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ISOMORPHISM OF SYLOW SUBGROUPS OF INFINITE GROUPS

B. H. NEUMANN

1. Known results. The classical theorems of Sylow are fundamental to the theory of finite groups, and it is, therefore, natural that attempts have been made to extend them to infinite groups.

The first step is to extend to infinite groups the notion of a Sylow subgroup: this is easy. Let p be a prime; then a p-group is a group all of whose elements have orders that are powers of p. Among the p-subgroups of a group G there are maximal ones, and these are called the Sylow p-subgroups of G; in fact every p-subgroup of G is, by an easy application of Zorn's Lemma, contained in a Sylow p-subgroup.

The next step is to try to formulate for infinite groups the propositions (irrespective so far of their validity) involving Sylow subgroups that are true for finite groups. Some such propositions, as e.g. "the order of a Sylow p-subgroup of G is the highest power of p that divides the order of G", have no natural extension to infinite groups. Other such propositions possess somewhat restricted extensions, as e.g. "the number of distinct Sylow p-subgroups of G is congruent to 1 modulo p"; this makes sense if the number of distinct Sylow p-subgroups is finite, and it is then in fact true (Dietzmann, Kurosch und Uzkow [4]; Kurosh [7, p. 163]; Specht [10, p. 383]). Again others can be formulated without change, for infinite groups as for finite groups, as e.g. "the normalizer of a Sylow subgroup is its own normalizer", which, moreover, is easily seen to be true. In this last class are also the propositions "every two Sylow p-subgroups are conjugate" and "every two Sylow p-subgroups are isomorphic"; clearly the former entails the latter, but not conversely. Neither of these propositions is generally valid, and not even such stringent assumptions as supersolubility coupled with the maximal condition for subgroups suffice to make them valid (Zappa [12]). On the other hand, the isomorphism of the Sylow p-subgroups, and an attenuated form of their conjugacy, have been proved (first by Baer [1]) for a class of groups called "locally finite" by Baer [1], "locally normal" by Kurosh [7] and

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by the Russian school of group theory generally, "ok" by Specht [10], and for which I propose yet another term, namely "periodic FC groups".

A group belongs to this class if every finite set of its elements is contained in a finite normal subgroup. In such a group every element must have finite order and finitely many conjugates only, that is to say, the group is, in a now widely accepted terminology, "periodic" and "FC". But the converse is also true, by virtue of the following theorem.

THEOREM OF DIETZMANN [3] (*cf. also* [9, *p.* 185]). *If an element of a group G has finite order and finitely many conjugates only, then it is contained in a finite normal subgroup of G; and, more generally, a finite set of elements of G is contained in a finite normal subgroup of G, if, and (trivially) only if, each of the elements of the set has finite order and finitely many conjugates only.*

Now the theorem of Baer adumbrated above can be stated thus:

THEOREM OF BAER [1]. *In a periodic* FC *group G every two Sylow p-subgroups S, S' are isomorphic.*

Two slightly different refinements, approximating to conjugacy of S and S', have been prove, namely:

COROLLARY OF BAER [1]. *There is an automorphism of G that maps S onto S' and that, moreover, maps every normal subgroup of G onto itself.*

COROLLARY OF GOL'BERG [5]. *There is an automorphism of G that maps S onto S' and that, moreover, coincides on every finite subset of G with an inner automorphism (depending on the subset) of G.*

Such an automorphism is called "locally inner"; a locally inner automorphism evidently maps each element on a conjugate element, and thus leaves normal subgroups invariant. It follows that Gol'berg's Corollary implies Baer's.

Kurosh [7, § 55] proves Baer's Theorem with Gol'berg's Corollary, Specht [10, Abschnitt 3.2.5] proves it with Baer's Corollary; their methods of proof are different. A further proof will be given here, again with Gol'berg's Corollary; it is fundamentally not unlike the known proofs, though it uses different tools. The reason for its inclusion here is that it extends the corollary to a larger and more interesting class of groups.

