by the Hausdorff-Bernstein theorem, there must exist a $d\mu_1(t) \geqq 0$ satisfying the case $\alpha = 1$ of (10). Finally, since (8) is not a constant, it follows from the case $\alpha = 1$ of (10) that the sign of equality cannot hold in $D^n L^*_1(x) \geqq 0$. This proves (i bis).

7. The end of the proof of (ii) depended on the fact that, if $\alpha(>0)$ is arbitrary, (10) is equivalent to (14) by virtue of (15); cf. (6) and (16). Hence, (ii) and (ii bis) together are equivalent to the first of the following assertions:

(iii) *If* $0<\alpha\leqq 1$, *then the Stieltjes moment problem*

$$\text{(17)} \qquad \int_0^\infty t^{n-1}d\mu_\alpha(t) = (n-1)!\,n^{-n\alpha}, \qquad \text{where } n = 1, 2, \dots ,$$

has a solution $d\mu_\alpha(t)\geqq 0$. *In addition,* (17) *is a determined moment problem. If* $\alpha=1$, *then* (17) *reduces to*

$$\text{(17 bis)} \qquad \int_0^{1/e} t^{n-1}d\mu_1(t) = (n-1)!\,n^{-n}, \qquad \text{where } n = 1, 2, \dots$$

(*and becomes therefore a Hausdorff moment problem, since* $e=2.71\ldots>1$), *whereas if* $0<\alpha<1$, *then* $d\mu_\alpha(t)\equiv 0$ *cannot hold for large* t.

By Stirling's formula, the reciprocal value of the n-th root of the moment (17) is asymptotically proportional to $n^{\alpha-1}$. It follows therefore from theorems II and I of Carleman [1, pp. 80–81] that not only the Stieltjes problem (17) but also the corresponding Hamburger problem is a determined moment problem if $0<\alpha\leqq 1$. The remaining assertions of (iii) follow from the circumstance that, as $n\to\infty$, the n-th root of the moment (17) tends to ∞ or $1/e$ according as $0<\alpha<1$ or $\alpha=1$.

If (15) and (13) are applied to $\alpha=1$, then the exceptional standing of the limiting case (17 bis) of (17) becomes understandable from the parenthetical remark made after (11).

8. Somewhat more than the first assertion of (iii) is contained in the following result concerning Mellin transforms:

(iv) *On the half-plane* $\mathrm{Re}\, z>0$, *the branch of the analytic function* $\Gamma(z)z^{-z}$ *which is positive for* $z>0$ *is the bilateral Laplace transform of a non-negative mass-distribution, and the same is true for* $\Gamma(z)z^{-\alpha z}$ *if* α *is restricted by* (5).

In other words, there belongs to every positive $\alpha\leqq 1$ a function $\psi_\alpha(u)$ which is monotone for $-\infty<u<\infty$ and has the property that

$$\text{(18)} \qquad \Gamma(z)z^{-\alpha z} = \int_{-\infty}^{\infty} e^{-zu}d\psi_\alpha(u), \qquad \text{where } d\psi_\alpha(u) \geqq 0 ,$$

holds on the half-plane $\operatorname{Re} z > 0$. In order to prove this, recourse must be had to the relation (14) which, in view of (15), contains somewhat more than (17).

First, (14) means that

$$\Gamma(z) z^{-\alpha z} = \int_0^\infty t^z d\lambda_\alpha(t), \qquad \text{where } d\lambda_\alpha(t) \geqq 0\,, \tag{19}$$

holds for $z = 1, 2, \ldots$. But the moment problem involved is a determined moment problem; cf. (iii). Hence, for reasons of analyticity, (19) must hold on the half-plane $\operatorname{Re} z > 1$. Actually, since the function on the left of (19) is regular for $\operatorname{Re} z > 0$, it follows from $d\lambda_\alpha(t) \geqq 0$ and from Landau's theorem, cited above, that (19) is valid (by convergence) for $\operatorname{Re} z > 0$. The assertion (18) of (iv) now follows by placing $t = e^{-u}$ in (19).

There is little doubt that residue methods, as applied in Mellin's work (cf. [6]), lead to some explicit determination of the $d\lambda_\alpha(t)$ or $d\psi_\alpha(t)$ in (19) or (18), where $0 < \alpha \leqq 1$.

9. According to the last part of (iii), the situation undergoes a drastic change when (5) is replaced by $\alpha = 1$. This indicates that the restriction $\alpha \leqq 1$, made above, is not merely *ad hoc*; in other words, that the following negation holds:

The assertions of (i)-(i bis), (ii)-(ii bis), (iii) *and* (iv) *are false for every* $\alpha > 1$.

In order to prove this, it is sufficient to show that for no $\alpha > 1$ can there exist on the half-line $0 \leqq t < \infty$ a monotone function $\mu_\alpha(t)$ satisfying (17). But if $\alpha > 1$, then it is clear from Stirling's formula that there belongs to every $\varepsilon > 0$ an N having the property that the n-th moment (17) is less than ε whenever $n > N$. Since this and $d\mu_\alpha(t) \geqq 0$ imply the identical vanishing of $\mu_\alpha(t)$, there results from (10) the contradiction that (6) vanishes identically.

NOTE. In refereeing this posthumous paper, L. Carleson has made the following remarks:

1° The assertion (iii) in section 7 is a consequence of the representation

$$\sum_{n=0}^\infty \frac{x^n}{\Gamma(n\alpha + 1)} = \int_0^\infty e^{ux} d\varphi_\alpha(u), \qquad d\varphi_\alpha(u) \geqq 0\,, \tag{P}$$

proved by Pollard in Bull. Amer. Math. Soc. 54 (1948), p. 1115. Differentiating (P) n times and putting $x = 0$ one finds (11) and the ensuing consequences. Assertions (i) and (i bis) in section 2 follow from (iii) on multiplication by $z^{n-1}/(n-1)!$ and addition.

2° The result (iv) in section 8 is connected with the fact that $\Gamma(z)z^{-\alpha z}$ $(\alpha \geqq 1)$ does not increase too fast with $\operatorname{Re} z$. The statement is correct, but it is not quite clear how it would follow directly from the determinateness of the moment problem.

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