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ON THE EQUIVALENCE OF BOUNDEDNESS FOR MULTIPLE HARDY-LITTLEWOOD AVERAGES AND RELATED OPERATORS

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Abstract

Necessary and sufficient conditions for the weight function u are obtained, which provide the boundedness for a class of averaging operators from L_p^+ to $L_{q,u}^+$. These operators include the multiple Hardy-Littlewood averages and related maximal operators, geometric mean operators, and geometric maximal operators. We show that, under suitable conditions, the boundedness of these operators are equivalent. Our theorems extend several one-dimensional results to multi-dimensional cases and to operators with multiple kernels. We also show that in the case $p < q$, some one-dimensional results do not carry over to the multi-dimensional cases, and the boundedness of T from L_p^+ to $L_{q,u}^+$ holds only if $u = 0$ almost everywhere.

1. Introduction

The well-known Hardy averages are defined by $H_1 f(x) = (1/x)\int_0^x f(t)\,dt$, where $x > 0$ and f is a nonnegative measurable function defined on $(0, \infty)$. The operator H_1 takes the average values of f over the interval $(0, x)$ for each $x > 0$, and it can be written in the form $H_1 f(x) = \int_0^1 f(tx)\,dt$. Weighted estimates for H_1 have been investigated by many authors. We refer the reader to the books [17], [29], and the references given there. In [43], Xiao consider the boundedness of weighted Hardy-Littlewood averaging operator H_ϕ on spaces $L^p(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$, where $H_\phi f(x) = \int_0^1 \phi(t) f(tx)\,dt$, $x \in \mathbb{R}^n$, and ϕ is a nonnegative measurable function defined on $(0, 1)$. The operator H_ϕ takes the weighted average values of f over the line segment from the origin to each point x in $\mathbb{R}^n$. In this paper we consider operators that take weighted average values of f over line segments, rectangles, rectangular cuboids, etc. We put these operators in the same form and investigate weighted estimates for them.

Let m and n be positive integers and $m \le n$. We split n into the sum of m positive integers n_i, $i = 1, 2, \dots, m$. For each $i = 1, 2, \dots, m$, let E_i be a subset of $\mathbb{R}^{n_i}$ defined at the end of this section. Define $\mathbf{E} = E_1 \times \cdots \times E_m$.

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Given a nonnegative measurable function f on $\mathbf{E}$ and $\mathbf{x} = (x_1, \dots, x_m) \in \mathbf{E}$, where $x_i \in E_i$, the multiple Hardy-Littlewood average $H_\phi f$ is defined as

$$H_\phi f(\mathbf{x}) := \int_{\mathbf{I}} \phi(\mathbf{t}) f(\mathbf{t} \circ \mathbf{x})\, d\mathbf{t}.$$

