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FREE PRODUCTS OF α-DISTRIBUTIVE BOOLEAN ALGEBRAS

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Introduction. In [12], Sikorski introduced the notion of the free product of a set of abstract algebras in a class $\mathfrak{A}$. He showed that these products exist under certain general conditions. In this paper we study the free product in the class of α-complete, α-distributive Boolean algebras, where α is an infinite cardinal number (see [6]). It is not obvious that Sikorski's existence proof applies to this class. Actually, it can be used, but to show this requires extension of Birkhoff's theorem in [1] on the existence of free algebras.

In section one, we provide a general theorem which can be used to establish directly the existence of free products of α-distributive Boolean algebras. Our theorem extensively overlaps Sikorski's, but its proof is different and the conditions for its application are somewhat simpler. Thus, its inclusion seems justified. Some general properties of free products of α-complete Boolean algebras are proved in section two. These are followed in section three by the central results of the paper: the existence and characterization of the free α-distributive product. This product is shown to be a generalization of the usual product of Borel fields of sets and it possesses most of the pleasant features of the latter. Finally, in section four, we examine the relation between the free products for the classes of α-complete, α-representable and α-distributive Boolean algebras. It is shown that if

$$\alpha \geqq \exp \aleph_0 ,$$

the free α-complete product of infinitely many non-trivial α-complete Boolean algebras is never the same as the free α-representable product of these algebras. However, we do prove that the free α-representable product coincides with the free α-distributive product for certain kinds of α-distributive algebras.

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Notation. The Greek alphabet is used as follows: α, β and γ denote infinite cardinal numbers, ξ, η, ζ, $\varkappa$, λ, μ and ν are used to represent ordinal numbers, ω, π, ϱ, σ and τ are reserved for indices, while φ, ψ and χ designate functions. In the Latin alphabet, a, b, c and d usually denote elements of a set or algebra, while g, h, i, j, k, l, p, q and r stand for mappings and particularly homomorphisms or isomorphisms. Capital letters represent sets or algebras and, in particular, capital German letters stand for sets of sets or classes. Set operations are denoted by rounded symbols in the usual way. The symbols Π and $\times$ represent the infinite and finite product symbols for sets. These same symbols will also be used in other technical senses which are explained below. For any set A, denote by $\mathfrak{P}(A)$ the set of all subsets of A. The set of all mappings of the set A into the set B is designated by the usual exponential notation B^A. If h is a mapping of the set A into the set B, then h induces mappings h of $\mathfrak{P}(A)$ into $\mathfrak{P}(B)$ and h^{-1} of $\mathfrak{P}(B)$ into $\mathfrak{P}(A)$ by the definitions

$$hC = \{hc \mid c \in C\}, \qquad \text{for} \quad C \subseteq A ,$$

and

$$h^{-1}D = \{a \in A \mid ha \in D\}, \qquad \text{for} \quad D \subseteq B .$$

The cardinality of a set A will be denoted by $|A|$. For typographical reasons, we denote

$$\exp \alpha = |\mathfrak{P}(A)|, \qquad \text{where} \quad |A| = \alpha .$$

Most of our notation and terminology for partially ordered sets and lattices is borrowed from [2]. The lattice operations of join, meet, complement and the inequality relation are designated $\vee$, $\wedge$, $(')$ and $\leqq$ (or $\geqq$) respectively. The symbols o and u (with subscripts or bars) always stand for the zero and unit of a Boolean algebra. The least upper bound of a set

$$A = \{a_\sigma \mid \sigma \in \Sigma\}$$

in a partially ordered set, when it exists, is designated either as

$$\text{l.u.b.}\, A \qquad \text{or} \qquad \bigvee_{\sigma \in \Sigma} a_\sigma .$$

A similar convention applies to greatest lower bounds.

A Boolean algebra B is called α-complete if every $A \subseteq B$ with $|A| < \alpha$ has a least upper bound in B. Throughout this paper, we are concerned with α-complete Boolean algebras almost exclusively. It is convenient therefore to suppress explicit reference to this fact. In particular, the terms homomorphisms, subalgebra and ideal will mean homomorphisms which preserve α-joins and subalgebras and ideals which are closed under α-joins. When exceptions to this convention occur, they will be so indicated.

A subset A of a Boolean algebra B is called an *α-partition* of B (or just partition, when its cardinality is immaterial) if l.u.b. $A = u$ (the unit of B), $|A| \leqq \alpha$ and A is *disjointed*, that is, if $a \neq b$ in A, then $a \wedge b = o$ (the zero of B). If $\{A_\sigma \mid \sigma \in \Sigma\}$ is a set of partitions of an α-complete Boolean algebra, and if $|\Sigma| \leqq \alpha$, then the product of these partitions, denoted

$$\Pi_{\sigma \in \Sigma} A_\sigma ,$$

is defined to be all distinct elements in B which are of the form

$$\bigwedge_{\sigma \in \Sigma} a_\sigma, \qquad \text{where} \quad a_\sigma \in A_\sigma .$$

An α-complete Boolean algebra B is α-distributive if and only if $\Pi_{\sigma \in \Sigma} A_\sigma$ is a partition for all choices of α-partitions A_σ, $\sigma \in \Sigma$, $|\Sigma| \leqq \alpha$ (see [6]). A Boolean algebra B is called α-representable if B is the homomorphic image of an α-field of sets.

1. Products of abstract algebras. The term abstract algebra will be used in the following (standard) sense. For any ordinal number $\varkappa$, a $\varkappa$-ary operation on a set A is a mapping

$$O\colon\ A^\varkappa \to A$$

of all well ordered sequences (of type $\varkappa$) of elements of A into A. Let

$$\langle \varkappa_0, \ldots, \varkappa_\xi, \ldots \rangle_{\xi < \lambda}$$

be a well ordered sequence of ordinal numbers. An *abstract algebra* (or just algebra) of type $\langle \varkappa_0, \ldots, \varkappa_\xi, \ldots \rangle_{\xi < \lambda}$ is a system

$$\langle A ; O_0, \ldots, O_\xi, \ldots \rangle_{\xi < \lambda} ,$$

where A is a non-empty set and O_ξ is a $\varkappa_\xi$-ary operation on A. As is customary, we will not distinguish between the abstract algebra $\langle A ; O_0, \ldots, O_\xi, \ldots \rangle_{\xi < \lambda}$ and the set A of its elements. The concepts of isomorphism, homomorphism, subalgebra and direct union of abstract algebras are defined in the usual way (see [2]). In this connection it should be noticed that the direct union in a class $\mathfrak{A}$ can be defined abstractly as the dual (in the sense of the theory of categories—see the appendix of [4]) of a free $\mathfrak{A}$-product, defined in 1.2 below. This procedure has two advantages. First, it extends the scope of the main existence theorem 1.5. More important however, it allows us to dualize this theorem: under conditions dual to those in 1.5 (omitting (iii) of course), the existence of a free $\mathfrak{A}$-product for all subsets of $\mathfrak{A}$ implies the existence of a direct union for all subsets of $\mathfrak{A}$. Of course an abstract direct union in $\mathfrak{A}$ may not be the same as the usual explicitly defined direct union.

If T is any non-empty subset of the algebra A, then there is a unique smallest subalgebra of A containing T, namely, the set intersection of all subalgebras containing T. This subalgebra is said to be generated by T. In the proof of the existence of free products (theorem 1.5 below), we need the following estimate of the cardinality of an algebra generated by a subset.

Lemma 1.1. *Let A be an abstract algebra of type $\langle \varkappa_0, \ldots, \varkappa_\xi, \ldots \rangle_{\xi<\lambda}$. Suppose T is a non-empty subset of A such that the subalgebra generated by T is all of A. Then*

$$|A| \leqq (|T| + 1)^\alpha ,$$

where $\alpha = \text{l.u.b. } \{\aleph_0, |\lambda|, |\varkappa_0|, \ldots, |\varkappa_\xi|, \ldots, \mid \xi < \lambda\}$.

Proof. Let $\beta = (|T|+1)^\alpha$. Then $\beta > \alpha$, $\alpha\beta = \beta$ and $\beta^\alpha = \beta$. Let μ be the least ordinal of cardinality $> \alpha$. By transfinite induction, define subsets $S_{(\eta,\xi)}$ of A, indexed by the well ordered lexicographic product $\mu \cdot \lambda$, as follows:

(a) $S_{(0,0)} = T \cup O_0(T)$,

(b) $S_{(\eta,\xi)} = T_{(\eta,\xi)} \cup O_\xi(T_{(\eta,\xi)})$, $\eta < \mu$, $\xi < \lambda$,

where

$$T_{(\eta,\xi)} = \bigcup\nolimits_{(\eta',\xi')<(\eta,\xi)} S_{(\eta',\xi')}$$

and $O_\xi(T_{(\eta,\xi)})$ is the set obtained by applying O_ξ to sequences from $T_{(\eta,\xi)}$. By induction,

$$|S_{(\eta,\xi)}| \leqq \beta \qquad \text{for all} \quad \eta < \mu \text{ and } \xi < \lambda .$$

Indeed,

$$|S_{(0,0)}| \leqq |T| + |T|^{|\varkappa_0|} \leqq \beta + \beta^\alpha = \beta .$$

Assuming $|S_{(\eta',\xi')}| \leqq \beta$ for all $(\eta', \xi') < (\eta, \xi)$ gives

$$|T_{(\eta,\xi)}| \leqq |\eta + 1| \cdot |\lambda| \cdot \beta \leqq \alpha \cdot \alpha \cdot \beta = \beta$$

and hence

$$|S_{(\eta,\xi)}| \leqq \beta + \beta^{|\varkappa_\xi|} \leqq \beta + \beta^\alpha = \beta .$$

Let

$$S = \bigcup\nolimits_{(\eta,\xi)\in\mu\cdot\lambda} S_{(\eta,\xi)} .$$

Then

$$|S| \leqq |\mu| \cdot |\lambda| \cdot \beta \leqq \beta^3 = \beta .$$

If $\langle a_0, \ldots, a_\zeta, \ldots \rangle_{\zeta < \varkappa_\xi}$ is a well ordered set of elements from S, then (since $|\varkappa_\xi| \leqq \alpha$) there is some $S_{(\eta,\nu)}$ containing all terms of this sequence. It follows that

$$O_\xi\langle a_0, \ldots, a_\zeta, \ldots \rangle \in S_{(\eta+1,\xi)} \subseteq S .$$

Hence, S is a subalgebra of A. But S contains T and T generates A, so that $S = A$.

