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ON TWO INEQUALITIES BY S. SELBERG

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S. Selberg has studied the sum

$$\sigma_k(x) = \sum_{\lambda=1}^{k} \left(x\left[\frac{\lambda}{x}\right] - (x+1)\left[\frac{\lambda}{x+1}\right]\right), \tag{1}$$

where k is a natural number, $x > 0$, and $[y]$ denotes the largest integer not exceeding y. In the papers [1] and [2], he has proved the following inequalities:

$$0 \leqq \sigma_k(x) \leqq k\,. \tag{2}$$

The purpose of this note is to give a simple proof of (2). In order to do this, we observe that (1) can be written as

$$\sigma_k(x) = \sum_{\lambda=1}^{k} \left(\left\{\lambda - (x+1)\left[\frac{\lambda}{x+1}\right]\right\} - \left\{\lambda - x\left[\frac{\lambda}{x}\right]\right\}\right), \tag{3}$$

or as

$$\sigma_k(x) = \sum_{\lambda=1}^{k} \left(\left\{\lambda - (x+1)\left[\frac{\lambda}{x+1}\right] - x\right\} + \left\{x - \lambda + x\left[\frac{\lambda}{x}\right]\right\}\right), \tag{4}$$

and that

$$\lambda - x\left[\frac{\lambda}{x}\right] \geqq 0, \qquad x - \lambda + x\left[\frac{\lambda}{x}\right] > 0\,. \tag{5}$$

If now

$$\lambda - (x+1)\left[\frac{\lambda}{x+1}\right] > 1\,,$$

that is

$$\lambda - (x+1)\left[\frac{\lambda}{x+1}\right] = 1 + \theta x, \qquad 0 < \theta < 1\,, \tag{6}$$

then

$$\lambda - \left[\frac{\lambda}{x+1}\right] - 1 = x\left[\frac{\lambda}{x+1}\right] + \theta x\,.$$

This means that

$$\lambda - \left[\frac{\lambda}{x+1}\right] - 1 - x\left[\frac{\lambda - \left[\frac{\lambda}{x+1}\right] - 1}{x}\right] = \theta x\,. \tag{7}$$

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As $\theta > 0$, it is seen from (6) that

$$\lambda - \left[\frac{\lambda}{x+1}\right] - 1 > 0 .$$

Obviously

$$\lambda - \left[\frac{\lambda}{x+1}\right] - 1 < \lambda \leqq k .$$

If

$$\lambda_1 - \left[\frac{\lambda_1}{x+1}\right] - 1 = \lambda_2 - \left[\frac{\lambda_2}{x+1}\right] - 1 ,$$

where λ_1 and λ_2 both satisfy (6), then

$$\left[\frac{\lambda_1}{x+1}\right] - \left[\frac{\lambda_2}{x+1}\right] = \theta_2 - \theta_1 . \tag{8}$$

As $-1 < \theta_2 - \theta_1 < 1$, (8) shows that

$$\theta_1 = \theta_2, \qquad \left[\frac{\lambda_1}{x+1}\right] = \left[\frac{\lambda_2}{x+1}\right],$$

whereupon (6) yields $\lambda_1 = \lambda_2$. Hence, if the l. h. s. of (6) is a term in (3), the l. h. s. of (7) will also be a term in (3), the contribution to (3) from these two terms being $+1$. We also showed that different terms (6) give rise to different terms (7). Taking (5) into consideration, we have thus proved the right half of (2).

To prove the other half of (2), we observe that a "bad" term of (4), with

$$\lambda - (x+1)\left[\frac{\lambda}{x+1}\right] - x < 0 ,$$

that is

$$\lambda - (x+1)\left[\frac{\lambda}{x+1}\right] - x = (\theta - 1)x, \qquad 0 \leqq \theta < 1 , \tag{9}$$

gives rise to the term

$$x - \left(\lambda - \left[\frac{\lambda}{x+1}\right]\right) + x\left[\frac{\lambda - \left[\frac{\lambda}{x+1}\right]}{x}\right] = (1-\theta)x . \tag{10}$$

(10) is derived from (9) in the same way as one derives (7) from (6). The net contribution of the terms (9) and (10) to (4) is 0. Exactly as above, one shows that (10) really is a term in (4), and that to different terms (9) correspond different terms (10). Finally, making use of (5), (2) is established in its entirety. It will be seen that the method of proof allows one to find all cases of equality taking place.

REFERENCES

1. S. Selberg, *On a conjecture by Ernst Jacobsthal*, Norske Vid. Selsk. Forh. 26 (1953), Nr. 21, 89–93.
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