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A NEW DERIVATION OF THE INFORMATION FUNCTION

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The purpose of this note is to prove the following

THEOREM. *Let a function, H, satisfy the conditions*

(i) *H is defined for any set of non-negative arguments with sum* 1, *and it is symmetric in all arguments.*

(ii) $H(x_1, x_2, \ldots, x_{n-1}, u, v) = H(x_1, x_2, \ldots, x_n) + x_n H\left(\frac{u}{x_n}, \frac{v}{x_n}\right)$, *whenever all terms of the equation have a meaning.*

(iii) *$H(x, 1-x)$ is integrable, in the sense of Lebesgue, on the interval* $0 \leqq x \leqq 1$.

Then H is determined up to a multiplicative constant.

Weaker forms of this theorem have been proved by Fadiev [1] and Khintchine [2]. They both assume, beside conditions (i) and (ii), the continuity of $H(x, 1-x)$. In addition, Khintchine assumes that

$$H(x_1, x_2, \ldots, x_n) \leqq H\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right).$$

In Shannon and Weaver [3] can be found a simple derivation of the form of H, the assumptions being those of Fadiev, and further that

$$H\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$$

is an increasing function of n.

If my weakening of the conditions is insignificant from an information-theoretic point of view, I do not think that it is so from a purely mathematical one.

The proof of the theorem is direct, by deducing the form of H: Conditions (i) and (ii) give

$$(1) \qquad H(x, u, v) = H(x, u+v) + (u+v) H\left(\frac{u}{u+v}, \frac{v}{u+v}\right)$$

$$= H(u, x+v) + (x+v) H\left(\frac{x}{x+v}, \frac{v}{x+v}\right),$$

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for $0 \leqq x < 1$, $0 \leqq u < 1$, $x+u \leqq 1$. With $f(x) = H(x, 1-x)$, (1) gives the functional equation

$$(2) \qquad f(x) + (1-x) f\left(\frac{u}{1-x}\right) = f(u) + (1-u) f\left(\frac{x}{1-u}\right).$$

Condition (iii) allows us to integrate (2) with respect to u between the limits 0 and $1-x$, and also to perform an appropriate change of variable in two of the integrals. The result is:

$$(3) \qquad (1-x)f(x) + (1-x)^2 \int_0^1 f(t)\,dt = \int_0^{1-x} f(t)\,dt + x^2 \int_x^1 t^{-3} f(t)\,dt.$$

Condition (iii) assures the continuity in x of all terms of this equation, except the first one, for $0 < x < 1$. We conclude that $f(x)$ is continuous, and then, by an analogous argument, differentiable for $0 < x < 1$. Upon differentiation, (3) yields

$$(4) \quad (1-x)f'(x) - f(x) - 2(1-x)\int_0^1 f(t)\,dt = -f(1-x) + 2x \int_x^1 t^{-3} f(t)\,dt - x^{-1} f(x).$$

Note that $f(1-x)$ cancels against $f(x)$, by condition (i). Then (4) shows the existence of $f''(x)$, and by differentiating (4) and then eliminating $\int_x^1 t^{-3} f(t)\,dt$, one gets

$$(5) \qquad f''(x) = -2x^{-1}(1-x)^{-1} \int_0^1 f(t)\,dt,$$

whence

$$(6) \qquad f(x) = ax + b - 2[x \log x + (1-x)\log(1-x)] \int_0^1 f(t)\,dt.$$

Symmetry shows that $a = 0$, and integration from 0 to 1 then gives $b = 0$.

Finally, one finds that $f(0) = f(1) = 0$ by letting $u = 1-x$ in equation (2), and (6) is seen to yield the general form of $H(x_1, x_2)$. By induction and use of conditions (i) and (ii), (6) is immediately extended to

$$(7) \qquad H(x_1, x_2, \ldots, x_n) = c\,(x_1 \log x_1 + \ldots + x_n \log x_n).$$

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3. C. G. Shannon and W. Weaver, *The mathematical theory of communication*, Urbana, Ill., 1949.

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