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AN EXTREMAL PROBLEM RELATED TO THE THEORY OF QUASI-ANALYTIC FUNCTIONS

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1. As is well known there are many problems intimately connected with the theory of quasi-analytic functions. We shall consider in this connection two (equivalent) extremal problems, the solutions of which we give in the following

THEOREM. *Let*

$$H(x) = \sum_{\nu=0}^{\infty} \frac{x^{2\nu}}{m_\nu^2}$$

with real m_ν, $m_0 = 1$, *be an even integral function. Let*

$$\alpha_1 = \inf \int_0^\infty H(t)\, f(t)^2\, dt \tag{1.1}$$

under the conditions

$$f \text{ real}, \qquad \int_0^\infty f(t)\, dt = (\tfrac{1}{2}\pi)^{\frac{1}{2}}, \qquad \int_0^\infty t^{2\nu} f(t)\, dt = 0, \qquad \nu \geqq 1\,, \tag{1.2}$$

and

$$\alpha_2 = \inf \sum_{\nu=0}^{\infty} \frac{1}{m_\nu^2} \int_0^\infty f^{(\nu)}(x)^2\, dx \tag{1.3}$$

under the conditions

$$f \text{ real}, \qquad f(0) = 1, \qquad f^{(\nu)}(0) = 0, \qquad \nu \geqq 1\,. \tag{1.4}$$

Then

$$\alpha_1 = \alpha_2 = \frac{1}{\pi} \int_0^\infty \frac{1}{x^2} \log H(x)\, dx \tag{1.5}$$

if one of the members is finite.

When we solve these problems we also get an elementary proof of the main theorem in the theory of quasi-analytic functions (Carleman [2]), formulated as follows:

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A necessary and sufficient condition that the class $C_A(-\infty, \infty)$ *of infinitely differentiable functions on* $(-\infty, \infty)$ *such that*

$$|f^{(\nu)}(x)| \leqq k^{\nu+1}A_\nu, \quad A_0 = 1, \qquad \nu = 0, 1, 2, \ldots,$$

should be quasi-analytic is that

$$\int_0^\infty \frac{1}{x^2} \log\left(\sum_{\nu=0}^{\infty} \frac{x^{2\nu}}{A_\nu^2}\right) dx = \infty . \tag{1.6}$$

2. We start by considering the following extremal problem: Let $\{m_\nu\}_0^n$ be a given sequence of positive numbers with $m_0 = 1$ and let C_n be the class of n times differentiable functions on $[0, \infty)$ with

$$f(0) = 1, \quad f^{(\nu)}(0) = 0, \qquad \nu = 1, 2, \ldots, n-1, \tag{2.1}$$

such that $f(x), f'(x), \ldots, f^{(n)}(x)$ belong to $L^2[0, \infty)$. Form the functional

$$F_n(f) = \sum_{\nu=0}^{n} \frac{1}{m_\nu^2} \int_0^\infty f^{(\nu)}(x)^2\, dx$$

and try to minimize it over the class C_n.

3. Let $f_n(x)$ be the solution of the differential equation

$$\sum_{\nu=0}^{n} \frac{(-1)^\nu}{m_\nu^2} D^{2\nu} y = 0 \tag{3.1}$$

which belongs to C_n. Since (3.1) contains derivatives of even orders only, the general solution has the form

$$y = \sum_{\substack{\nu=-nk \\ \nu \neq 0}}^{nk} P_\nu(x)\, e^{r_\nu x}$$

with $r_{-\nu} = -r_\nu$, $\mathrm{Re}\{r_\nu\} > 0$ for $\nu > 0$, where $P_\nu(x)$ are polynomials containing in all $2n$ arbitrary constants. Obviously, in $f_n(x)$ the n constants of $P_\nu(x)$, $\nu > 0$, must be zero and the others are determined by the conditions (2.1). Since every r_ν occurring in the exponents of $f_n(x)$ has a negative real part, $f_n(x)$ and all its derivatives obviously belong to $L^2[0, \infty)$.