The concept of the free product of a set of abstract algebras is defined relative to a class $\mathfrak{A}$ of algebras. It will always be assumed that the algebras in $\mathfrak{A}$ are of the same type. Explicit mention of this assumption will usually be omitted.

DEFINITION 1.2. *Let $\mathfrak{A}$ be a class of abstract algebras. Suppose $\{A_\omega \mid \omega \in \Omega\}$ is a subset of $\mathfrak{A}$. A system $\langle A ; i_\omega \rangle_{\omega \in \Omega}$ consisting of an algebra $A \in \mathfrak{A}$ and a family of isomorphisms*

$$i_\omega \colon A_\omega \to A$$

is called a free $\mathfrak{A}$-product of the set $\{A_\omega \mid \omega \in \Omega\}$ if the following extension property is satisfied:

(E) *if $\{h_\omega \mid \omega \in \Omega\}$ is a set of homomorphisms $h_\omega \colon A_\omega \to B$, where $B \in \mathfrak{A}$, then there is a unique homomorphism $h \colon A \to B$ such that $h_\omega = h \circ i_\omega$ for all ω.*

It is often convenient to speak of the algebra A as the free $\mathfrak{A}$-product of the set $\{A_\omega \mid \omega \in \Omega\}$. This abuse of terminology causes no trouble, since the context of the usage always makes the meaning clear.

It is by no means certain that free $\mathfrak{A}$-products exist. However, if the product of a set of algebras does exist, then it is unique in the following sense: for any two free $\mathfrak{A}$-products

$$\langle A ; i_\omega \rangle_{\omega \in \Omega} \qquad \text{and} \qquad \langle \bar{A} ; \bar{\imath}_\omega \rangle_{\omega \in \Omega}$$

of a set $\{A_\omega \mid \omega \in \Omega\} \subseteq \mathfrak{A}$, there exist unique inverse isomorphisms

$$i \colon \bar{A} \to A \qquad \text{and} \qquad \bar{\imath} \colon A \to \bar{A}$$

such that

$$\bar{\imath} \circ i_\omega = \bar{\imath}_\omega, \qquad i \circ \bar{\imath}_\omega = i_\omega .$$

Indeed, the existence and uniqueness of i and $\bar{\imath}$ comes immediately from 1.2. Since

$$i \circ \bar{\imath} \circ i_\omega = i_\omega \qquad \text{and} \qquad \bar{\imath} \circ i \circ \bar{\imath}_\omega = \bar{\imath}_\omega$$

the uniqueness in (E) requires that $i \circ \bar{\imath}$ and $\bar{\imath} \circ i$ be the identity mappings of A and $\bar{A}$ respectively.

The definition 1.2 of free $\mathfrak{A}$-products is somewhat different from Sikorski's (in [12]). Instead of assuming the existence of specific isomorphisms i_ω of the algebras A_ω into A, Sikorski postulates the existence of subalgebras $\bar{A}_\omega$ of A which are isomorphic to the given A_ω. He assumes that the set $\bigcup_{\omega\in\Omega}\bar{A}_\omega$ generates A, but he does not require that the extending homomorphism h in 1.2 be unique. However, in important cases, these two conditions are equivalent.

LEMMA 1.3. *Let* $\langle A; i_\omega\rangle_{\omega\in\Omega}$ *be a system having all the properties of* 1.2, *except possibly the uniqueness of the homomorphism* h *in* (E). *Assume that* $\bigcup\{i_\omega A_\omega \mid \omega\in\Omega\}$ *generates* A. *Then* h *is unique.*

Conversely, if the class $\mathfrak{A}$ *is closed under the formation of subalgebras and* $\langle A; i_\omega\rangle_{\omega\in\Omega}$ *is a free* $\mathfrak{A}$*-product, then* $\bigcup\{i_\omega A_\omega \mid \omega\in\Omega\}$ *generates* A.

PROOF. If h and $\bar{h}$ are homomorphisms of A such that $h\circ i_\omega=\bar{h}\circ i_\omega$ for all $\omega\in\Omega$, then $\{a\in A \mid ha=\bar{h}a\}$ is a subalgebra of A containing $\bigcup\{i_\omega A_\omega \mid \omega\in\Omega\}$. Since the latter set is a generator of A, this implies $h=\bar{h}$.

To prove the converse, let C be the subalgebra of A generated by $\bigcup\{i_\omega A_\omega \mid \omega\in\Omega\}$. By (E), there is a homomorphism $i\colon A\to C$ such that $i\circ i_\omega=i_\omega$. The uniqueness in (E) requires that i be the identity mapping on A. Hence, $C=A$.

COROLLARY 1.4. *Let* $\mathfrak{A}$ *be a class of algebras which is closed under formation of subalgebras. Suppose* $\langle A; i_\omega\rangle_{\omega\in\Omega}$ *is the free* $\mathfrak{A}$*-product of* $\{A_\omega \mid \omega\in\Omega\}$. *Assume* $\{h_\omega \mid \omega\in\Omega\}$ *is a set of homomorphisms*

$$h_\omega\colon\ A_\omega\to B,\qquad \text{where}\qquad B\in\mathfrak{A}\,.$$

Then the homomorphism

$$h\colon\ A\to B\qquad \text{satisfying}\qquad h_\omega = h\circ i_\omega$$

for all $\omega\in\Omega$ *is onto if and only if* $\bigcup\{h_\omega A_\omega \mid \omega\in\Omega\}$ *generates* B.

PROOF. Let C be the subalgebra of B generated by $\bigcup\{h_\omega A_\omega \mid \omega\in\Omega\}$. Then since

$$h_\omega A_\omega = h(i_\omega A_\omega)\subseteq hA\quad \text{for all } \omega\in\Omega\,,$$

it follows that $C\subseteq hA$. Also, $h^{-1}C$ is a subalgebra of A which contains all $i_\omega A_\omega$, so by 1.3, $h^{-1}C=A$. Thus

$$C = hh^{-1}C = hA\,.$$

We conclude this section by proving the existence theorem for free $\mathfrak{A}$-products. It is necessary of course to impose fairly strong restrictions on $\mathfrak{A}$.

Theorem 1.5. *Let* $\mathfrak{A}$ *be a class of abstract algebras of type*

$$\langle \varkappa_0, \ldots, \varkappa_\xi, \ldots \rangle_{\xi < \lambda}$$

with the following properties:

(i) *any algebra isomorphic to an algebra of* $\mathfrak{A}$ *is in* $\mathfrak{A}$;
(ii) *any subalgebra of an algebra of* $\mathfrak{A}$ *is in* $\mathfrak{A}$;
(iii) *any direct union of algebras of* $\mathfrak{A}$ *is in* $\mathfrak{A}$.

Let $\{A_\omega \mid \omega \in \Omega\}$ *be a subset of* $\mathfrak{A}$. *Suppose that for each* $\sigma \in \Omega$ *there is an algebra* $B_\sigma \in \mathfrak{A}$ *and a family* $h_{\sigma\omega}$ *of homomorphisms*

$$h_{\sigma\omega}\colon\ A_\omega \to B_\sigma$$

with $h_{\sigma\sigma}$ *an isomorphism. Then the free* $\mathfrak{A}$*-product of* $\{A_\omega \mid \omega \in \Omega\}$ *exists.*

Proof. (1) Let $\alpha = \text{l.u.b.}\ \{\aleph_0, |\lambda|, |\varkappa_0|, \ldots, |\varkappa_\xi|, \ldots \mid \xi < \lambda\}$ and

$$\beta = (1 + \Sigma_{\omega \in \Omega} |A_\omega|)^\alpha$$

as in 1.1. Choose a set M of cardinality β. Let

$$\mathfrak{B} = \{\langle B_\varrho, g_{\varrho\omega}\rangle_{\omega \in \Omega} \mid \varrho \in P\}$$

be the collection of all systems in which B_ϱ is an algebra of $\mathfrak{A}$, the elements of which are in M, and for each $\omega \in \Omega$, $g_{\varrho\omega}$ is a homomorphism of A_ω into B_ϱ. Note that the class $\mathfrak{B}$ is actually a set, in fact, a subset of

$$\mathfrak{P}(M) \times \Pi_{\xi < \lambda} \mathfrak{P}(M^{\varkappa_\xi + 1}) \times \Pi_{\omega \in \Omega}(M \times A_\omega)\ .$$

(2) The key to the proof of 1.5 is the following property of the collection $\mathfrak{B}$: if

$$h_\omega\colon\ A_\omega \to B \in \mathfrak{A}$$

is a set of homomorphisms, then there is a $\varrho \in P$ and an ismorphism

$$g\colon\ B_\varrho \to B$$

such that

$$h_\omega = g \circ g_{\varrho\omega} \quad \text{for all } \omega \in \Omega\ .$$