2. New theorems. We note some immediate extensions of Baer's Theorem; first we recall that an FC group is a group in which the classes of conjugate elements are finite. These groups have been much studied in recent years.

THEOREM A. *In an* FC *group* G *every two Sylow* p*-subgroups* S, S' *are isomorphic.*

PROOF. The periodic elements of G form a subgroup P (see [9, Theorem 5.1]), which must contain S and S', and must contain them as Sylow subgroups. Clearly P is periodic and FC, and application of Baer's Theorem completes the proof.

Another, independent proof will be given in the next sections. Here we extend Baer's Theorem further to groups in which the periodic elements have finite classes of conjugate elements. Let us introduce the name PFC for this group property: thus we define G to be a PFC group if the periodic elements of G each have finitely many conjugates only.

THEOREM B. *In a* PFC *group* G *every two Sylow* p*-subgroups* S, S' *are isomorphic.*

PROOF. The periodic elements of G again form a subgroup P; for by Dietzmann's Theorem every pair of periodic elements a, b is contained in a finite normal subgroup of G, and this then also contains $a^{-1}b$; hence $a^{-1}b$ is periodic. As before, P is periodic and FC, and S and S' are Sylow p-subgroups of P, hence isomorphic by Baer's Theorem.

If instead of assuming finite conjugacy classes for all periodic elements one assumes them only for the elements of prime power orders, the resulting generalization is only apparent; for the groups with this property are still the PFC groups. We only have to remark that every periodic element a can be written as a product

$$a = bc \dots d$$

where the factors on the right-hand side have prime powers orders, for various primes; if each of these has finitely many conjugates only, then the same is true of a.

On the other hand, if we assume finite conjugacy classes only for the elements whose orders are powers of a single fixed prime p, then we get a true, albeit slight, generalization of our results. We introduce the (I hope ephemeral) name pPPFC for this group property: thus the group G is pPPFC, where p stands for a prime number, if the elements of G whose orders are powers of p have finite classes of conjugate elements.

THEOREM C. *In a* pPPFC *group* G *every two Sylow* p*-subgroups* S, S' *are isomorphic.*

Proof. Let H be the subgroup of G generated by the elements of order a power of p. Then H is an FC group because it is generated by elements each with finitely many conjugates only (cf. [9, Lemma 2.2]), and S and S' clearly are Sylow p-subgroups of H. Theorem A now completes the proof.

In fact even this result can still be sharpened a little: one need assume only that each element of the set union $S \cup S'$ has finitely many conjugates *in the subgroup generated by* $S \cup S'$, so that this subgroup is FC.

3. Some lemmas. In order to find out how far the corollaries of Baer and Gol'berg are capable of a parallel generalization, we have to provide a direct proof of Theorem A. The proof that follows is basically the same as the known proofs, but in place of an *ad hoc* transfinite induction or a theorem on "projection sets" (Kurosh [7, § 55]) it uses Steenrod's Theorem. We begin with some lemmas; they were proved by Baer ([1, § 4]; cf. Specht [10, p. 391]) for periodic FC groups, but will be required here more generally for FC groups. We state and prove them even more generally for pPPFC groups; the method is Baer's.

Lemma 1. *If F is a finite group, N a normal subgroup of F, and S a Sylow p-subgroup of F, then $S \cap N$ is a Sylow p-subgroup of N.*

This fact is well known. For a proof see e.g. Baer [1, (4.1.1)], Kurosh [7, p. 161], or Zassenhaus [13, p. 136].

Lemma 2. *Let G be a pPPFC group, T a p-subgroup of G, and N a finite normal subgroup of G. Then there is a p-subgroup S of G that contains T and intersects N in a Sylow p-subgroup of N.*