We now consider $F_n(f_n + \eta)$, where $\eta(x)$ satisfies the conditions in C_n with the only exception $\eta(0) = 0$, $(\eta \not\equiv 0)$:

$$F_n(f_n + \eta) = F_n(f_n) + F_n(\eta) + 2 \sum_{\nu=0}^{n} \frac{1}{m_\nu^2} \int_0^\infty f_n^{(\nu)}(x)\, \eta^{(\nu)}(x)\, dx .$$

Partial integrations give for $\nu = 1, 2, \ldots, n$

$$\int_0^\infty f_n^{(\nu)}(x)\,\eta^{(\nu)}(x)\,dx = (-1)^\nu \int_0^\infty f_n^{(2\nu)}(x)\,\eta(x)\,dx\,,$$

and hence

$$\sum_{\nu=0}^{n} \frac{1}{m_\nu^2}\int_0^\infty f_n^{(\nu)}(x)\,\eta^{(\nu)}(x)\,dx = \int_0^\infty \eta(x)\Big(\sum_{\nu=0}^{n}\frac{(-1)^\nu}{m_\nu^2} f_n^{(2\nu)}(x)\Big)\,dx = 0$$

since $f_n(x)$ satisfies (3.1). Consequently

$$F_n(f_n+\eta) = F_n(f_n) + F_n(\eta) > F_n(f_n)\,.$$

Every f in C_n can evidently be expressed in the form $f_n+\eta$, and therefore f_n realizes

$$\min_{f\in C_n} F_n(f) = F_n(f_n) = \mu_n\,. \tag{3.2}$$

4. We compute the minimum value μ_n. For $\nu \geqq 1$ partial integrations give

$$\int_0^\infty f_n^{(\nu)}(x)^2\,dx = (-1)^\nu f_n^{(2\nu-1)}(0) + (-1)^\nu \int_0^\infty f_n^{(2\nu)}(x)\,f_n(x)\,dx\,.$$

Hence

$$\begin{aligned}\mu_n &= \sum_{\nu=0}^{n}\frac{1}{m_\nu^2}\int_0^\infty f_n^{(\nu)}(x)^2\,dx\\ &= \sum_{\nu=1}^{n}\frac{(-1)^\nu}{m_\nu^2} f_n^{(2\nu-1)}(0) + \int_0^\infty f_n(x)\Big(\sum_{\nu=0}^{n}\frac{(-1)^\nu}{m_\nu^2} f_n^{(2\nu)}(x)\Big)\,dx \qquad (4.1)\\ &= \sum_{\nu=[\frac{1}{2}n]+1}^{n}\frac{(-1)^\nu}{m_\nu^2} f_n^{(2\nu-1)}(0)\,.\end{aligned}$$

Suppose first that the characteristic equation corresponding to (3.1) has only simple roots. Then

$$f_n(x) = \sum_{\nu=1}^{n} c_\nu e^{-r_\nu x}\,.$$

The boundary conditions (2.1) are

$$\left\{\begin{aligned} &\sum_{\nu=1}^{n} c_\nu &&= 1\\ &\sum_{\nu=1}^{n} c_\nu r_\nu &&= 0\\ &\cdots\cdots\cdots\cdots\\ &\sum_{\nu=1}^{n} c_\nu r_\nu^{\,n-1} &&= 0\,.\end{aligned}\right. \tag{4.2}$$

The determinant of (4.2) is of the Vandermonde type and we find

$$c_\nu = \frac{\prod_{j=1}^{n} r_j}{r_\nu \prod_{j \neq \nu}^{1,n} (r_j - r_\nu)} ;$$

and since $\prod_{j=1}^{n} r_j = m_n/m_0 = m_n$,

$$f_n(x) = m_n \sum_{\nu=1}^{n} \frac{e^{-r_\nu x}}{r_\nu \prod_{j \neq \nu}^{1,n} (r_j - r_\nu)} .$$

