MATH. SCAND. 6 (1958), 293–296

MODELS OF PROPOSITIONAL CALCULI IN RECURSIVE ARITHMETIC

R. L. GOODSTEIN

The familiar model of two-valued logic in recursive arithmetic, in which the *tertium-non-datur* is expressed by the provable equation $x(1 \dot{-} x) = 0$, appears to confer upon the two-valued logic a special validity which disappears when one recognises that the equation $x(1 \dot{-} x) = 0$ proves the *tertium non datur* only with respect to the model in question. The object of this note is to construct within recursive arithmetic models of some finitely and infinitely many valued logics similar in form to the familiar two-valued model.

A model for a (Post) $N+1$ valued logic is exhibited in the following table

proposition	representing function
$p \vee q$	$p \dot{-} (p \dot{-} q)$
$p \mathbin{\&} q$	$p + (q \dot{-} p)$
$\neg p$	$\{1 \dot{-} (1 \dot{-} (N \dot{-} p))\}(p+1)$
$p \rightarrow q$	$(N \dot{-} p) \dot{-} (N \dot{-} q)$

with 0 as the designated value. In this model $p \rightarrow p$ is translated by the provable equation

$$(N \dot{-} p) \dot{-} (N \dot{-} p) \;=\; 0$$

but $p \vee \neg p$ becomes

$$p \dot{-} \{p \dot{-} [1 \dot{-} (1 \dot{-} (N \dot{-} p))](p+1)\}\,,$$

which simplifies to

$$p\{1 \dot{-} (1 \dot{-} (N \dot{-} p))\}\,,$$

taking the value p when $p < N$ and the value 0 for $p \geqq N$.

The disjunction

$$A_r(p) \equiv p \vee \neg p \vee \neg\neg p \vee \ldots \vee \neg\neg \ldots \neg p$$

Received September 8, 1958.

with $r+1$ negations in the last disjunctand, is expressed by the formula

$$p\{1 \dot{-} (1 \dot{-} (N \dot{-} (p+r))\};$$

we prove this by an induction over r.

The formula holds for $r=0$, as we have already observed. Let P denote

$$\{1 \dot{-} (1 \dot{-} (N \dot{-} p))\}(p+1),$$

let Q denote

$$1 \dot{-} \{1 \dot{-} (N \dot{-} (P+r))\}$$

and let

$$\Phi(r, p) = p\{1 \dot{-} (1 \dot{-} (N \dot{-} (p+r)))\};$$

then $\Phi(r, P) = PQ$ and

$$p \dot{-} (p \dot{-} \Phi(r, P)) = p \dot{-} (p \dot{-} PQ).$$

If $p \geqq N$, $P=0$ and so

$$p \dot{-} (p \dot{-} PQ) = 0;$$

if $p<N$ and $Q=0$ then

$$p \dot{-} (p \dot{-} PQ) = 0$$

and if $p<N$ and $Q=1$ then

$$p \dot{-} (p \dot{-} PQ) = p \dot{-} (p \dot{-} (p+1)) = p.$$

However, if $P+r \geqq N$ then $Q=0$ and if $P+r<N$ then $Q=1$, hence if $p<N$ (so that $P=p+1$) and $p+r+1 \geqq N$ then $Q=0$, and if $p+r+1<N$ then $Q=1$, and therefore

$$p \dot{-} (p \dot{-} PQ) = p\{1 \dot{-} (1 \dot{-} (N \dot{-} (p+r+1)))\}.$$

Since $A_{r+1}(p)$ is equivalent to $p \vee A_r(\neg p)$, and P represents $\neg p$, it follows that if $A_r(p)$ is represented by $\Phi(r, p)$ for a certain r, then $A_{r+1}(p)$ is represented by $\Phi(r+1, p)$.

In particular, therefore, the representing function for the disjunction

$$p \vee \neg p \vee \neg \neg p \vee \ldots$$

with $N+1$ disjunctands, and N negations in the last disjunctand, is

$$p\{1 \dot{-} (1 \dot{-} (1 \dot{-} p))\} = p(1 \dot{-} p) = 0.$$

The derivation rule

$$\frac{\begin{array}{c} P \\ P \to Q \end{array}}{Q}$$

in this $N+1$ valued logic is expressed by the schema

$$\frac{\begin{array}{c} P = 0 \\ (N \dot{-} P) \dot{-} (N \dot{-} Q) = 0 \end{array}}{Q = 0}$$

To prove this schema, write $N=M+1=M'$, then consider

$$f(r) = \left[1 \dot- \{M' \dot- (M' \dot- r)\}\right] r\,.$$

We have $f(0)=0$ and if $r'=r+1$

$$\begin{aligned} f(r') &= \left[1 \dot- \{M' \dot- (M \dot- r)\}\right] r' \\ &= \left[1 \dot- \{1 + (M \dot- (M \dot- r))\}\right] r' \\ &= 0 \end{aligned}$$

proving $f(r)=0$. From $P=0$ and $(N \dot- P) \dot- (N \dot- Q)=0$ we derive

$$N \dot- (N \dot- Q) = 0\,,$$

and thence, from $f(Q)=0$, follows $Q=0$.

