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EXTENSIONS OF LIOUVILLE'S THEOREM TO n DIMENSIONS

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1. Liouville's Theorem states that any bounded function analytic in the entire plane is constant. I wish to give here two theorems concerning differentiable vector maps of regions in n-space, E^n, which generalize this result. These theorems are:

THEOREM A. *Let $f: E^n \to E^n$ be a vector map of class C', and having a non-negative Jacobian, $J(f)$. Suppose $\lim_{p\to\infty} f(p)$ exists. Then $J(f)$ is identically* 0.

THEOREM B. *Let D be a domain in E^n, and let $f: \bar{D} \to E^n$ be a vector function that is continuous in $\bar{D}$, and in D is of class C', with non-negative Jacobian. Let R be a spherical region lying with its closure in D and of radius r. Suppose that on R the function f is one-to-one and that there $J(f) \geqq k > 0$. Then the oscillation of f on the boundary of D is not less than $2k^{1/n}r$.*

An analytic function has non-negative Jacobian. If the function is bounded at ∞, it has a removable singularity there and so has a limit. Its Jacobian being identically zero implies that so is its derivative. Hence Theorem A reduces to Liouville's theorem in this specialization.

2. Loewner gave in [3] a lemma for maps with non-negative Jacobians in an annular plane region. I give here a formulation and a proof in n dimensions, the essential ideas being in his paper. For the general principles of degree needed here, see [1] or [4].

LEMMA. *Let D be a domain with compact boundary in E^n, and let R be a subdomain of D. Let $f: \bar{D} \to E^n$ be continuous in $\bar{D}$ and of class C' in $D - \bar{R}$. Suppose that at each point of $D - \bar{R}$, the Jacobian of f is non-negative. Then if p is a point of E^n which is neither in $f(\bar{D} - D)$ nor in $f(\bar{R} - R)$,*

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$$\deg(f, p) \geqq \deg(f|R, p) .$$

The symbol $f|R$ stands for the restriction of f to R; that is, f considered on the domain R only.

PROOF. If p is not in $f(D-\bar{R})$, then certainly $\deg(f, p)=\deg(f|R, p)$. Suppose then that p is in $f(D-\bar{R})$. Let K be the set of points of $D-\bar{R}$ at which $J(f)=0$. By a theorem of Sard [5], the n-measure of $f(K)$ is zero, so that every neighborhood of p contains points not in $f(K)$. Let U be a connected open set containing p, but not meeting $f(\bar{R}-R)$ or $f(\bar{D}-D)$. It is a well-known result of degree theory [1, p. 473] that both $\deg(f, x)$ and $\deg(f|R, x)$ are constant on such an open set. Therefore, if there is any point q in U that is not in $f(D-\bar{R})$, we have

$$\deg(f, q) = \deg(f|R, q) = \deg(f, p) .$$

If every point in U is in $f(D-\bar{R})$, some point q of U is not in $f(K)$, so that

$$f^{-1}(q) \cap (D-\bar{R}) \quad \text{is in} \quad (D-\bar{R})-K .$$

The set $f^{-1}(q)\cap(D-\bar{R})$ can have no limit point in $\bar{R}-R$, in $\bar{D}-D$, or in K. Each point x of $f^{-1}(q)\cap(D-\bar{R})$ lies in a neighborhood on which f is a sense-preserving homeomorphism, since $J(f(x))>0$. It follows from these two statements that $f^{-1}(q)\cap(D-\bar{R})$ is finite, and that $\deg(f, q)-\deg(f|R, q)$ is equal to the number of points in $f^{-1}(q)\cap(D-\bar{R})$, which proves the lemma.

It is possible for R and D to be concentric circular regions in E^2, to have f several times differentiable over $\bar{D}$, to have $J(f)\geqq 0$ in D, to have $J(f)=0$ on $D-\bar{R}$, but yet not have $f(D-\bar{R})$ be a subset of $f(\bar{R})$. I do not have a simple expression for such a map, and will not take the space to describe one.

PROOF OF THEOREM B. From the change-of-variables formula, the n-measure of $f(R)$ is not less than kV. The diameter of $f(R)$ is therefore not less than the smallest possible diameter of an open set with volume kV. The sets with n-measure kV with the least diameter are the spherical regions [2, p. 278, Satz XVII][2], and simple proportionality shows that this least diameter is $2k^{1/n}r$. Now suppose that

$$\operatorname{osc}(f, \bar{D}-D) < 2k^{1/n}r .$$

Then there is a point of $f(R)$ in the unbounded component C of $E^n-f(\bar{D}-D)$, since the diameter of $f(\bar{D}-D)$ is equal to the diameter of

[2] I am indebted to Professor G. Polya for this reference.

the union of $f(\bar{D}-D)$ and its bounded complementary domains. The hypothesis on R insures that $\deg(f|R)=1$. Loewner's lemma shows that $\deg(f|D, p) \geqq 1$. We know that $\deg(f, x)$ is constant over C; also that if q is a point for which $\deg(f, q) \neq 0$, then q is in $f(D)$. It follows that C is contained in $f(\bar{D})$. But $f(\bar{D})$ is compact, whereas C is unbounded. This contradiction establishes the theorem.

PROOF OF THEOREM A. If $J(f)$ is not identically 0, there is a spherical region R of radius r on which, for some $k>0$, $J(f) \geqq k$. Let q be $\lim_{p\to\infty} f(p)$, and let U denote a spherical neighborhood of the origin in E^n so large that (i) U contains R; and (ii) for any point p in $E^n - R$,

$$\operatorname{dist}\big(f(p), q\big) < \tfrac{1}{2}k^{1/n}r .$$

Then

$$\operatorname{osc}(f|\bar{U}, \bar{U}-U) \leqq k^{1/n}r ,$$

contradicting Theorem B, applied to U and R.

3. Theorem B may have applications in uniqueness proofs, for functions with prescribed boundary conditions. Suppose that D is a domain in E^n and that f, g are two vector functions each continuous in $\bar{D}$ and of class C' in D, mapping $\bar{D}$ into E^n. Suppose that on $\bar{D}-D$, f and g are identical. If $J(f-g) \geqq 0$ in D, then by Theorem B, $J(f-g) \equiv 0$. If we knew that this implied $f \equiv g$, we would have uniqueness.

It is perfectly possible, however, to have $f-g$ vanish on the boundary, to have $J(f-g) \equiv 0$, but yet have $f \neq g$. For example, in the unit disk in E^2, let

$$f-g = \big(\cos 2\pi(x^2+y^2) - 1,\ \sin 2\pi(x^2+y^2)\big) .$$

4. In [7], Titus and I have announced theorems on mappings with non-negative Jacobians which show that some of these have a maximum-modulus principle. Theorem B does not replace this result. We will give in that paper an example of a function with a non-negative Jacobian which takes its maximum modulus at an interior point only, and which satisfies the hypothesis of Theorem B. Some related work can be found in [6].

There are two generalizations of Theorems A and B possible. The sets may be replaced by open sets of n-dimensional orientable differentiable manifolds, with maps into another one. The proof is the same, except for minor details. The other is to let the mapping f be a map of E^n into E^m requiring that one of the submatrices of greatest rank in $J(f)$ be non-negative.

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