Inserting in (4.1) gives

$$\mu_n = m_n \sum_{k=1}^{n} \frac{1}{\prod_{j \neq k}^{1,n} (r_j - r_k)} \sum_{\nu=[\frac{1}{2}n]+1}^{n} \frac{(-1)^{\nu+1} r_k^{2\nu-2}}{m_\nu^2}$$

$$= m_n \sum_{k=1}^{n} \frac{1}{\prod_{j \neq k}^{1,n} (r_j - r_k)} \sum_{\nu=0}^{[\frac{1}{2}n]} \frac{(-1)^{\nu} r_k^{2\nu-2}}{m_\nu^2}$$

where the last step is justified since r_k satisfies the characteristic equation. We now state that

$$\mu_n = \sum_{k=1}^{n} \frac{1}{r_k} \tag{4.3}$$

and prove this by comparison with Lagrange's interpolation formula

$$P(z) = \sum_{k=1}^{n} \frac{P(r_k)}{\prod_{j \neq k}^{1,n} (r_k - r_j)} \cdot \frac{\prod_{j=1}^{n} (z - r_j)}{z - r_k}$$

giving a polynomial $P(z)$ assuming at n given points r_k given values $P(r_k)$, $k=1, 2, \ldots, n$. Obviously $\mu_n = P(0)$ if

$$P(r_k) = \sum_{\nu=0}^{[\frac{1}{2}n]} \frac{(-1)^{\nu} r_k^{2\nu-1}}{m_\nu^2} = \frac{1}{r_k} + \sum_{\nu=1}^{[\frac{1}{2}n]} \frac{(-1)^{\nu} r_k^{2\nu-1}}{m_\nu^2} .$$

$P(0)$ must be equal to the value for $z=0$ of another polynomial $P_1(z)$ assuming in the points r_k the values

$$P_1(r_k) = 1/r_k .$$

Furthermore $P_1(0) = P_2'(0)$, with

$$P_2(z) = z P_1(z) .$$

The polynomial $P_2(z)$ must consequently fulfil the conditions

$$P_2(0) = 0, \qquad P_2(r_k) = 1, \quad k = 1, 2, \ldots, n .$$

But this polynomial is easily written down directly:

$$P_2(z) = 1 - \frac{\prod_{k=1}^{n} (r_k - z)}{\prod_{k=1}^{n} r_k}$$

and we find

$$\mu_n = P_2'(0) = \sum_{k=1}^{n} \frac{1}{r_k}.$$

The assumption that the r_k should be simple roots is no restriction. For $\{m_k\}_1^n$ can be approximated arbitrarily closely by another sequence $\{m_k'\}_1^n$ so that all the roots r_k' of the corresponding characteristic equation are simple, and furthermore both μ_n' and $\{r_k'\}_1^n$ are continuous functions of $\{m_k'\}_1^n$.

5. In order to get a connection with the condition in the main theorem on quasi-analytic functions we now compute

$$\int_0^\infty \frac{1}{x^2} \log\left(\sum_{\nu=0}^{n} \frac{x^{2\nu}}{m_\nu^2}\right) dx . \tag{5.1}$$

Since the equation

$$\sum_{\nu=0}^{n} \frac{x^{2\nu}}{m_\nu^2} = 0$$

has the roots $\pm i r_\nu$, $\nu = 1, 2, \ldots, n$, we have

$$\sum_{\nu=0}^{n} \frac{x^{2\nu}}{m_\nu^2} = \prod_{\nu=1}^{n} \left(1 + \frac{x^2}{r_\nu^2}\right)$$

and (5.1) becomes

$$\sum_{\nu=1}^{n} \int_0^\infty \frac{1}{x^2} \log\left(1 + \frac{x^2}{r_\nu^2}\right) dx = \pi \cdot \sum_{\nu=1}^{n} \frac{1}{r_\nu}.$$

Hence

$$\mu_n = \frac{1}{\pi} \int_0^\infty \frac{1}{x^2} \log\left(\sum_{\nu=0}^{n} \frac{x^{2\nu}}{m_\nu^2}\right) dx . \tag{5.2}$$

6. Suppose now that the integral in (1.6) converges. Then, if we put $m_{\nu+1} = A_\nu$, $\nu \geqq 0$, $m_0 = 1$, it is easy to see that the integral

$$\int_0^\infty \frac{1}{x^2} \log\left(\sum_{\nu=0}^{\infty} \frac{x^{2\nu}}{m_\nu^2}\right) dx$$

is also convergent, i.e.

