MATH. SCAND. 7 (1959), 49–56

FINITE BOUNDARY VALUE PROBLEMS SOLVED BY GREEN'S MATRIX

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0. Introduction. This paper proposes to study the problem of inverting a non-singular symmetric matrix

$$c = [c_{ik}] \tag{1}$$

whose components satisfy

$$c_{ik} = 0 \quad \text{for} \quad |k-i| > 1 .$$

This matrix arises naturally in many technical problems (Clapeyron's equations etc.). The first and last rows of c may be associated with (finite) homogeneous boundary conditions of such problems. In the limit when the number of equations under proper conditions tends to infinity, the matrix tends to a linear self-adjoint differential operator with in general variable coefficients, together with homogeneous boundary conditions. Therefore the theory of such differential equations

$$(ey')' + fy = p$$

and especially the solution of their boundary problems by Green's function may be simply founded upon properties of the finite matrices c.

But the type of matrices in question, which are called band matrices in the sequel, are also of a more general interest. Givens [4] has shown that every symmetric matrix can be transformed into a band matrix by a finite number of rotations. Givens uses this result to compute the eigenvalues and eigenfunctions of a symmetric matrix. By what is here called Green's matrix, Egerváry [3] has recently treated simple cases of beams and suspension bridge girders on hinged supports but with no general treatment of boundary conditions.

The method of inverting a band matrix which is given here is modeled on the ordinary method of solving a linear differential equation by means of a Green's function. Another method has been proposed by W. J. Berger and E. Saibel [2]. The method proposed in this paper is capable of considerable generalization, which will be carried out in a forthcoming paper by E. Asplund [1].

Received August 4, 1958.

Band equations. The system of equations $cy = z$ or

$$\sum_{i=0}^{n} c_{ki} y_i = z_k, \qquad k = 0, 1, \ldots, n\,,$$

with $\det c_{ki} \neq 0$, has "band form" if

$$c_{ki} = 0 \qquad \text{for} \qquad |k-i| > e\,.$$

The coefficients of the unknowns y can be in order the same in each equation (except in the e first and e last equations); the system of equations can be said to have "constant coefficients". The coefficients may also vary from equation to equation so that one may speak of "variable coeffficients". Linear differential equations with constant or variable coefficients give rise to band equations when they are transformed into difference equations.

One- and two-point boundary problems. Among the e first unknowns, e further linear relations can be prescribed at the same time as the e last equations are dropped. Then all the unknowns can be found by considering the equations successively, starting from above. The problem is then called a "*one-point*" or "*starting value*" boundary problem of order $2e$.

The e first equations can also be interpreted, as prescribed homogeneous or inhomogeneous relations, "boundary conditions", between the unknowns

$$y_0, y_1, \ldots, y_{2e-1}$$

involved and analoguously for the e last equations. Then a "two-point boundary problem" is at hand or, simply, a "boundary value problem". The $n+1-2e$ middle pure band equations with $n+1$ unknowns then as a rule can be directly solved: $y = c^{-1}z$.

Another method to solve this boundary value problem is by Green's matrix as will be demonstrated presently.

1. Inversion of a band matrix. In a system of band equations for a finite boundary value problem of the second order ($e=1$) (arising e.g. when approximating a boundary problem of a second order linear differential equations by a difference equation):

$$cy = z, \qquad \begin{bmatrix} c_{00} & c_{01} & 0 & 0 & . & 0 \\ c_{10} & c_{11} & c_{12} & 0 & . & 0 \\ 0 & c_{21} & c_{22} & c_{23} & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & c_{ml} & c_{mm} & c_{mn} \\ 0 & 0 & . & 0 & c_{nm} & c_{nn} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ . \\ y_m \\ y_n \end{bmatrix} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ . \\ z_m \\ z_n \end{bmatrix}, \qquad m = n-1\,,$$

the first and last equations are interpreted as *inhomogeneous boundary conditions*

$$c_{00}y_0 + c_{01}y_1 = z_0, \qquad c_{nm}y_m + c_{nn}y_n = z_n \,. \tag{2}$$

When $z_0 = z_n = 0$ the boundary conditions are *homogeneous.*

The system of equations $cy = z$ can be solved by a fitting together of the solutions c_a, c_b of two homogeneous "starting value boundary problems" and subsequent superposition:

Consider the homogeneous equations

$$(cc_a)_i = 0, \qquad i = 0, 1, \ldots, n-1;$$

cancelling $(cc_a)_n = 0$. For an arbitrarily chosen starting value c_{a0} compute c_{a1} from the first equation

$$(cc_a)_0 = c_{00}c_{a0} + c_{01}c_{a1} = 0$$

and $c_{a2}, \ldots, c_{an}$ from $(cc_a)_i = 0$ in order $i = 1, \ldots, n-1$. (None of c_{01}, c_{12} etc. is supposed to be equal to zero. If any c_{01}, c_{12} etc. vanishes, the matrix c can be split in parts which can be inverted separately by this same method.)