Proof. We show that there is a Sylow p-subgroup U of N that together with T generates again a p-group $\{T, U\} = S$. Denote by $U_1, U_2, \ldots, U_m$ those p-subgroups of N that fail to do this: thus none of

$$\{T, U_1\}, \{T, U_2\}, \ldots, \{T, U_m\}$$

is a p-group, and for all other p-subgroups U of N, if any, $\{T, U\}$ is a p-group. Choose $h_i \in \{T, U_i\}$ of order not a power of p, and express each h_i $(i = 1, 2, \ldots, m)$ in terms of elements of T and elements of U_i. Only a finite number of elements of T are required for this, say

$$t_1, t_2, \ldots, t_n .$$

Their orders are powers of p, and as G is assumed to be pPPFC, they have finite classes of conjugate elements. By Dietzmann's Theorem they are contained in a finite normal subgroup M of G. Then $L = MN$ is also a finite normal subgroup of G. Let V be a Sylow p-subgroup of L that contains the intersection $T \cap L$. Now V contains $t_1, t_2, \ldots, t_n$, because they are in $T \cap L$. Then V can not contain any of the U_i, $i=1, 2, \ldots, m$; for

$$h_i \in \{t_1, t_2, \ldots, t_n, U_i\} ,$$

and the order of h_i is not a power of p, whereas V is a p-group. Consider $U = V \cap N$. By Lemma 1 this is a Sylow p-subgroup of N, and it is distinct from all the U_i, $i=1, 2, \ldots, m$; thus $\{T, U\}$ is a p-group, and the lemma follows.

LEMMA 3. *The p-subgroup S of the pPPFC group G is a Sylow subgroup of G if, and only if, $S \cap N$ is a Sylow p-subgroup of N for every finite normal subgroup N of G.*

PROOF. If T is a p-subgroup of G and N a finite normal subgroup of G, and if $T \cap N$ fails to be a Sylow p-subgroup of N, then T is properly contained in another p-subgroup S of G, by Lemma 2; thus T is not a Sylow p-subgroup of G, and the necessity of the stated condition follows. Conversely, assume that T is a p-subgroup of G which intersects each finite normal subgroup N of G in a Sylow p-subgroup of N, and let S be a Sylow p-subgroup of G containing T. We show that then $T=S$; for let $g \in S$. Then the order of g is a power of p and, as G is pPPFC, g has finitely many conjugates only; thus by Dietzmann's Theorem, g is contained in a finite normal subgroup N of G. Now $N \cap T$ is a Sylow p-subgroup of N by our assumption on T, and $N \cap S$ is a p-subgroup of N containing $N \cap T$; hence $N \cap S = N \cap T$, and thus $g \in N \cap S$ implies $g \in N \cap T$. It follows that $S \leq T$, and thus $S=T$. This shows also the sufficiency of the condition, completing the proof of the lemma.

4. Proof of the extended corollary. Throughout this section we assume G to be an FC group. We then note an easy consequence of known results:

LEMMA 4. *Let F be a finitely generated subgroup of the* FC *group G. Then the periodic elements of F form a finite subgroup N, and if F is normal in G, then so is N.*

The periodic elements of F form a subgroup because F is FC, and a finite subgroup because F is, moreover, finitely generated ([9, Theorem 5.1]). This subgroup is clearly characteristic in F, hence normal in G if F is normal in G.

We now let $\mathfrak{F}$ stand for the set of all finitely generated normal subgroups of G, and D, E, F for typical elements of $\mathfrak{F}$, that is for typical finitely generated normal subgroups of G. We order $\mathfrak{F}$ by inclusion; then it becomes a directed set, for if E, F are finitely generated normal subgroups of G, then both are contained in EF, and this is again finitely generated and normal in G.

Let S and S' be two Sylow p-subgroups of G; our task is to construct a locally inner automorphism of G that maps S onto S'. To this end we define to every $F \in \mathfrak{F}$ the "space" X_F whose "points" are those automorphisms of F that

(i) map $F \cap S$ into $F \cap S'$, and

(ii) are induced by inner automorphisms of G.

Thus $x \in X_F$ if, and only if, (i) x is an automorphism of F mapping $F \cap S$ into $F \cap S'$, and (ii) there is an element $t \in G$ such that x is (the restriction to F of) the transformation by t.