To prove this, let C be the subalgebra of B generated by $\bigcup \{h_\omega A_\omega \mid \omega \in \Omega\}$. Since $B \in \mathfrak{A}$, assumption (ii) implies $C \in \mathfrak{A}$. By lemma 1.1, it follows that $|C| \leqq \beta$. Thus, there is a one-to-one mapping φ of C into M. Let $N = \varphi C$. Clearly φ induces unique operations

$$O_0, \ldots, O_\xi, \ldots \qquad (\xi < \lambda)$$

on N in such a way that φ becomes an isomorphism. With these operations N is an algebra of $\mathfrak{A}$ because of (i), and the mappings $\varphi \circ h_\omega$ are homomorphisms of A_ω into N. Since $N \subseteq M$, there is a $\varrho \in P$ such that N is the algebra B_ϱ and $g_{\varrho\omega} = \varphi \circ h_\omega$. Finally, define $g = \varphi^{-1}$. Then g is an isomorphism of B_ϱ into B satisfying

$$h_\omega = g \circ g_{\varrho\omega} .$$

(3) Let $D = \Sigma_{\varrho \in P} B_\varrho$ be the direct union of the algebras of $\mathfrak{B}$. Assumption (iii) implies that $D \in \mathfrak{A}$. Denote by p_ϱ the projection homomorphism of D on B_ϱ. Define

$$i_\omega\colon\ A_\omega \to D \qquad \text{by} \qquad i_\omega(a) = (\ldots\, g_{\varrho\omega}(a)\, \ldots) ,$$

that is, i_ω is the unique homomorphism such that

$$p_\varrho \circ i_\omega = g_{\varrho\omega} .$$

By (2) and the last hypothesis of the theorem, there exists for each $\omega \in \Omega$, some ϱ such that $g_{\varrho\omega}$ is one-to-one. Thus, i_ω is an isomorphism of A_ω into D. Let A be the subalgebra of D which is generated by $\bigcup \{i_\omega A_\omega \mid \omega \in \Omega\}$. Then i_ω determines an isomorphism of A_ω into A (which will still be denoted by i_ω). The proof is completed by showing that $\langle A\,;\, i_\omega \rangle_{\omega \in \Omega}$ is the free $\mathfrak{A}$-product of $\{A_\omega \mid \omega \in \Omega\}$.

Suppose $\{h_\omega \mid \omega \in \Omega\}$ is a set of homomorphisms

$$h_\omega\colon\ A_\omega \to B \in \mathfrak{A} .$$

By (2), there exists $\varrho \in P$ and an isomorphism

$$g\colon\ B_\varrho \to B \qquad \text{satisfying} \qquad h_\omega = g \circ h_{\varrho\omega} .$$

Define

$$h\colon\ A \to B \qquad \text{by} \qquad h = g \circ q_\varrho ,$$

where q_ϱ is the restriction of p_ϱ to A. Then

$$h \circ i_\omega = g \circ q_\varrho \circ i_\omega = g \circ p_\varrho \circ i_\omega = g \circ g_{\varrho\omega} = h_\omega .$$

Since $\bigcup \{i_\omega A_\omega \mid \omega \in \Omega\}$ generates A, lemma 1.3 implies that h is unique. This completes the proof.

COROLLARY 1.6. *If the class $\mathfrak{A}$ of abstract algebras satisfies conditions* (i), (ii) *and* (iii) *of* 1.5, *and if, in addition, every algebra of $\mathfrak{A}$ contains a one element subalgebra, then every subset of $\mathfrak{A}$ has a free $\mathfrak{A}$-product.*

For in this case, we can satisfy the last condition of 1.5 by taking $B_\sigma = A_\sigma$, with $h_{\sigma\sigma}$ the identity on A_σ and $h_{\sigma\omega}$ the unique homomorphism of A_ω onto the one element subalgebra of A_σ (for $\sigma \neq \omega$). Another situation in which the last condition of 1.5 is obviously satisfied is where all the algebras A_ω are isomorphic. In particular (see Sikorski [12, p. 215]):

COROLLARY 1.7. *Let* $\mathfrak{A}$ *be a class of algebras satisfying* (i), (ii) *and* (iii) *of* 1.5. *Let* A_0 *be a free* $\mathfrak{A}$*-algebra with one generator and let* γ *be any cardinal number. Then the free* $\mathfrak{A}$*-algebra with* γ *generators exists and is the free* $\mathfrak{A}$*-product of* γ *replicas of* A_0. *More generally, the free* $\mathfrak{A}$*-product of any set of free* $\mathfrak{A}$*-algebras exists and is a free* $\mathfrak{A}$*-algebra.*

The free $\mathfrak{A}$-algebra with γ-generators is an algebra $B\in\mathfrak{A}$ containing a subset G of cardinality γ with the property that any mapping of G into an algebra $A\in\mathfrak{A}$ can be uniquely extended to a homomorphism of B into A.

PROOF OF COROLLARY 1.7. The existence of a free $\mathfrak{A}$-product of free $\mathfrak{A}$-algebras comes from 1.5 and the observation that a free $\mathfrak{A}$-algebra can be mapped homomorphically into any algebra of the class $\mathfrak{A}$. The fact that such a product is a free algebra is a consequence of the following easily verified associativity property (see [12, p. 214]): let $\{A_\omega \mid \omega\in\Omega\}$ be a set of $\mathfrak{A}$-algebras; suppose $\Omega=\bigcup_{\tau\in T}\Omega_\tau$, where the Ω_τ's are disjoint non-empty sets; assume that for each $\tau\in T$, the system $\langle\bar{A}_\tau;\, \bar{\imath}_{\tau\omega}\rangle_{\omega\in\Omega_\tau}$ is a free $\mathfrak{A}$-product of $\{A_\omega \mid \omega\in\Omega_\tau\}$ and that $\langle\bar{A};\, \bar{\imath}_\tau\rangle_{\tau\in T}$ is a free $\mathfrak{A}$-product of $\{\bar{A}_\tau \mid \tau\in T\}$. Then

$$\langle\bar{A};\, \bar{\imath}_\tau\circ\bar{\imath}_{\tau\omega}\rangle_{\omega\in\Omega_\tau,\,\tau\in T}$$

is a free $\mathfrak{A}$-product of $\{A_\omega \mid \omega\in\Omega\}$.

This corollary contains Rieger's theorem (in [9]) on the existence of a free α-complete Boolean algebra with γ generators, since the four element Boolean algebra is a free algebra with one generator. It is easy to modify the proof of 1.5 to establish the existence of free $\mathfrak{A}$-algebras directly. Indeed, this was the method of proof used by Rieger in the paper cited above. The same idea is also used in Birkhoff's paper [1].

2. Products of Boolean algebras. An α-complete Boolean algebra is an abstract algebra of type $\langle 1, \varkappa\rangle$, where $\varkappa$ is the least ordinal of cardinality α. Our interest will be directed toward free $\mathfrak{A}$-products, where $\mathfrak{A}$ is a subclass of the class of all α-complete Boolean algebras. Ultimately, we will concentrate on the class of α-complete, α-distributive algebras. However, it is possible to establish some interesting properties of free products of Boolean algebras in a more general setting.

If B is any Boolean algebra and h is a homomorphism of B, then h will be called a *principal homorphism* if its kernel is a principal ideal. Thus, if a is a non-zero element of B, the mapping

$$b\to b\wedge a$$

is a principal homomorphism of B onto

$$(a) = \{c \in B \mid c \leqq a\}$$

(with kernel (a')), and every principal homomorphism (onto) is equivalent to one of this form. The following restriction on the class $\mathfrak{A}$ will be needed in this section:

(iv) if B is in $\mathfrak{A}$ and h is a principal homomorphism of B onto C, then C is in $\mathfrak{A}$.

As noted in the proof of 1.7, free $\mathfrak{A}$-products satsify an infinite associative law. We will now prove that free products in certain classes of Boolean algebras also satisfy a distributive law.

PROPOSITION 2.1. *Let $\mathfrak{A}$ be a class of α-complete Boolean algebras satisfying conditions* (i), (ii), (iii) (*of* 1.5) *and* (iv). *Suppose*

$$\{B_\varrho \mid \varrho \in P\} \subseteq \mathfrak{A} \qquad \textit{and} \qquad |P| \leqq \alpha .$$

Let $B = \Sigma_{\varrho \in P} B_\varrho$ be the direct union of this set of Boolean algebras. Suppose that $B' \in \mathfrak{A}$ and for each $\varrho \in P$ that $\langle \bar{B}_\varrho; i'_\varrho, i_\varrho \rangle$, where

$$i'_\varrho:\ B' \to \bar{B}_\varrho \qquad \textit{and} \qquad i_\varrho:\ B_\varrho \to \bar{B}_\varrho ,$$

is a free $\mathfrak{A}$-product of $\{B', B_\varrho\}$. Put $\bar{B} = \Sigma_{\varrho \in P} \bar{B}_\varrho$. Then there exist isomorphisms

$$i':\ B' \to \bar{B} \qquad \textit{and} \qquad i:\ B \to \bar{B}$$

such that $\langle \bar{B}; i', i \rangle$ is a free $\mathfrak{A}$-product of $\{B', B\}$. In less precise terms:

$$B' \times (\Sigma_{\varrho \in P} B_\varrho) = \Sigma_{\varrho \in P} B' \times B_\varrho .$$

PROOF. (1) Let $q_\varrho: B \to B_\varrho$ and $p_\varrho: \bar{B} \to \bar{B}_\varrho$ be the component projection homomorphisms. Define $i': B' \to \bar{B}$, $i: B \to \bar{B}$ to be the unique homomorphisms satisfying

$$p_\varrho \circ i' = i'_\varrho, \qquad p_\varrho \circ i = i_\varrho \circ q_\varrho .$$

Thus,

$$i'(b) = (\ldots i'_\varrho(b) \ldots), \qquad i(c) = (\ldots i_\varrho(q_\varrho c) \ldots) .$$

It is clear that i' and i are isomorphisms.