The schema of mathematical induction

$$\frac{\begin{array}{c} f(0) = 0 \\ f(x) = 0 \underset{N}{\rightarrow} f(x+1) = 0 \end{array}}{f(x) = 0\,,}$$

where

$$' \underset{N}{\rightarrow} '$$

is the implication connective in the $N+1$ valued logic, is expressed in the model by the schema

$$\frac{\begin{array}{c} f(0) = 0 \\ \{N \dot- f(x)\} \dot- \{N \dot- f(x+1)\} = 0 \end{array}}{f(x) = 0}$$

To prove this schema, we observe that from the hypotheses $f(x)=0$ and

$$\{N \dot- f(x)\} \dot- \{N \dot- f(x+1)\} = 0$$

follows $f(x+1)=0$ and so (by the deduction theorem for 2-valued logic) from

$$\{N \dot- f(x)\} \dot- \{N \dot- f(x+1)\} = 0$$

we may derive

$$\{f(x) = 0\} \underset{1}{\rightarrow} \{f(x+1)\} = 0$$

whence by the induction schema for 2-valued logic, we derive $f(x)=0$ from the given hypotheses.

This result shows that, by means of the model, many-valued logics have the same power to reveal mathematical connections as classical logic.

The substitution theorem for $N+1$-valued logic

$$x = y \underset{N}{\rightarrow} \{f(x) = f(y)\}$$

however, fails for $N>1$, since for instance, if $N=M+2$, and $f(x)=2x$,

$$\{N \dot{-} \delta(x, y)\} \dot{-} \{N \dot{-} \delta(f(x), f(y))\}$$

where $\delta(u, v)=(u \dot{-} v)+(v \dot{-} u)$, takes the value

$$(M+1) \dot{-} M = 1$$

when $x=1$, $y=2$. It follows that the deduction theorem, that

$$P \underset{N}{\rightarrow} Q$$

holds if Q is derivable from P (in the model), also fails for $N>1$ since we can certainly derive $f(x)=f(y)$ from $x=y$.

As an example of a system with infinitely many truth values we consider the system LC with truth values

$$0, 1, 2, 3, \ldots, \omega$$

and connectives &, v, $\neg$, $\rightarrow$ having the following truth tables:

$$a \,\&\, b = \max(a, b), \qquad a \vee b = \min(a, b),$$
$$\neg a = \omega, \text{ if } a < \omega, \qquad a \rightarrow b = 0, \text{ if } b \leqq a,$$
$$= 0, \text{ if } a = \omega \qquad = b, \text{ if } b > a.$$

The system LC has recently (in a forthcoming paper in the Journal of Symbolic Logic) been shown by Michael Dummet to be equivalent to the intuitionistic calculus augmented by the axiom

$$(p \rightarrow q) \vee (q \rightarrow p).$$

A model for system LC is given in the following table *with unity as the designated value.*

proposition	representing function
$\neg p$	$1 \dot{-} p$
$p \,\&\, q$	$\{1 \dot{-} (1 \dot{-} p)\}\{1 \dot{-} (1 \dot{-} q)\}\{p+(q \dot{-} p)\}$
$p \vee q$	$\{1 \dot{-} (1 \dot{-} pq)\}\{p \dot{-} (p \dot{-} q)\}+(1 \dot{-} p)q+(1 \dot{-} q)p$
$p \rightarrow q$	$(1 \dot{-} p)+[1 \dot{-} \{(q \dot{-} p)+(1 \dot{-} q)\}]+[1 \dot{-} \{(1 \dot{-} p)+(1 \dot{-} (q \dot{-} p))\}]q$

The schema of mathematical induction is valid with LC-implication but the substitution theorem and deduction theorems are not.

I have been unable to find a model of this kind for the intuitionistic calculus itself.

THE UNIVERSITY, LEICESTER, ENGLAND