We note that $F \cap S$ and $F \cap S'$ are finite groups; for they are clearly subgroups of the group N of all periodic elements of F, and this is finite. Moreover, $F \cap S = N \cap S$ and $F \cap S' = N \cap S'$ are Sylow p-subgroups of N, by Lemma 3. It follows that every $x \in X_F$ maps $F \cap S$ isomorphically onto $F \cap S'$. We further deduce that X_F is not empty; for $N \cap S$ and $N \cap S'$ are, as Sylow p-subgroups of N, conjugate in N; hence there is an element $t \in N$ such that transformation by t maps $N \cap S$ onto $N \cap S'$. But transformation by t induces an automorphism of F because F is normal in G; hence it defines a point in X_F. On the other hand each X_F is finite; for let

$$f_1, f_2, \ldots, f_d$$

be a set of generators of F; each of these has finitely many distinct conjugates only, because G is FC; thus there are only finitely many sets

$$f_1', f_2', \ldots, f_d'$$

conjugate to $f_1, f_2, \ldots, f_d$, and that means the inner automorphisms of G can induce only finitely many automorphisms of F; and X_F consists of some of these finitely many automorphisms.

If $E \leq F$ and if $x \in X_F$, then the restriction y of x to E is a point in X_E; for E is normal in G and thus transformation by an element $t \in G$ which induces x in F induces also an automorphism of E; and this must clearly map $E \cap S$ into $E \cap S'$. We can, therefore, define a mapping φ_{FE} of X_F into X_E by putting $x\varphi_{FE} = y$. Clearly φ_{FF} is the identity mapping of X_F, and if $D \leq E \leq F$ then

$$\varphi_{FD} = \varphi_{FE}\varphi_{ED} .$$

Thus the directed set $\mathfrak{F}$, the spaces X_F, and the mappings φ_{FE} (for $E \leq F \in \mathfrak{F}$) define an inverse mapping system, and we can form the inverse limit X_*, together with its (natural) projections φ_{*F} of X_* into X_F; these have the property that if $E \leq F$ then

$$\varphi_{*E} = \varphi_{*F}\varphi_{FE} \cdot$$

As we have seen, the spaces X_F are finite. We give them the discrete topology; then they are compact. and the mappings φ_{FE} are (trivially) continuous. We have also seen that the spaces X_F are non-empty. By Steenrod's Theorem ([11, Theorem 2.1]; cf. also Lefschetz [8, p. 32]) the inverse limit X_* then is not empty. Let x_* be a point of X_*. We define from this a mapping α of G into G as follows. Let $g \in G$, and let F be a finitely generated normal subgroup of G containing g; such subgroups exist, for the finitely many conjugates of g generate one. Then $x_*\varphi_{*F} = x$ is an automorphism of F, and we put $g\alpha = gx$. This appears to depend on the F chosen, but in fact it does not; for if E denotes the group generated by g and its conjugates, then $E \in \mathfrak{F}$ and $E \leq F$, and denoting by y the restriction of $x = x_*\varphi_{*F}$ to E, then

$$y = x\varphi_{FE} = x_*\varphi_{*F}\varphi_{FE} = x_*\varphi_{*E};$$

and further

$$g\alpha = gx = gy = g(x_*\varphi_{*E}),$$

which is seen to depend on g and the particular point x_* only, not on F.

Next we show that α is a locally inner automorphism of G. Let

$$g_1, g_2, \ldots, g_n$$

be a finite set of elements of G, and F a finitely generated normal subgroup containing them, e.g. the subgroup generated by all the conjugates of these elements. Then α has the same effect on the elements of F as $x_*\varphi_{*F}$; but this is a point of X_F, that is an automorphism of F induced by transformation by some element $t \in G$. Thus we have

$$(*) \qquad g_1\alpha = t^{-1}g_1t, \quad g_2\alpha = t^{-1}g_2t, \quad \ldots, \quad g_n\alpha = t^{-1}g_nt\,.$$

We choose $g_3 = g_1g_2$ to deduce the homomorphism property

$$(g_1g_2)\alpha = g_1\alpha g_2\alpha\,.$$

Next, if $g \in G$ is arbitrarily given, we choose $g = g_1, g_2, \ldots, g_n$ as the conjugates of g in G—these are finite in number because G is FC—and we deduce from (*) that α permutes these conjugates; it follows that α is one-to-one and onto, and thus an automorphism of G. Finally we see from (*) that it is locally inner.