(2) We will next show that $i'B' \cup iB$ generates $\bar{B}$. Let $\bar{C}$ be the subalgebra of $\bar{B}$ generated by $i'B' \cup iB$. For each $\tau \in P$, $p_\tau \bar{C}$ is a subalgebra of $\bar{B}_\tau$ with the property

$$p_\tau \bar{C} \supseteq p_\tau(i'B' \cup iB) = p_\tau(i'B') \cup p_\tau(iB) = i'_\tau B' \cup i_\tau(q_\tau B) = i'_\tau B' \cup i_\tau B_\tau .$$

By 1.3, this implies $p_\tau \bar{C} = \bar{B}_\tau$. Hence, if $\bar{a} \in \bar{B}$, there exists $\bar{c}_\tau \in \bar{C}$ such that

$$p_\tau \bar{c}_\tau = p_\tau \bar{a} \qquad \text{for each} \qquad \tau \in P .$$

Let $b_\varrho = (\ldots o_\sigma \ldots u_\varrho \ldots o_\tau \ldots) \in B$, so that $q_\tau b_\varrho = o_\tau$ if $\tau \neq \varrho$ and $q_\varrho b_\varrho = u_\varrho$. Set

$$\bar{c} = \bigvee_{\varrho \in P} (ib_\varrho \wedge \bar{c}_\varrho) .$$

Then $\bar{c} \in \bar{C}$, because $|P| \leqq \alpha$. Moreover, for all $\tau \in P$,

$$p_\tau \bar{c} = \bigvee_{\varrho \in P} \left(p_\tau(ib_\varrho) \wedge p_\tau \bar{c}_\varrho\right) = \bigvee_{\varrho \in P} \left(i_\tau(q_\tau b_\varrho) \wedge p_\tau \bar{c}_\varrho\right) = p_\tau \bar{c}_\tau = p_\tau \bar{a} .$$

Hence, $\bar{a} = \bar{c} \in \bar{C}$.

(3) Let $A \in \mathfrak{A}$ and suppose $h' \colon B' \to A$, $h \colon B \to A$ are homomorphisms. Define $b_\varrho \in B$ as in (2). Then $\bigvee_{\varrho \in P} b_\varrho = u$ and $b_\varrho \wedge b_\tau = o$ if $\varrho \neq \tau$. Moreover, there is an isomorphism $j_\varrho \colon B_\varrho \to (b_\varrho)$ satisfying

$$j_\varrho(q_\varrho b) = b \wedge b_\varrho \qquad \text{for all} \quad b \in B .$$

Let $a_\varrho = hb_\varrho \in A$. By (iv), $(a_\varrho) \in \mathfrak{A}$, provided $a_\varrho \neq o$. Let $r_\varrho \colon A \to (a_\varrho)$ be the corresponding principal homomorphism. For any $b \in B$,

$$(r_\varrho \circ h)(b) = a_\varrho \wedge hb = hb_\varrho \wedge hb = (h \circ j_\varrho \circ q_\varrho)(b) .$$

Thus,

$$r_\varrho \circ h = h \circ j_\varrho \circ q_\varrho .$$

By the extension property of the free product $\langle \bar{B}_\varrho ; i'_\varrho, i_\varrho \rangle$, there exists (for each ϱ such that $a_\varrho \neq o$) a homomorphism $\bar{h}_\varrho \colon \bar{B}_\varrho \to (a_\varrho)$ satisfying

$$h \circ j_\varrho = \bar{h}_\varrho \circ i_\varrho \qquad \text{and} \qquad r_\varrho \circ h' = \bar{h}_\varrho \circ i'_\varrho .$$

The composition $\bar{h}_\varrho \circ p_\varrho$ maps $\bar{B}$ into (a_ϱ). Define $\bar{h} \colon \bar{B} \to A$ by

$$\bar{h}\bar{b} = \text{l.u.b.}\ \{\bar{h}_\varrho(p_\varrho \bar{b}) \mid \varrho \in P,\ a_\varrho \neq o\} .$$

Since $\bigvee_{\varrho \in P} a_\varrho = u$ and $a_\varrho \wedge a_\tau = o$ for $\varrho \neq \tau$, the mapping $\bar{h}$ is a homomorphism satisfying

$$r_\varrho \circ \bar{h} = \bar{h}_\varrho \circ p_\varrho .$$

Thus,

$$r_\varrho \circ \bar{h} \circ i = \bar{h}_\varrho \circ p_\varrho \circ i = \bar{h}_\varrho \circ i_\varrho \circ q_\varrho = h \circ j_\varrho \circ q_\varrho = r_\varrho \circ h$$

and

$$r_\varrho \circ \bar{h} \circ i' = \bar{h}_\varrho \circ p_\varrho \circ i' = \bar{h}_\varrho \circ i'_\varrho = r_\varrho \circ h'$$

for all ϱ such that $a_\varrho \neq o$. This implies

$$\bar{h} \circ i = h \qquad \text{and} \qquad \bar{h} \circ i' = h'.$$

It follows from 1.3, (2) and (3) that $\langle \bar{B} ; i', i \rangle$ is a free product of $\{B', B\}$.

DEFINITION 2.3. *Let B be an α-complete Boolean algebra and let $\mathfrak{P} = \{A_\omega \mid \omega \in \Omega\}$ be a set of subalgebras of B. The set $\mathfrak{P}$ is called α-independent in B if for any subset $\Sigma \subseteq \Omega$ with $|\Sigma| \leqq \alpha$ and any choice of $a_\sigma \in A_\sigma$ with $a_\sigma \neq o$ for all $\sigma \in \Sigma$, the greatest lower bound $\bigwedge_{\sigma \in \Sigma} a_\sigma$ is not zero.*

PROPOSITION 2.4. *Let $\mathfrak{A}$ be a class of α-complete Boolean algebras satisfying* (i), (ii), (iii), (iv) *and having the property that every subset of $\mathfrak{A}$ admits a free $\mathfrak{A}$-product. Suppose $\langle B; i_\omega\rangle_{\omega\in\Omega}$ is a free $\mathfrak{A}$-product of the set $\{B_\omega \mid \omega\in\Omega\}$. Then the collection of subalgebras $\{i_\omega B_\omega \mid \omega\in\Omega\}$ is α-independent in B.*

PROOF. Let $\Sigma\subseteq\Omega$, $|\Sigma|\leqq\alpha$ and $i_\sigma a_\sigma\neq o$ in B for all $\sigma\in\Sigma$. The proposition is proved by showing

$$\bigwedge_{\sigma\in\Sigma} i_\sigma a_\sigma \neq o\,.$$

Define $A_\omega=B_\omega$ if $\omega\notin\Sigma$ and $A_\sigma=(a_\sigma)$ for $\sigma\in\Sigma$. By (iv), $A_\omega\in\mathfrak{A}$ for all $\omega\in\Omega$. Denote by p_ω the principal homomorphism of B_ω on A_ω (the identity if $\omega\notin\Sigma$). Let $\langle A; j_\omega\rangle_{\omega\in\Omega}$ be a free $\mathfrak{A}$-product of $\{A_\omega \mid \omega\in\Omega\}$. This exists by assumption. Since $\langle B; i_\omega\rangle_{\omega\in\Omega}$ is a free product, there exists a homomorphism $h\colon B\to A$ satisfying $h\circ i_\omega=j_\omega\circ p_\omega$ for all $\omega\in\Omega$. Then

$$h\Big(\bigwedge_{\sigma\in\Sigma} i_\sigma a_\sigma\Big) = \bigwedge_{\sigma\in\Sigma} j_\sigma(p_\sigma a_\sigma) = \bigwedge_{\sigma\in\Sigma} j_\sigma u_\sigma = u \neq o$$

in A. Thus, $\bigwedge_{\sigma\in\Sigma} i_\sigma a_\sigma\neq o$.

3. Free products of α-distributive Boolean algebras. To prove the existence of free α-distributive products it is sufficient to show that the last condition of 1.5 is satisfied. We will prove a slightly stronger result: for any set $\{B_\omega \mid \omega\in\Omega\}$ of α-distributive Boolean algebras, there is an α-distributive Boolean algebra B containing subalgebras isomorphic to the B_ω's.

Let $\{B_\omega \mid \omega\in\Omega\}$ be a given set of Boolean algebras. Define Φ to be the set of all functions φ on Ω such that

(a) $\varphi(\omega)$ is a non-zero element of B_ω,
(b) $\{\omega \mid \varphi(\omega)\neq u_\omega\}$ has cardinality at most α.

In addition, let Φ contain the symbol o. Define $\varphi\leqq\omega$ if $\varphi(\omega)\leqq\psi(\omega)$ for all $\omega\in\Omega$. Set $o\leqq\varphi$ for all $\varphi\in\Phi$. Then Φ becomes a meet closed partially ordered set with

(c) $$(\varphi\wedge\psi)(\omega) = \begin{cases} \varphi(\omega)\wedge\psi(\omega) & \text{if } \varphi(\omega)\wedge\varphi(\omega)\neq o \text{ for all } \omega\in\Omega\,, \\ o & \text{otherwise.}\end{cases}$$

Moreover Φ is disjunctive. Indeed, $\varphi\nleqq\psi$ implies $\varphi\neq o$ and $\varphi(\omega_0)\nleqq\psi(\omega_0)$ for some ω_0, or else $\psi=o$. Define χ by $\chi(\omega)=\varphi(\omega)$ for $\omega\neq\omega_0$ and $\chi(\omega_0)=\varphi(\omega_0)\wedge(\psi(\omega_0))'$, or $\chi=\varphi$ if $\psi=o$. Then

$$o\neq\chi\leqq\varphi \quad\text{and}\quad \chi\wedge\psi=o\,.$$

The disjunctive property implies that it is possible to embed Φ as a dense sub-semi-lattice in a complete Boolean algebra $\bar{B}$ (see [3]). Denote by $\Pi_{\omega\in\Omega}B_\omega$ the subalgebra of $\bar{B}$ which is generated by Φ.

