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INVERSES OF MATRICES $\{a_{ij}\}$ WHICH SATISFY $a_{ij} = 0$ FOR $j > i+p$

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0. Introduction. It is well known that a non-singular matrix over a field k (not necessarily commutative) is triangular (i.e., all elements above the main diagonal are equal to zero) if and only if its inverse is triangular. The purpose of this paper is to give a corresponding characterization of invertible matrices whose elements are zero above some other diagonal parallel to the main diagonal.

This investigation was initiated by an article of S. O. Asplund [1]. The first section deals with a geometric lemma. The next section states and proves the main results, and in a concluding section we examine the possibilities of extending the result to more general rings.

1. A geometric lemma. Let f be an automorphism of a finite dimensional right vector space V over a (non-commutative) field k. Let V be decomposed in two ways as a direct sum of a pair of supplementary subspaces,

$$V = R \oplus S = T \oplus U ,$$

and denote the corresponding projections by r, s, t and u respectively. The identity automorphism on V is denoted by e. We have the following relations

$$r^2 = r, \quad s^2 = s, \quad t^2 = t, \quad u^2 = u ,$$

$$r+s = t+u = e, \qquad rs = sr = tu = ut = 0$$

and the

LEMMA. *The equation $tfs=0$ holds, if and only if*

$$\dim rf^{-1}(U) = \operatorname{rank}(rf^{-1}u) \leqq \dim R - \dim T .$$

It is to be noted that the statement of the lemma is vacuous if $\dim R < \dim T$, since $tfs=0$ is equivalent to $tfrf^{-1}=t$ and this implies that $\dim R \geqq \dim T$.

To prove the lemma, we first suppose that $tfs=0$. From the equivalent

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relation $tfrf^{-1}=t$ we infer that tf maps R onto T. The nullity of tf, regarded as a mapping from R to T, is thus $\dim R-\dim T$. As $tfrf^{-1}u=tu=0$, we conclude that

$$\text{rank}\,(rf^{-1}u) \leqq \dim R-\dim T\,.$$

Conversely, let

$$\text{rank}\,(rf^{-1}u) = \dim rf^{-1}(U) \leqq \dim R-\dim T\,.$$

This means that the nullity of rf^{-1} when regarded as a mapping on U, is at least equal to $\dim S$. However, the nullity of rf^{-1} regarded as a mapping on V, is equal to $\dim fs(V)=\dim S$. Thus the null space of rf^{-1} is contained in U. But the null space of rf^{-1} is identical with the range of fs, hence $tfs=0$. The statements of the lemma are now proved, and by the proof it is also clear that $\text{rank}(rf^{-1}u)$ must be equal to $\dim R-\dim T$.

2. Band matrices and Green's matrices. We define two types of square matrices over a field k.

DEFINITION 1. A band matrix of order p is a square matrix $\{a_{ij}\}$ over a field k, whose elements satisfy $a_{ij}=0$ for $j>i+p$.

DEFINITION 2. A Green's matrix of order p is a square matrix $\{a_{ij}\}$ over a field k, whose submatrices have rank $\leqq p$ if their elements belong to the part of $\{a_{ij}\}$ for which $j+p>i$.

Triangular matrix of order 0 and Green's matrix of order 0 are identical concepts for square matrices over a field. For band matrices and Green's matrices of arbitrary order we now state an extension of the property of inverses of triangular matrices mentioned in the introduction.

THEOREM 1. *A non-singular square matrix over a field is a band matrix of order p if and only if its inverse is a Green's matrix of order p.*

The proof of theorem 1 is a straightforward application of the lemma. Suppose that $\{a_{ij}\}$ is a non-singular band matrix of order p over k and consider the corresponding automorphism f of the right vector space V spanned by the base vectors b_i, $i=1, 2, \ldots, n$. Take some submatrix of $\{a_{ij}\}^{-1}$, whose elements all are from $j+p>i$. We shall prove that its rank is $\leqq p$. There must be some submatrix including the given one whose row-index i ranges over the interval $1\leqq i\leqq p+q$ and whose column-index j ranges over $q+1\leqq j\leqq n$, and we will prove that this later submatrix has rank $\leqq p$. We may also suppose that q is in the interval $1\leqq q\leqq n-p-1$, since any submatrix from the first p rows or the last p

columns of $\{a_{ij}\}^{-1}$ obviously has rank $\leqq p$. Denote the subspaces of V spanned by