Consider also the homogeneous equations

$$(cc_b)_i = 0, \qquad i = 1, \ldots, n;$$

cancelling $(cc_b)_0 = 0$. For an arbitrarily chosen starting value c_{bn} compute c_{bm} from the last equation $(cc_b)_n = 0$ and $c_{b,\,n-2}, \ldots, c_{b0}$ from $(cc_b)_i = 0$ in reversed order $i = m, \ldots, 1$. Writing

$$k-1 = j, \qquad k+1 = l$$

we have

$$\begin{aligned} (c_a)_k &= 0 \quad \text{is} \quad c_{kj}c_{aj} + c_{kk}c_{ak} + c_{kl}c_{al} = 0\,, \\ (c_b)_k &= 0 \quad \text{is} \quad c_{kj}c_{bj} + c_{kk}c_{bk} + c_{kl}c_{bl} = 0\,. \end{aligned} \tag{3}$$

Multiplication by c_{bk}, c_{ak} and subtraction yields

$$c_{kj}\begin{vmatrix} c_{aj} & c_{bj} \\ c_{ak} & c_{bk} \end{vmatrix} = c_{kl}\begin{vmatrix} c_{ak} & c_{bk} \\ c_{al} & c_{bl} \end{vmatrix} = W\,. \tag{4}$$

For a *symmetrical* matrix c it follows inductively (write (4) for $k = k+1$) that W takes on the same value for all k from 1 to m. This W will correspond to Wronski's determinant in the theory of Green's functions.

The solution of (1) will be obtained after further simplifications: Make $z_k = 1$ and all other $z = 0$, including $z_0 = z_n = 0$, that is, $z = I_{.k}$ = the (k)th column of the unit matrix. Then the solution of the system $cy^{(k)} = I_{.k}$ can be written

(5) $$y_i^{(k)} = \begin{cases} ac_{ai} & \text{for} \quad i \leqq k\,, \\ bc_{bi} & \text{for} \quad k \leqq i\,, \end{cases}$$

and thus

(6) $$y_k^{(k)} = ac_{ak} = bc_{bk}\,.$$

The $(k+1)$st equation in (1) is then $(cy^{(k)})_k = 1$, or

(7) $$c_{kj}ac_{aj} + c_{kk}bc_{bk} + c_{kl}bc_{bl} = 1\,.$$

Subtract b times (3b):

$$c_{kj}(ac_{aj} - bc_{bj}) = 1\,.$$

Multiply by c_{bk} and apply (6):

$$c_{kj}(ac_{aj}c_{bk} - ac_{ak}c_{bj}) = c_{bk}\,,$$

or by (4):

$$a = c_{bk}/W\,,$$

and (6):

$$b = c_{ak}/W\,.$$

Thus by (5)

(8b) $$y_i^{(k)} = c_{ik}^{g} = \begin{cases} c_{ai}c_{bk}/W & \text{for} \quad i \leqq k\,, \\ c_{ak}c_{bi}/W & \text{for} \quad k \leqq i\,. \end{cases}$$

For

$$cy^{(k)} = z = I_{.k}z_k$$

instead of $z = I_{.k}$, the solution will be

$$y_i^{(k)} = c_{ik}^{g}z_k\,.$$

When several elements on the right hand side z differ from zero and when the inhomogeneous boundary conditions z_0, z_n are entered, superposition of solutions $y_i^{(k)}$ will yield the solution of $cy = z$:

(8a) $$y = c^{g}z = c^{-1}z\,,$$

where c^{-1} may be called "Green's matrix"

(8c) $$c^{-1} = \begin{bmatrix} c_{a0}c_{b0} & c_{a0}c_{b1} & \cdot & c_{a0}c_{bn} \\ c_{a0}c_{b1} & c_{a1}c_{b1} & \cdot & c_{a1}c_{bn} \\ \cdot & \cdot & \cdot & \cdot \\ c_{a0}c_{bn} & c_{a1}c_{bn} & \cdot & c_{an}c_{bn} \end{bmatrix} / W\,.$$

The homogeneous boundary conditions $y_0 = y_n = 0$, $z_0 = z_n = 0$ require that $c_{01} = 0$, $c_{nm} = 0$ in (42) $c_{a0} = c_{bn} = 0$. Then Green's matrix c^{-1} will be entirely bordered by zero elements that, together will all 'border elements" in y and z can be omitted for convenience ("abbreviated" matrix, cf. Egerváry [3]).