It remains to show that α maps S onto S'. Now if $g \in S$ and if again $g \in F \in \mathfrak{F}$, then

$$g\alpha = g(x_*\varphi_{*F}) \in F \cap S'$$

because $x_*\varphi_{*F}$, being a point of X_F, maps $F \cap S$ into $F \cap S'$. Thus $S\alpha \leqq S'$. Conversely, if $g' \in S'$ and if $g' \in F \in \mathfrak{F}$, then there is an element $g \in F \cap S$ that is mapped on g' by $x_*\varphi_{*F}$ and thus by α; for we have already seen that every $x \in X_F$ maps $F \cap S$ isomorphically onto $F \cap S'$. Thus also $S' \leq S\alpha$, and it follows that $S\alpha = S'$. This completes the proof of Theorem A, together with its corollary:

COROLLARY A. *If S, S' are Sylow p-subgroups of the* FC *group G, then there is a locally inner automorphism of G that maps S onto S'.*

5. A counterexample. It might be thought that a similar corollary must be true under the assumptions of Theorem B, that is for PFC groups; this is, however, not the case, as the following example shows.

We take an infite index set I and for each $i \in I$ a copy A_i of the tetrahedral group, that is the group generated by two elements a_i, b_i with the defining relations

$$a_i^3 = b_i^2 = (a_ib_i)^3 = 1$$

(cf. Coxeter and Moser [2, p. 134]). We form the restricted direct product H of all these A_i; thus H is generated by all a_i, b_i and is defined by the above relations together with

$$a_ia_j = a_ja_i, \qquad a_ib_j = b_ja_i, \qquad b_ib_j = b_jb_i$$

for all $i \neq j \in I$. To obtain G we add a further generator c and the relations

$$c^{-1}a_ic = a_i^{-1}, \qquad c^{-1}b_ic = b_i$$

for all $i \in I$. Then c induces an outer automorphism (of order 2) of each A_i, and thus of H, and c^2 is an element of the centre of G. Clearly H consists of all periodic elements of G, and all elements of H have finitely many conjugates only; hence G is a PFC group. Now consider the group S generated by all a_i, $i \in I$; this is a Sylow 3-subgroup of H, hence also of G; so also is the group S' generated by all a_ib_i, $i \in I$. They are both elementary abelian 3-groups of the same order, and thus isomorphic—as indeed they are bound to be, by Theorem B. By Corollary A there is in fact a locally inner automorphism of H that maps S onto S'; but there is no automorphism of G, and *a fortiori* no locally inner automorphism of G, that maps S onto S'. To see this we note that there is an element of G, namely c, that transforms each element of S into its inverse; but there is no element of G that does the same to S': we write an arbitrary element of G in the form

$$g = c^mh, \qquad h \in H\,,$$

and consider the effect of transforming S' by g. Transformation by h

affects only a finite number of generators $a_i b_i$ of S'; transformation by c^m affects none if m is even, and all if m is odd; but in this latter case $a_i b_i$ is transformed not into its inverse, but into $a_i^{-1} b_i$. Hence no $g \in G$ can transform each element of S' into its inverse. Now if there were an automorphism of G mapping S onto S', then it would have to map c on an element c', say, that transforms each element of S' into its inverse; as this has just been shown to be impossible, there is no such automorphism of G.

Finally we remark that the Sylow p-subgroups are unique in nilpotent and even in locally nilpotent groups (Kurosh [7, p. 229]); but they are not even necessarily isomorphic in FC-nilpotent (Haimo [6]) groups, nor in metabelian groups. This is shown by the example constructed by Zappa [12, § 5], and also by the free metabelian product of two cyclic groups, one of order p and one of order p^2. The verification, which is not difficult, is omitted.

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THE UNIVERSITY, MANCHESTER, ENGLAND