For a later application we define another Boolean algebra by replacing (b) above by

(b′) $\{\omega \mid \varphi(\omega) \neq u_\omega\}$ is finite.

Denote the Boolean algebra obtained in this way by $\Pi'_{\omega\in\Omega}B_\omega$.

We now define mappings $j_\pi\colon B_\pi \to \Pi_{\omega\in\Omega}B_\omega$. For $a \neq o_\pi$ in B_π let $j_\pi a$ be the function on Φ defined by

$$(j_\pi a)(\pi) = a, \qquad (j_\pi a)(\omega) = u_\omega \quad \text{if} \quad \omega \neq \pi, \qquad \text{and} \qquad j_\pi o_\pi = o\,.$$

Clearly j_π is one-to-one. It preserves finite meets by (c). Suppose $A \subseteq B_\pi$ and l.u.b. $A = b$. Evidently $j_\pi b$ is an upper boound of the set $\{j_\pi a \mid a \in A\}$. If $j_\pi b$ were not the least upper bound there would exist some $\psi \neq o$ in Φ such that $\psi \leqq j_\pi b$, but $\psi \wedge j_\pi a = o$ for all $a \in A$. If $\omega \neq \pi$,

$$(j_\pi a)(\omega) = u_\omega, \qquad \text{so} \qquad \psi(\omega) \wedge j_\pi a(\omega) \neq o_\omega\,.$$

Hence $\psi(\pi) \wedge (j_\pi a)(\pi) = o_\pi$, that is,

$$\psi(\pi) \wedge a = o \qquad \text{for all} \quad a \in A\,.$$

Therefore $\psi(\pi) \wedge b = o_\pi$ and consequently $\psi \wedge j_\pi b = o$. This contradicts $o \neq \psi \leqq j_\pi b$. Hence

$$j_\pi b = \text{l.u.b.}\,\{j_\pi a \mid a \in A\}\,.$$

This shows that j_π is an isomorphism which preserves all existing least upper bounds in B_π.

In the same way we can define isomorphisms $j'_\pi\colon B_\pi \to \Pi'_{\omega\in\Omega}B_\omega$. These isomorphisms also preserve any bounds which exist in B_π.

Definition 3.1. *Let $\{B_\omega \mid \omega\in\Omega\}$ be a set of α-complete Boolean* alge-bras. *The system*

$$\langle \Pi_{\omega\in\Omega}B_\omega; j\rangle_{\omega\in\Omega}$$

will be called the minimal α-product of this set.

The concept of a minimal α-product generalizes to arbitrary cardinal α the minimal σ-product introduced in Sikorski's paper [10]. However Sikorski defines this concept in a somewhat different way. The choice of terminology is justified by 3.10 below.

For convenience we summarize some evident properties of the products defined above.

Proposition 3.2. *Let $\{B_\omega \mid \omega\in\Omega\}$ be a set of α-complete Boolean* *algebras. Then* $\Pi_{\omega\in\Omega}B_\omega$ *and* $\Pi'_{\omega\in\Omega}B_\omega$ *are* α*-complete Boolean algebras and the mappings* j_ω *and* j'_ω *are* α*-homomorphisms. Moreover,*

(a) $\bigcup\{j_\omega B_\omega \mid \omega\in\Omega\}$ *generates* $\Pi_{\omega\in\Omega}B_\omega$ *and* $\bigcup\{j'_\omega B_\omega \mid \omega\in\Omega\}$ *generates* $\Pi'_{\omega\in\Omega}B_\omega$;

(b) *the subalgebras* $\{j_\omega B_\omega \mid \omega\in\Omega\}$ *are* α*-independent in* $\Pi_{\omega\in\Omega}B_\omega$;

(c) $\{\bigwedge_{\sigma\in\Sigma} j_\sigma a_\sigma \mid \Sigma\subseteq\Omega,\ |\Sigma|\leqq\alpha,\ a_\sigma\in B_\sigma\}$ *is dense in* $\Pi_{\omega\in\Omega}B_\omega$;

(d) *if* T *is an infinite subset of* Ω *and if, for each* $\tau\in T$, a_τ *is an element of* B_τ *distinct from* u_τ, *then* $\bigwedge_{\tau\in T} j'_\tau a_\tau = o$ *in* $\Pi'_{\omega\in\Omega}B_\omega$.

The above construction, together with 1.5, establishes the existence of free products in the class of α-complete Boolean algebras (see [12]). In section 4, we will show that if each B_ω above is α-representable, then $\Pi_{\omega\in\Omega}B_\omega$ is also α-representable. By this means, Sikorski's theorem on the existence of the free product in the class of α-representable Boolean algebras is obtained. Our present aim is to show that if all B_ω are α-distributive, then $\Pi_{\omega\in\Omega}B_\omega$ is also α-distributive, so that the free product exists in the class of α-distributive algebras as well. The following lemma, together with 3.2, proves this result.

LEMMA 3.3. *Let* B *be an* α*-complete Boolean algebra. Suppose* $\{B_\omega \mid \omega\in\Omega\}$ *is an* α*-independent set of subalgebras of* B *such that each* B_ω *is* α*-distributive and* $\bigcup\{B_\omega \mid \omega\in\Omega\}$ *is an* α*-generator of* B. *Then the following are equivalent:*

(a) B *is* α*-distributive;*

(b) *the set* $\{\bigwedge_{\sigma\in\Sigma} a_\sigma \mid \Sigma\subseteq\Omega,\ |\Sigma|\leqq\alpha,\ a_\sigma\in B_\sigma\}$ *is dense in* B.

PROOF. Let $\bar{B}$ be the normal completion of B, i.e., the unique complete Boolean algebra containing B as a dense subalgebra. It is well known (see [2, p. 58]) that the least upper bounds (when they exist) in B coincide with those in $\bar{B}$. Let $\mathfrak{P}$ be the collection of all sets of the form

$$A = \prod_{\sigma\in\Sigma}\{a_{\sigma 1}, a_{\sigma 2}\} = \left\{\bigwedge_{\sigma\in\Sigma} a_{\sigma\varphi(\sigma)} \mid \varphi\in 2^{\Sigma}\right\},$$

where $a_{\sigma 2} = (a_{\sigma 1})'$ belongs to $\bigcup\{B_\omega \mid \omega\in\Omega\}$ and $|\Sigma|\leqq\alpha$. The elements of every $A\in\mathfrak{P}$ are pairwise disjoint, since if $\varphi\neq\psi$ there exists $\tau\in\Sigma$ with $\varphi(\tau)\neq\psi(\tau)$. Hence

$$a_{\tau\varphi(\tau)} = (a_{\tau\psi(\tau)})'$$

and

$$\left(\bigwedge_{\sigma\in\Sigma} a_{\sigma\varphi(\sigma)}\right)\wedge\left(\bigwedge_{\sigma\in\Sigma} a_{\sigma\psi(\sigma)}\right) \leqq a_{\tau\varphi(\tau)}\wedge a_{\tau\psi(\tau)} = o\ .$$

We will show that under either hypothesis (a) or (b) each $A\in\mathfrak{P}$ is a partition, that is,

$$\text{l.u.b. } A = u .$$

If B is α-distributive this is clear:

$$\text{l.u.b. } A = \bigvee_{\varphi \in 2^{\Sigma}} \bigwedge_{\sigma \in \Sigma} a_{\sigma\varphi(\sigma)} = \bigwedge_{\sigma \in \Sigma} (a_{\sigma_1} \vee a_{\sigma_2}) = u$$

in B and hence also in $\bar{B}$. To prove that $\Pi_{\sigma \in \Sigma}\{a_{\sigma_1}, a_{\sigma_2}\}$ is a partition under hypothesis (b), let Σ be written as a disjoint union

$$\Sigma = \bigcup_{\tau \in T'} \Sigma_\tau, \qquad \text{where} \qquad T' \subseteq \Omega \quad \text{and} \quad \sigma \in \Sigma_\tau$$

if and only if

$$\{a_{\sigma_1}, a_{\sigma_2}\} \subseteq B_\tau .$$

(Note that since the algebras B_ω are independent, $B_\omega \cap B_\tau = \{o, u\}$ for $\omega \neq \tau$. There is no loss of generality in assuming that $a_{\sigma_1} \neq o$, u for all $\sigma \in \Sigma$.) For each $\tau \in T'$, the product $\Pi_{\sigma \in \Sigma_\tau}\{a_{\sigma_1}, a_{\sigma_2}\}$ is a partition of B_τ by the α-distributivity. To prove that l.u.b. $A = u$, it suffices by (b) to show that every non-zero $\bigwedge_{\tau \in T} a_\tau$ ($T \subseteq \Omega$, $|T| \leqq \alpha$, $a_\tau \in B_\tau$) has non-zero meet with some element of A. If $\tau \in T \cap T'$, then a_τ has non-zero meet with an element of the partition $\Pi_{\sigma \in \Sigma_\tau}\{a_{\sigma_1}, a_{\sigma_2}\}$, say

$$b_\tau = a_\tau \wedge \bigwedge_{\sigma \in \Sigma_\tau} a_{\sigma\varphi_\tau(\sigma)} \neq o, \qquad \text{where} \qquad \varphi_\tau \in 2^{\Sigma_\tau}.$$

For $\tau \in T - (T \cap T')$, let $b_\tau = a_\tau$. For $\tau \in T' - (T \cap T')$, choose an arbitrary $\varphi_\tau \in 2^{\Sigma_\tau}$ such that

$$\bigwedge_{\sigma \in \Sigma_\tau} a_{\sigma\varphi_\tau(\sigma)} \neq o$$

and let b_τ be this non-zero element of B. Since the family $\{B_\omega \mid \omega \in \Omega\}$ is α-independent and the b_τ are not zero,

$$b = \bigwedge_{\tau \in T \cup T'} b_\tau \neq o .$$

By construction,

$$b \leqq \bigwedge_{\tau \in T} a_\tau$$

and

$$b \leqq \bigwedge_{\tau \in T'} \bigwedge_{\sigma \in \Sigma_\tau} a_{\sigma\varphi_\tau(\sigma)} = \bigwedge_{\sigma \in \Sigma} a_{\sigma\varphi(\sigma)} ,$$

where $\varphi \in 2^{\Sigma}$ is defined by $\varphi(\sigma) = \varphi_\tau(\sigma)$ for $\sigma \in \Sigma_\tau$. Hence

$$\bigwedge_{\tau \in T} a_\tau \wedge \bigwedge_{\sigma \in \Sigma} a_{\sigma\varphi(\sigma)} \neq o ,$$

which is the required conclusion.