$$\lim_{n\to\infty} \mu_n = c < \infty .$$

Then, if n is an arbitrary positive integer, there exists a function $f_n(x)$ such that

$$\sum_{\nu=0}^{n} \frac{1}{m_\nu^2} \int_0^\infty f_n^{(\nu)}(x)^2 \, dx = \mu_n < c .$$

Hence for an arbitrary $a > 0$

$$\int_0^a f_n^{(\nu)}(x)^2 \, dx < c m_\nu^2, \qquad \nu = 0, 1, \ldots, n ,$$

and Schwarz's inequality gives for $0 \leqq x \leqq a$

$$|f_n^{(\nu)}(x)| \leqq (ac)^{\frac{1}{2}} \cdot A_\nu, \qquad \nu = 0, 1, \ldots, n-1 .$$

Then, by a standard argument, we can select a subsequence $\{f_{n_k}(x)\}$ converging to a function $g(x)$ and such that $f_{n_k}^{(\nu)}(x) \to g^{(\nu)}(x)$, uniformly in $[0, a]$. Now in $g(x)$ we have an example showing that $C_A[0, a]$ is non-quasi-analytic. The transformation $x = at/(1+t)$ then gives a function in $C_A(-\infty, \infty)$ (see [2, pp. 22–23]) which shows that this class is non-quasi-analytic.

7. To prove the sufficiency of the condition (1.6) we show that if $C_A(-\infty, \infty)$ is non-quasi-analytic then (1.6) cannot hold. We may assume that the sequence $\{\log A_\nu\}$ is convex (see [1, p. 15]).

If $C_A(-\infty, \infty)$ is non-quasi-analytic, it contains a function $f(x)$ with

$$f^{(\nu)}(0) = f^{(\nu)}(c) = 0, \qquad \nu \geqq 0 ,$$
$$f(x) > 0 \quad \text{for} \quad 0 < x < c$$

and $f(x) \equiv 0$ elsewhere (see [1, p. 53]). Let, for $0 \leqq x \leqq c$,

$$\varphi(x) = \frac{\int_x^c f(t)\, dt}{\int_0^c f(t)\, dt} .$$

Then $\varphi(0) = 1$, $\varphi^{(\nu)}(0) = 0$, $\nu \geqq 1$, $\varphi^{(\nu)}(c) = 0$, $\nu \geqq 0$. Let $\varphi(x) \equiv 0$ for $x > c$. $\varphi(x)$ belongs to $C_A[0, \infty)$, for integration is permitted within the class C_A

when $\{\log A_\nu\}$ is convex (Bang [1]). Consequently we have for some constant $a \geqq 1$

$$|\varphi^{(\nu)}(x)| \leqq ak^\nu A_\nu, \qquad 0 \leqq x < \infty, \qquad \nu \geqq 0\,,$$

where we may assume $k < 1$. But since $\varphi(x)$ belongs to the class C_n (see 2 above) we have $\mu_n \leqq F_n(\varphi)$ for every n, and therefore

$$\mu_n \leqq \sum_{\nu=0}^{n} \frac{1}{A_\nu^2} \int_0^\infty \varphi^{(\nu)}(x)^2\, dx \leqq a^2 c \cdot \sum_{\nu=0}^{n} k^{2\nu} < \frac{a^2 c}{1-k^2},$$

and we conclude the convergence of the integral in (1.6).

8. We now return to the extremal problem (1.3). From (3.2) and the uniform convergence of $f_{n_k}^{(\nu)}(x)$ to $g^{(\nu)}(x)$ in every finite interval $[0, a]$ we conclude that

$$\lim_{n\to\infty} F_n(f_n) = \sum_{\nu=0}^{\infty} \frac{1}{m_\nu^2} \int_0^\infty g^{(\nu)}(x)^2\, dx = F(g)$$

and moreover that $g(x)$ realizes

$$\alpha_2 = \min F(f) = F(g)$$

which then according to (5.2) has the value given in (1.5).

Finally, having made $f(x)$ even, we pass from (1.4) to (1.2) by means of a Fourier transform; use of Parsevals formula gives at once the solution (1.5) of (1.1).

REFERENCES

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