2. Finite boundary value problems of higher order. Finite boundary value problems of a higher order $2e$ than the second are solved analoguously. The band matrix in question has

$$c_{ik} = 0 \qquad \text{for} \qquad |k-i| > e .$$

If it is symmetric, the solution can always be written in terms of a Green's matrix of higher order.

The classical theory for Green's functions follows by a passage to the limit from these finite cases. Symmetry in the matrix c of (1), will result in self-adjoint differential expressions. For symmetric band matrices of order two the differential expression will be of the form

$$(ey')' + fy = p .$$

3. Band matrices of divergence type. If the row sums of a band matrix vanish, except possibly for the sums of the first and last rows, the matrix will be said to be of divergence type, which name is borrowed from the corresponding type of differential equation. If all row sums vanish, the matrix is obviously singular. Berger and Saibel have discussed in [2] the case when only one row sum differs from zero, which they call the gnomonic symmetry case. It will here be supposed that both the first and the last rows have non-vanishing sums, and also, according to a remark in section 1, that all elements immediately above and below the main diagonal are non-zero. Such a divergence type band matrix, belonging to a "divergence type" second order boundary value problem $Ny = z$, may thus be written

$$N = \begin{bmatrix} -n_{a0}^{-1} - n_{01}^{-1} & n_{01}^{-1} & 0 & . & 0 \\ n_{01}^{-1} & -n_{01}^{-1} - n_{12}^{-1} & n_{12}^{-1} & . & 0 \\ 0 & n_{12}^{-1} & -n_{12}^{-1} - n_{23}^{-1} & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & -n_{mn}^{-1} - n_{nb}^{-1} \end{bmatrix} . \tag{9}$$

By adjusting n_{a0}, n_{nb}, z_0, z_n, the first and last equations $Ny = z$ will represent any prescribed homogeneous or inhomogeneous boundary conditions.

The symbols N_a, N_b will be used to denote the counterparts of c_a, c_b in section 1. If one assumes that $N_{a0} = n_{a0}$, the equations for N_a, $(NN_a)_0 = 0$, $(NN_a)_1 = 0$ etc., yield by induction

$$-N_{a0}/n_{a0} + (N_{a1} - N_{a0})/n_{01} = 0, \qquad -1 + (N_{a1} - n_{a0})/n_{01} = 0 ,$$

$$N_{a1} = n_{a0} + n_{01} = n_{a1};$$

$$-(N_{a1}-N_{a0})/n_{01} + (N_{a2}-N_{a1})/n_{12} = 0, \qquad -1 + (N_{a2}-n_{a1})/n_{12} = 0,$$

$$N_{a2} = n_{a1}+n_{12} = n_{a2};$$

etc., inductively.

(10) $$N_{ak} = n_{ak} = n_{a,k-1}+n_{k-1,k}$$
$$= n_{a0}+n_{01}+\dots n_{k-1,k} \qquad \text{for} \qquad k = 1, 2, \dots, n .$$

Analoguously assuming $N_{bn}=n_{nb}$,

$$N_{bk} = n_{kb} = n_{k,k+1}+n_{k+1,b}$$
$$= n_{k,k+1}+\dots+n_{mn}+n_{nb} \qquad \text{for} \qquad k = 0, 1, \dots, m .$$

By (4), writing $n_{a0}+n_{a1}+\dots+n_{mn}+n_{mn}+n_{nb}=n_{ab}$,

$$W = c_{01}\begin{vmatrix} N_{a0} & N_{b0} \\ N_{a1} & N_{b1} \end{vmatrix} = \begin{vmatrix} n_{a0} & n_{0b} \\ n_{a1} & n_{1b} \end{vmatrix}/n_{01} = \begin{vmatrix} n_{ab} & n_{0b} \\ n_{ab} & n_{1b} \end{vmatrix}/n_{01} = \begin{vmatrix} 0 & n_{01} \\ n_{ab} & n_{1b} \end{vmatrix}/n_{01},$$

(11) $$W = -n_{ab} .$$

Entering (10) and (11) into (8a, b, c) yields the solution

(12a) $$y = N^{-1}z ,$$

(12b) $$N_{ik}^{-1} = \begin{cases} -n_{ai}n_{kb}/n_{ab} & \text{for} \quad i \leqq k , \\ -n_{ak}n_{ib}/n_{ab} & \text{for} \quad k \leqq i , \end{cases}$$