We next observe that the family $\mathfrak{P}$ has the α-refinement property, that is, if

$$\{A_\varrho \mid \varrho \in P\} \subseteq \mathfrak{P} \qquad \text{and} \qquad |P| \leqq \alpha ,$$

there exists $A \in \mathfrak{P}$ such that A refines every A_ϱ: for each $a \in A$ there exists $a_\varrho \in A_\varrho$ with $a \leqq a_\varrho$. Indeed, suppose

$$A_\varrho = \Pi_{\sigma \in \Sigma_\varrho} \{a^\varrho_{\sigma_1}, a^\varrho_{\sigma_2}\} \qquad \text{for each} \quad \varrho \in P$$

(where $a^\varrho_{\sigma_1} = (a^\varrho_{\sigma_2})'$ and $|\Sigma_\varrho| \leqq \alpha$). Then clearly

$$A = \Pi_{\varrho \in P} \Pi_{\sigma \in \Sigma_\varrho} \{a^\varrho_{\sigma_1}, a^\varrho_{\sigma_2}\}$$

is in $\mathfrak{P}$ and refines every A_ϱ.

Let C be the set of all elements of $\bar{B}$ which are the least upper bounds of subsets of the members of $\mathfrak{P}$. By [7, lemma 3.2], C is an α-complete, α-distributive Boolean algebra containing all B_ω and such that the set of all elements

$$\left\{ \bigwedge_{\sigma \in \Sigma} a_\sigma \mid \Sigma \subseteq \Omega, \ |\Sigma| \leqq \alpha, \ a_\sigma \in B_\sigma \right\}$$

is dense in C. Since $\bigcup\{B_\omega \mid \omega \in \Omega\}$ generates B, it follows that $B \subseteq C$. This conclusion is true under either of the hypotheses (a) or (b). Thus, if (b) holds, it follows that B is α-distributive (since it is a subalgebra of an α-distributive Boolean algebra). If (a) holds, then the set

$$\left\{ \bigwedge_{\sigma \in \Sigma} a_\sigma \mid \Sigma \subseteq \Omega, \ |\Sigma| \leqq \alpha, \ a_\sigma \in B_\sigma \right\}$$

is contained in B and is dense in C. Hence, this set is also dense in B.

Corollary 3.4. *The minimal α-product algebra $\Pi_{\omega \in \Omega} B_\omega$ of α-complete, α-distributive Boolean algebras is α-distributive.*

Theorem 3.5. *The free α-distributive product of any set of α-complete, α-distributive Boolean algebras exists.*

Proof. This follows from 3.4 and 1.5.

Lemma 3.6. *The free α-distributive product $\langle B; i_\omega \rangle_{\omega \in \Omega}$ of a set $\{B_\omega \mid \omega \in \Omega\}$ of α-distributive Boolean algebras has the property that the set*

$$\left\{ \bigwedge_{\sigma \in \Sigma} i_\sigma b_\sigma \mid \Sigma \subseteq \Omega, \ |\Sigma| \leqq \alpha, \ b_\sigma \in B_\sigma \right\}$$

is dense in B.

Proof. By 1.3, 2.4 and the assumed α-distributivity the hypotheses of 3.3 are satisfied.

Proposition 3.7. *Let $\langle B; i_\omega \rangle_{\omega \in \Omega}$ be a free α-distributive product of the set $\{B_\omega \mid \omega \in \Omega\}$ of α-distributive Boolean algebras. Suppose A is α-distributive and $h_\omega \colon B_\omega \to A$ are homomorphisms for each $\omega \in \Omega$. Let $h \colon B \to A$ be the homomorphism such that $h_\omega = h \circ i_\omega$. Then h is one-to-one if and only* *if each h_ω is one-to-one and $\{h_\omega B_\omega \mid \omega \in \Omega\}$ is an α-independent set of subalgebras of A.*

Proof. Suppose h is one-to-one. Then since $h_\omega = h \circ i_\omega$, each h_ω is also one-to-one. Moreover, if $\Sigma \subseteq \Omega$, $|\Sigma| \leqq \alpha$ and $a_\sigma \in B_\sigma$ are non-zero elements for every $\sigma \in \Sigma$, then

$$\bigwedge_{\sigma \in \Sigma} h_\sigma a_\sigma = \bigwedge_{\sigma \in \Sigma} h(i_\sigma a_\sigma) = h\Big(\bigwedge_{\sigma \in \Sigma} i_\sigma a_\sigma\Big) \neq o$$

by 2.4 and the fact that h is one-to-one. Thus $\{h_\omega B_\omega \mid \omega \in \Omega\}$ is an α-independent set of subalgebras of A.

Conversely, suppose each h_ω is one-to-one and $\{h_\omega B_\omega \mid \omega \in \Omega\}$ is an α-independent set. Let $b \neq o$ in B. By 3.6 there is a set $\Sigma \subseteq \Omega$ of cardinality at most α and for each $\sigma \in \Sigma$ an element $b_\sigma \in B_\sigma$ such that

$$o \neq \bigwedge_{\sigma \in \Sigma} i_\sigma b_\sigma \leqq b .$$

Then clearly $b_\sigma \neq o$ for all σ. Thus,

$$o \neq \bigwedge_{\sigma \in \Sigma} h_\sigma b_\sigma = h\Big(\bigwedge_{\sigma \in \Sigma} i_\sigma b_\sigma\Big) \leqq hb .$$

This shows that the kernel of h is zero so that h is one-to-one.

Theorem 3.8. *Let $\{B_\omega \mid \omega \in \Omega\}$ be a set of α-distributive Boolean algebras. A system $\langle A; j_\omega \rangle_{\omega \in \Omega}$ consisting of an α-complete, α-distributive Boolean algebra A and a family of isomorphisms j_ω: $B_\omega \to A$ (for each $\omega \in \Omega$) is a free α-distributive product of the set $\{B_\omega \mid \omega \in \Omega\}$ if and only if*

(a) *$\{j_\omega B_\omega \mid \omega \in \Omega\}$ is an α-independent set of subalgebras of A, and,*
(b) *$\bigcup \{j_\omega B_\omega \mid \omega \in \Omega\}$ generates B.*

Proof. By 1.3 and 2.4 a free α-distributive product has properties (a) and (b). Conversely suppose $\langle A; j_\omega \rangle_{\omega \in \Omega}$ has properties (a) and (b). Let $\langle B; i_\omega \rangle_{\omega \in \Omega}$ be a free α-distributive product of $\{B_\omega \mid \omega \in \Omega\}$. Then there is a homomorphism j: $B \to A$ such that $j_\omega = j \circ i_\omega$ for all $\omega \in \Omega$. By 1.4 and (b), j maps B onto A. By 3.7 and (a), j is one-to-one. It follows easily that $\langle A; j_\omega \rangle_{\omega \in \Omega}$ is a free α-distributive product of $\{B_\omega \mid \omega \in \Omega\}$.

Corollary 3.9. *The minimal α-product of α-distributive Boolean algebras is a free α-distributive product of these algebras.*

Proof. By 3.2, 3.4 and 3.8.

Corollary 3.10. *Let $\mathfrak{A}$ be a class of α-distributive Boolean algebras satisfying conditions* (i), (ii), (iii) *of* 1.5, (iv) *of section* 2, *and such that* *every subset of $\mathfrak{A}$ has a free $\mathfrak{A}$-product. Then a free $\mathfrak{A}$-product of a subset of $\mathfrak{A}$ is also a free α-distributive product of this subset.*

PROOF. By 1.4, 2.4 and 3.8.

In particular it follows from 3.10 that the usual α-product of a set of α-fields is a free α-distributive product of these fields (considered as α-distributive Boolean algebras).

4. Relations between free products. Let $\{B_\omega \mid \omega\in\Omega\}$ be a set of α-distributive Boolean algebras. Then it is possible to form various free products of these algebras, according to the class of algebras in which they are considered. Important cases are the free α-complete product, the free α-representable product, and the free α-distributive product. In this section we consider the relations among these various products and in particular the conditions under which they are the same.

LEMMA 4.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be classes of α-complete Boolean algebras satisfying conditions* (i) *and* (ii) *of* 1.5. *Assume $\mathfrak{A}\subseteq\mathfrak{B}$. Let $\{B_\omega \mid \omega\in\Omega\}$ be a subset of $\mathfrak{A}$ and suppose that $\langle B; i_\omega\rangle_{\omega\in\Omega}$, $\langle B'; i'_\omega\rangle_{\omega\in\Omega}$ are free $\mathfrak{A}$- and $\mathfrak{B}$-products respectively of this set of algebras. Then there is a unique homomorphism h of B' onto B such that*

$$i_\omega = h\circ i'_\omega \quad \text{for all} \quad \omega\in\Omega .$$

Moreover the following are equivalent:

(a) *h is one-to-one,*
(b) $B'\in\mathfrak{A}$.