$$(b_1 \ldots b_{p+q}), \quad (b_{p+q+1} \ldots b_n), \quad (b_1 \ldots b_q) \quad \text{and} \quad (b_{q+1} \ldots b_n)$$

by

$$R, \quad S, \quad T \quad \text{and} \quad U,$$

respectively. Introducing the corresponding projections r, s, t and u as in section 1, we have $tfs=0$ since $a_{ij}=0$ for $j>i+p$. Thus by the lemma,

$$\operatorname{rank}(rf^{-1}u) \leqq \dim R - \dim T = p .$$

Thus, $\{a_{ij}\}^{-1}$ is a Green's matrix of order p. On the other hand, suppose that $\{a_{ij}\}^{-1}$ is a Green's matrix of order p and choose some integer q in the interval $1 \leqq q \leqq m-p-1$. Using the lemma in the same way as above, we find that $a_{ij}=0$ for $1 \leqq i \leqq q$, $p+q+1 \leqq j \leqq n$. But every pair (i,j) of indices for which $j>i+p$, lies in an interval $1 \leqq i \leqq q$, $p+q+1 \leqq j \leqq n$ for some q, $1 \leqq q \leqq n-p-1$. This finishes the proof of theorem 1.

In a non-singular band matrix of order 0 all diagonal elements must be different from zero. The highest non-vanishing diagonal in a general non-singular band matrix can contain zeros. However, in the opposite case the following qualification of theorem 1 is valid.

THEOREM 2. *A non-singular square matrix over a field is a band matrix of order p with non-vanishing elements in the p:th diagonal above the main diagonal if and only if its inverse is the sum of a matrix of rank p and a band matrix of order $-p$ (and hence also a Green's matrix of order p).*

Let X be a matrix satisfying the conditions of the first half of the theorem. We partition off the last p rows and the first p columns of X and obtain the partitioned matrix

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

where X_{12} is a $(n-p)$-square invertible triangular matrix. By the usual theory for inverting block matrices, we find that the inverse

$$(X_{21} - X_{22}X_{12}^{-1}X_{11})^{-1} = Y$$

exists, and the block partitioning of X^{-1} which conforms with the given partitioning of X is

$$X^{-1} = \begin{pmatrix} 0 & 0 \\ X_{12}^{-1} & 0 \end{pmatrix} + \begin{pmatrix} E \\ -X_{12}^{-1}X_{11} \end{pmatrix} Y\,(-X_{22}X_{12}^{-1}, E) ,$$

where E denotes a unit matrix of order p. Thus X^{-1} satisfies the conditions of the latter half of theorem 2. Conversely, suppose that X^{-1} is the sum of a band matrix of order $-p$ and a matrix of rank p:

$$X^{-1} = \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} + \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}.$$

Here Z is a $(n-p)$-square triangular matrix and the second block matrix above has rank p. As

$$Z_{11}X_{11}+Z_{12}X_{21} = E ,$$

the matrix $(Z_{11}Z_{12})$ also has rank p and consequently the rows of $(Z_{21}Z_{22})$ are left linear combinations of the rows of $(Z_{11}Z_{12})$. Thus

$$Z_{11}X_{12}+Z_{12}X_{22} = 0 \quad \text{implies} \quad Z_{21}X_{12}+Z_{22}X_{22} = 0 ,$$

and hence $ZX_{12}=E'$, a unit matrix of order $n-p$. Since Z is triangular, X_{12} must be triangular and have non-vanishing diagonal elements. This proves the "if" part of theorem 2.

3. Extension to more general rings. With due modifications of the definitions, theorem 1 is valid also when the matrix elements belong to certain other types of rings, e.g. matrix rings over a field, or arbitrary commutative rings. It may be noted that in an arbitrary ring with unit not even the original statement about the inverse of a triangular matrix is necessarily true. Take a ring with two elements a, b which satisfy $ab=1$ but $ba \neq 1$. Then the two matrices

$$\begin{pmatrix} a & 0 \\ 1-ba & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b & 1-ba \\ 0 & a \end{pmatrix}$$

are inverses of each other, but the second matrix is not triangular in the sense used here.

In a ring with the property that every right or left inverse is also a two-sided inverse, the statement that the inverse of a triangular matrix (if it exists) is triangular, is true. The author does not know if this condition is also sufficient for the general validity of theorem 1, nor indeed if there is a suitable definition of Green's matrix in this case.

REFERENCE

1. S. O. Asplund, *Finite boundary value problems solved by Green's matrix*, Math. Scand. 7 (1959), 49–56.

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