(12c) $$N^{-1} = \begin{bmatrix} n_{a0}n_{0b} & n_{a0}n_{1b} & \cdot & n_{a0}n_{nb} \\ n_{a0}n_{1b} & n_{a1}n_{1b} & \cdot & n_{a1}n_{nb} \\ \cdot & \cdot & \cdot & \cdot \\ n_{a0}n_{nb} & n_{a1}n_{nb} & \cdot & n_{an}n_{nb} \end{bmatrix}/(-n_{ab}) .$$

4. A geometric application: Deviation-angles in polygons. The polygon of Fig. 1 has consecutive corners (x_j, y_j), (x_k, y_k), (x_l, y_l) on the abscissa intervals $x_{jk}=x_k-x_j$ and x_{kl}. The corner deviation angle A_k (as defined in Fig. 1) at x_k is

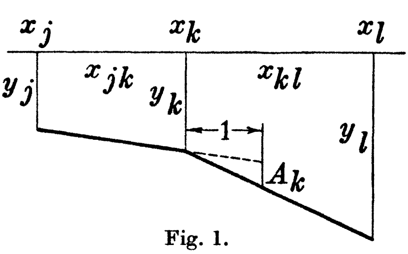

Fig. 1.

(13) $$A_k = (y_l-y_k)/x_{kl} - (y_k-y_j)/x_{jk}$$

or

$$A_k = x_{jk}^{-1}y_j +$$
$$+ (-x_{jk}^{-1}-x_{kl}^{-1})y_k + x_{kl}^{-1}y_l .$$

Adding equations for $k=0$ and n, representing boundary conditions, all corner deviation angles may be written

(14)

$$\begin{bmatrix} -x_{a0}^{-1}-x_{01}^{-1} & x_{01}^{-1} & 0 & 0 & . & 0 \\ x_{01}^{-1} & -x_{01}^{-1}-x_{12}^{-1} & x_{12}^{-1} & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & -x_{mn}^{-1}-x_{nb}^{-1} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ . \\ y_n \end{bmatrix} = \begin{bmatrix} A_0 \\ A_1 \\ . \\ A_n \end{bmatrix}$$

or

(15) $$Xy = A .$$

The matrix X may be called the "deviation-angle matrix". The solution of (15) is given by (12) with $n_{ik}=x_{ik}$:

(16a) $$y = X^{-1}A ,$$

(16b) $$X_{ik}^{-1} = \begin{cases} -x_{ai}x_{kb}/x_{ab} & \text{for} \quad i \leqq k , \\ -x_{ak}x_{ib}/x_{ab} & \text{for} \quad k \leqq i , \end{cases}$$

(16c) $$X^{-1} = \begin{bmatrix} x_{a0}x_{0b} & x_{a0}x_{1b} & . & x_{a0}x_{nb} \\ x_{a0}x_{1b} & x_{a1}x_{1b} & . & x_{a1}x_{nb} \\ . & . & . & . \\ x_{a0}x_{nb} & x_{a1}x_{nb} & . & x_{an}x_{nb} \end{bmatrix} /(-x_{ab}) .$$

A correct choice of x_{a0} and A_0 in (14) expresses any prescribed homogeneous or inhomogeneous boundary condition.

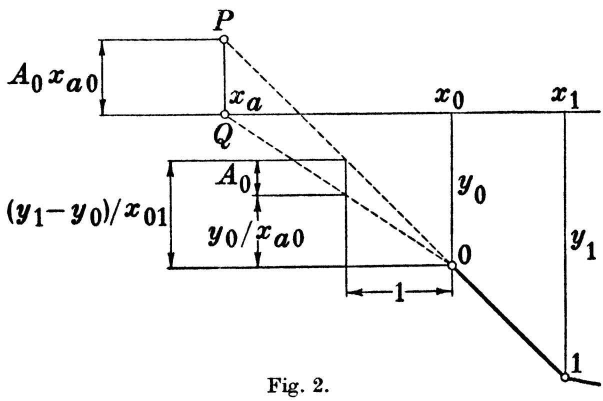

Fig. 2.

The inhomogeneous boundary condition in (14),

$$-y_0/x_{a0} + (y_1-y_0)/x_{01} = A_0 ,$$

is equivalent to the condition in Fig. 2, that the polygon side 01 shall pass through P. For homogeneous boundary conditions $A_0=0$ and thus the side 01 passes through Q.

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