If $\mathfrak{A}$ is closed under the formation of homomorphic images, then (b) *is equivalent to*

(c) *if $\bar{B}\in\mathfrak{B}$ contains subalgebras $\bar{B}_\omega$ isomorphic to B_ω such that $\bigcup_{\omega\in\Omega}\bar{B}_\omega$ generates $\bar{B}$, then $\bar{B}\in\mathfrak{A}$.*

This lemma is an elementary consequence of definition 1.2 and we omit its proof.

The theorem of Loomis implies that the classe of $\aleph_0$-complete Boolean algebras coincides with the class of $\aleph_0$-representable algebras. This is not true for $\alpha\geqq\exp(\aleph_0)$, so we may ask whether the α-complete product of α-representable algebras differs from the α-representable product of these algebras. The following theorem settles this question for infinite products.

THEOREM 4.2. *Let $\{B_\omega \mid \omega\in\Omega\}$ be a set of α-representable Boolean* *algebras, where* $\alpha \geqq \exp(\aleph_0)$. *Assume that infinitely many* B_ω*'s contain at least four elements. Then the free* α*-complete product of this set is not the* α*-representable product.*

PROOF. Let $\bar{B}$ be the α-complete algebra $\Pi'_{\omega \in \Omega} B_\omega$ defined in section 3. Let $\bar{B}_\omega = j'_\omega B_\omega$. By 3.2 the algebra $\bar{B}_\omega$ is an α-subalgebra of $\bar{B}$ and $\bigcup_{\omega \in \Omega} \bar{B}_\omega$ generates $\bar{B}$. Hence 4.2 follows from 4.1 (c) by showing that $\bar{B}$ is not α-representable. If $\bar{B}$ were α-representable, it would be $\aleph_0$-distributive because $\alpha \geqq \exp(\aleph_0)$. Choose $\omega_1, \omega_2, \ldots, \omega_n, \ldots$, an infinite subset of Ω, and elements $a_n \in B_{\omega_n}$ which are neither the zero nor the unit. Denote

$$\bar{a}_{n1} = j_{\omega_n} a_n \quad \text{and} \quad \bar{a}_{n2} = j_{\omega_n}(a_n)' .$$

Then by 3.2,

$$u = \bigwedge_{n=1}^{\infty} (\bar{a}_{n1} \vee \bar{a}_{n2}) \neq o = \bigvee_{\varphi} \bigwedge_{n=1}^{\infty} \bar{a}_{n\varphi(n)} ,$$

where φ runs through all functions from the natural numbers to $\{1, 2\}$. Therefore $\bar{\mathfrak{B}}$ is not $\aleph_0$-distributive.

This theorem gives no information about the free products of finite sets of Boolean algebras. The following result shows that finite products in the class of α-complete Boolean algebras may even be α-distributive products.

THEOREM 4.3. *Let* $B_1, \ldots, B_n$ *be* α*-complete Boolean algebras all of which satisfy the following condition: if* $\{A_\sigma \mid \sigma \in \Sigma\}$ *is a set of* α*-partitions of* B_i *and* $|\Sigma| \leqq \alpha$, *then there is an* α*-partition of* B_i *refining all* A_σ. *If* $\mathfrak{A}$ *is a class of* α*-complete Boolean algebras closed under the formation of subalgebras, and if* $\langle B; i_j \rangle_{j=1,2,\ldots,n}$ *is a free* $\mathfrak{A}$*-product of* $\{B_1, \ldots, B_n\}$, *then* B *is* α*-distributive.*

PROOF. By the general distributive law ([2, p. 165]), the sets of the form

$$i_1 A_1 \wedge \ldots \wedge i_n A_n = \{i_1 a_1 \wedge \ldots \wedge i_n a_n \mid a_j \in A_j\} ,$$

with A_j an α-partition of B_j, are α-partitions of B. Let $\mathfrak{P}$ be the class of all such patitions of B. It is clear from the refinement property of the α-partitions of each B_j that any subset of $\mathfrak{P}$ which has cardinality not exceeding α will have a common refinement in $\mathfrak{P}$. Let $\bar{B}$ be the subalgebra of B composed of joins of subsets of the α-partitions in $\mathfrak{P}$. By [7, lemma 3.2] $\bar{B}$ is an α-complete, α-distributive subalgebra of B which clearly contains $i_j B_j$ for $j = 1, \ldots, n$. Thus according to 1.3, $\bar{B} = B$, so that B is α-distributive.

A class of Boolean algebras to which 4.3 applies is obtained as follows.

Let X be a set of cardinality β. Let $G_{\alpha\beta}$ consist of all $Y \subseteq X$ such that either Y or $X - Y$ has cardinality at most α. Then $G_{\alpha\beta}$ is an α-field. Any α-partition of $G_{\alpha\beta}$ consists of one set Y with $|X - Y| \leqq \alpha$, together with at most α sets of cardinality $\leqq \alpha$. It is clear that a set $\mathfrak{P}$ of partitions of this kind has a common refinement of the same type, provided $|\mathfrak{P}| \leqq \alpha$. Note that if $\beta \leqq \alpha$, then $G_{\alpha\beta}$ is a complete atomic Boolean algebra.

Now let us consider the relation between the free α-representable and free α-distributive products. First note the following general fact.

LEMMA 4.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be classes of α-complete Boolean algebras with $\mathfrak{A} \subseteq \mathfrak{B}$. Assume that the free $\mathfrak{A}$-product of any subset of $\mathfrak{A}$ exists and that $\mathfrak{A}$ is closed under the formation of subalgebras. Then a free $\mathfrak{A}$-product of a set $\{A_\omega \mid \omega \in \Omega\} \subseteq \mathfrak{A}$ is a free $\mathfrak{B}$-product of the set if and only if*

(L) *for any set $\{h_\omega \mid \omega \in \Omega\}$ of homomorphisms $h_\omega\colon A_\omega \to B$, where $B \in \mathfrak{B}$, there exists $A \in \mathfrak{A}$, a homomorphism $k\colon A \to B$ and a set of homomorphisms $g_\omega\colon A_\omega \to A$ such that $h_\omega = k \circ g_\omega$ for all $\omega \in \Omega$.*

PROOF. Suppose that the free $\mathfrak{A}$-product $\langle C; i_\omega\rangle_{\omega\in\Omega}$ of $\{A_\omega \mid \omega \in \Omega\}$ is a free $\mathfrak{B}$-product as well. Then there is a homomorphism $h\colon C \to B$ satisfying

$$h_\omega = h \circ i_\omega \qquad \text{for all} \quad \omega ,$$

so (L) is satisfied with $A = C$, $k = h$, and $g_\omega = i_\omega$. Conversely, let (L) be satisfied. Then there is a homomorphism $g\colon C \to A$ such that

$$g_\omega = g \circ i_\omega .$$

The homomorphism $k \circ g\colon C \to B$ satisfies

$$(k \circ g) \circ i_\omega = k \circ g_\omega = h_\omega .$$

Thus by 1.3 the product $\langle C; i_\omega\rangle_{\omega\in\Omega}$ is also a free $\mathfrak{B}$-product.

In particular, if $\mathfrak{A}$ is the class of α-distributive algebras and $\mathfrak{B}$ is the class of α-representable algebras, then 1.7, 4.4 and 3.10 give the following theorem due to Sikorski (see [11]).

COROLLARY 4.5. *A free α-representable Boolean algebra is isomorphic to an α-field of sets.*

It is now possible to give a simple proof of Sikorski's theorem (see [12]) on the existence of the free α-representable product of any set of α-representable Boolean algebras. By 1.5 and 3.2, we only have to show that the minimal α-product $\Pi_{\omega\in\Omega} B_\omega$ of α-representable Boolean algebras is α-representable. Let $k_\omega\colon F_\omega \to B_\omega$ be homorphisms of the free α-representable algebra F_ω onto B_ω. Let $\langle F; i_\omega\rangle_{\omega\in\Omega}$ be the free α-representable product of $\{F_\omega \mid \omega \in \Omega\}$. That this product exists and is in fact a free α-representable Boolean algebra follows from 1.7. Let $k\colon F \to \Pi_{\omega\in\Omega} B_\omega$ be the homomorphism satisfying

$$k \circ i_\omega = j_\omega \circ k_\omega .$$

By 1.4, k maps F onto $\Pi_{\omega\in\Omega} B_\omega$. Since F is an α-field by 4.5, we conclude that $\Pi_{\omega\in\Omega} B_\omega$ is α-representable.

Let $\mathfrak{A}$ be a class of α-complete Boolean algebras satisfying the conditions (i), (ii) and (iii) of 1.5. Following the terminology of homological algebra (see [4]), we will say that A is $\mathfrak{A}$-projective (or projective with respect to $\mathfrak{A}$) if $A \in \mathfrak{A}$, and if, for any homomorphisms $h\colon A \to B$ and $p\colon C \to B$, where B, $C \in \mathfrak{A}$ and p is onto, there exists a homomorphism $g\colon A \to C$ such that $h = p \circ g$. The following properties of projective algebras are easy consequences of this definition and 1.2 (see [4, Ch. I, 2.1 and 2.2).

(4.6) *Any free $\mathfrak{A}$-product of $\mathfrak{A}$-projective Boolean algebras is $\mathfrak{A}$-projective.*

(4.7) *An α-complete Boolean algebra A is $\mathfrak{A}$-projective if and only if there is a free $\mathfrak{A}$-algebra F and homomorphisms $q\colon F \to A$ and $k\colon A \to F$ such that $q \circ k$ is the identity mapping on A.*

Now let $\mathfrak{A}$ be the class of α-representable algebras. By 4.5 and 4.7, any $\mathfrak{A}$-projective algebra is α-distributive and therefore projective with respect to the class of α-distributive Boolean algebras. The converse is also true by 4.7. Hence, by 4.6:

THEOREM 4.8. *Let $\{B_\omega \mid \omega \in \Omega\}$ be a set of α-complete Boolean algebras, each of which is projective with respect to the class of α-distributive Boolean algebras. Then the free α-distributive product of $\{B_\omega \mid \omega \in \Omega\}$ is also a free α-representable product.*

By 4.7 and 4.5, any α-distributive algebra is projective and any projective α-distributive Boolean algebra is an α-field. However, it is not easy to decide whether or not a specific α-field is projective in the class of α-representable Boolean algebras. Hence the scope of 4.8 is somewhat obscure. Our next theorem singles out a more tangible class of α-distributive algebras in which the α-representable and α-distributive products coincide, namely the class considered in 4.3.

THEOREM 4.9. *Let $\{B_\omega \mid \omega \in \Omega\}$ be a set of α-complete Boolean algebras, each of which has the following property: if $\{A_\sigma \mid \sigma \in \Sigma\}$ is a set of α-partitions of B_ω and $|\Sigma| \leqq \alpha$, then there is an α-partition of B_ω which refines all*

A_σ, $\sigma\in\Sigma$. Then the free α-distributive product of $\{B_\omega \mid \omega\in\Omega\}$ is also a free α-representable product.

PROOF. Let $\langle B; i_\omega\rangle_{\omega\in\Omega}$ be a free α-representable product of

$$\{B_\omega \mid \omega\in\Omega\}.$$

Then B is the homomorphic image of an α-field. Moreover, it is possible (see [8]) to find a representation of B which "splits" in the following sense: there is an α-field F, a homomorphism p of F onto B preserving α-joins (that is, a homomorphism of F considered as an α-complete Boolean algebra) and an isomorphism j of B into F (preserving only finite joins) such that $p\circ j$ is the identity mapping of B onto itself. Let D be the α-field in F generated by jB. Clearly p maps D onto B. Define J to be the α-ideal of D generated by the elements of the form

$$I(A_\omega) = \left(\bigcup\{j(i_\omega a) \mid a\in A_\omega\}\right)^c,$$

where A_ω is an α-partition of B_ω. Let $\bar{B}=D/J$ and define $q\colon D\to\bar{B}$ to be the natural projection of D onto its quotient algebra D/J. We will prove the following facts:

(a) J is contained in the kernel of p, so there exists a homomorphism $l\colon \bar{B}\to B$ satisfying $p=l\circ q$;

(b) $\langle\bar{B}, j_\omega\rangle_{\omega\in\Omega}$ is a free α-distributive product of $\{B_\omega \mid \omega\in\Omega\}$, where $j_\omega=q\circ j\circ i_\omega$.

From these it will follow that $\langle\bar{B}; j_\omega\rangle_{\omega\in\Omega}$ is a free α-representable product. Indeed, given homomorphisms $h_\omega\colon B_\omega\to B_0$ into an α-representable Boolean algebra B_0, there exists a homomorphism $h\colon B\to B_0$ such that $h_\omega=h\circ i_\omega$. Then $h\circ l\colon\bar{B}\to B_0$ satisfies

$$h\circ l\circ j_\omega = h\circ l\circ q\circ j\circ i_\omega = h\circ p\circ j\circ i_\omega = h\circ i_\omega = h_\omega,$$

for all $\omega\in\Omega$.

To prove (a), note that

$$p(I(A_\omega)) = \left(\bigvee\{p(j(i_\omega a)) \mid a\in A_\omega\}\right)' = \left(\bigvee\{i_\omega a \mid a\in A_\omega\}\right)' = u' = o.$$

Hence, every $I(A_\omega)$ is in the kernel of p. Since the kernel of p is an α-ideal containing the set which generates J, it must contain J.

The proof of (b) will be based on 3.3 and 3.8. First observe that since $\bigcup\{i_\omega B_\omega \mid \omega\in\Omega\}$ generates B and jB generates D, it follows that $\bigcup\{j_\omega B_\omega \mid \omega\in\Omega\}$ generates $\bar{B}$. Next, suppose that for each σ in an index set $\Sigma\subseteq\Omega$, with $|\Sigma|\leqq\alpha$, the element a_σ is a non-zero element of B_σ. Then

$$\bigwedge_{\sigma\in\Sigma} j_\sigma a_\sigma \neq o \quad \text{in} \quad \bar{B}.$$

Indeed,

$$l\Big(\bigwedge_{\sigma\in\Sigma} j_\sigma a_\sigma\Big) = \bigwedge_{\sigma\in\Sigma} l(j_\sigma a_\sigma) = \bigwedge_{\sigma\in\Sigma} l\big(q(j(i_\sigma a_\sigma))\big)$$
$$= \bigwedge_{\sigma\in\Sigma} p\big(j(i_\sigma a_\sigma)\big) = \bigwedge_{\sigma\in\Sigma} i_\sigma a_\sigma \neq o$$

by 2.4. In particular, every j_ω is one-to-one.

We wish to show next that the elements of the form $\bigwedge_{\sigma\in\Sigma} j_\sigma a_\sigma$ are dense in $\bar{B}$. For this purpose, consider the collection $\mathfrak{P}$ of all partitions A of D for which there is a collection $\{A_\sigma \mid \sigma\in\Sigma\}$, $|\Sigma|\leqq\alpha$, of α-partitions of B_σ such that every $a\in A$ is either contained in $\bigcup_{\sigma\in\Sigma} I(A_\sigma)$ or equal to a set of the form $\bigcap_{\sigma\in\Sigma} j(i_\sigma a_\sigma)$, where $a_\sigma\in A_\sigma$. It is easy to see that since every collection of α-partitions of B_σ with cardinality $\leqq\alpha$ has a common refining α-partition, the set $\mathfrak{P}$ has the α-refinement property. Hence, by [7, lemma 3.2], the set of all unions of subsets of the partitions of $\mathfrak{P}$ form an α-field which contains $\bigcup_{\omega\in\Omega} j(i_\omega B_\omega)$ for all $\omega\in\Omega$, and therefore also contains D. This implies that every $a\in D$ is either contained in some $\bigcup_{\sigma\in\Sigma} I(A_\sigma)$, or contains a set of the form $\bigcap_{\sigma\in\Sigma} j(i_\sigma a_\sigma)$, with $a_\sigma\neq o_\sigma$ in B_σ. Consequently, if

$$\bar{a} = pa \neq o \quad \text{in} \quad \bar{B}\,,$$

there exists $\Sigma\subseteq\Omega$ with $|\Sigma|\leqq\alpha$ and $a_\sigma\neq o_\sigma$ in B_σ for each $\sigma\in\Sigma$ such that

$$\bar{a} \geqq \bigwedge_{\sigma\in\Sigma} j_\sigma a_\sigma \neq o\,.$$

That is, elements of the form $\bigwedge_{\sigma\in\Sigma} j_\sigma a_\sigma$ are dense, as claimed.

To complete the proof of (b), it suffices to show that the mappings j_ω are isomorphisms (preserving α-joins). We have already seen that the j_ω's are one-to-one. Let A_ω be an α-partition of B_ω. Then

$$\bigcup\{j(i_\omega a) \mid a\in A_\omega\}\cup I(A_\omega)$$

is the unit of D. Hence,

$$\bigvee\{j_\omega a \mid a\in A_\omega\} = \bigvee\{q(j(i_\omega a)) \mid a\in A_\omega\}$$
$$= q\big(\bigcup\{j(i_\omega a) \mid a\in A_\omega\}\big)\cup q\big(I(A_\omega)\big)$$

is the unit of $\bar{B}$. Consequently (see [8, 3.2 and 3.3]) j_ω is an isomorphism.

It is possible to show that neither of the classes of algebras in 4.8 and 4.9 contains the other. Clearly, the free α-representable Boolean algebra with α generators does not satisfy the hypotheses of 4.9. On tne other hand, the algebras $G_{\alpha\beta}$ in the example following 4.3 above are not projective in the class of α-distributive algebras, provided

$$\beta > \exp(\exp \alpha) .$$

To see this, note that $G_{\alpha\beta}$ has β distinct atoms, so $\delta(G_{\alpha\beta}) = \beta$ (where $\delta(B)$ denotes the smallest cardinal γ such that every disjointed set of non-zero elements of B has cardinality at most γ). However, if F is a free α-representable Boolean algebra, it is possible to show, using 1.7, 4.5 and 3.9, that

$$\delta(F) \leqq \exp(\exp \alpha) .$$

Thus, $G_{\alpha\beta}$ cannot be isomorphic to a subalgebra of any free α-representable algebra (hence, by 4.7, cannot be projective) if $\beta > \exp(\exp \alpha)$.

We have given no examples of α-distributive products which are not also α-representable products. Such an example, for the case $\alpha = \aleph_0$, was constructed by Sikorski in [13]. If Sikorski's example is translated into our notation and generalized slightly, the following result is obtained.

PROPOSITION 4.10. *Let B_1 be an α-complete Boolean algebra whose normal completion is α-distributive. Suppose B_2 is an α-complete and α-distributive Boolean algebra. Let $\langle B; i_1, i_2 \rangle$ be a free α-distributive product of B_1 and B_2. Assume that an element $a_0 \in B$ exists such that the ideal*

$$J_1 = \{b \in B_1 \mid i_1 b \wedge a_0 = o\}$$

is not principal. Denote by $J_1{}^*$ the ideal

$$J_1{}^* = \{b \in B_1 \mid b \wedge a = o \text{ for all } a \in J_1\} .$$

Then $B_1/J_1{}^*$ is α-complete, α-distributive and the free α-distributive prvduct of $B_1/J_1{}^*$ and B_2 is not a free α-representable product.

By combining this result with the existence theorem for projective sets of class 1 which are not of class 0 (see [5, pp. 360–368]) in a separable metric space, one obtains Sikorski's example. The attempt to generalize Sikorski's procedure to higher orders of completeness leads to a new realm of problems lying beyond the intended scope of this paper.

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