Here $\mathbf{I} := (0,1)^m$, $\mathbf{t} = (t_1, \dots, t_m)$, $\mathbf{t} \circ \mathbf{x} = (t_1 x_1, \dots, t_m x_m)$, $d\mathbf{t} = dt_m \dots dt_1$, and ϕ is a nonnegative measurable function defined on $\mathbf{I}$. In the following we give an example for the case $\mathbf{E} \subseteq \mathbb{R}^3$ and $n = 3$. If $m = 1$ then $n_1 = 3$, $\mathbf{I} = (0,1)$, $\mathbf{t} = t_1$, $\mathbf{x} = x_1 = (x_{11}, x_{12}, x_{13}) \in E_1 \subseteq \mathbb{R}^3$, and $\mathbf{t} \circ \mathbf{x} = (t_1 x_{11}, t_1 x_{12}, t_1 x_{13})$. In this case $H_\phi f(\mathbf{x})$ takes the weighted average value of f over the line segment from the origin to the point $\mathbf{x}$. If $m = 2$, and $n_1 = 2$, $n_2 = 1$, then $\mathbf{I} = (0,1) \times (0,1)$, $\mathbf{t} = (t_1, t_2)$, and $x_1 = (x_{11}, x_{12}) \in E_1 \subseteq \mathbb{R}^2$, $x_2 \in E_2 \subseteq \mathbb{R}$, $\mathbf{x} = (x_1, x_2) = (x_{11}, x_{12}, x_2)$, and $\mathbf{t} \circ \mathbf{x} = (t_1 x_{11}, t_1 x_{12}, t_2 x_2)$. In this case $H_\phi f(\mathbf{x})$ takes the weighted average value of f over the rectangle with vertices $(0,0,0)$, $(x_{11}, x_{12}, 0)$, $(0,0,x_2)$, and (x_{11}, x_{12}, x_2). If $m = 3$, then $n_1 = n_2 = n_3 = 1$, $\mathbf{I} = (0,1) \times (0,1) \times (0,1)$, $\mathbf{t} = (t_1, t_2, t_3)$, $\mathbf{x} = (x_1, x_2, x_3)$, where $x_i \in E_i \subseteq \mathbb{R}$ for $i = 1, 2, 3$, and $\mathbf{t} \circ \mathbf{x} = (t_1 x_1, t_2 x_2, t_3 x_3)$. In this case $H_\phi f(\mathbf{x})$ takes the weighted average value of f over the rectangular cuboid with the origin and the point $\mathbf{x}$ as the endpoints of a space diagonal. This example shows that the multiple averaging operator H_ϕ defined in this paper can take weighted average values of f over different types of regions.

If $\Phi = \int_{\mathbf{I}} \phi(\mathbf{t})\, d\mathbf{t} = 1$, the geometric mean operator G_ϕ is defined as

$$G_\phi f(\mathbf{x}) := \exp\left(\int_{\mathbf{I}} \phi(\mathbf{t}) \log f(\mathbf{t} \circ \mathbf{x})\, d\mathbf{t} \right).$$

We also consider the maximal operator $\mathcal{M}_{gh}^-$:

$$\mathcal{M}_{gh}^- f(\mathbf{x}) := \sup_{\mathbf{z} \in \mathbf{I}} h(\mathbf{z}) \int_{\mathbf{I_z}} g(\mathbf{t}) f(\mathbf{t} \circ \mathbf{x})\, d\mathbf{t},$$

where g and h are positive measurable functions defined on $\mathbf{I}$ and $\mathbf{I_z} = (1 - z_1, 1) \times \cdots \times (1 - z_m, 1)$ for $\mathbf{z} = (z_1, \dots, z_m) \in \mathbf{I}$. If $\tilde{G}(\mathbf{z}) = \int_{\mathbf{I_z}} g(\mathbf{t})\, d\mathbf{t} < \infty$ for all $\mathbf{z} \in \mathbf{I}$, the geometric maximal operator $\mathcal{G}_{gh}^-$ is defined as

$$\mathcal{G}_{gh}^- f(\mathbf{x}) := \sup_{\mathbf{z} \in \mathbf{I}} h(\mathbf{z}) \exp\left(\frac{1}{\tilde{G}(\mathbf{z})} \int_{\mathbf{I_z}} g(\mathbf{t}) \log f(\mathbf{t} \circ \mathbf{x})\, d\mathbf{t} \right).$$

Moreover, if $\tilde{G}(\mathbf{z})^{-1} \int_{\mathbf{I_z}} g(\mathbf{t}) f(\mathbf{t} \circ \mathbf{x})^\epsilon\, d\mathbf{t}$ is finite for some $\epsilon > 0$, then

$$\lim_{\epsilon \searrow 0} \left(\frac{1}{\tilde{G}(\mathbf{z})} \int_{\mathbf{I_z}} g(\mathbf{t}) f(\mathbf{t} \circ \mathbf{x})^\epsilon\, d\mathbf{t} \right)^{1/\epsilon} = \exp\left(\frac{1}{\tilde{G}(\mathbf{z})} \int_{\mathbf{I_z}} g(\mathbf{t}) \log f(\mathbf{t} \circ \mathbf{x})\, d\mathbf{t} \right).$$

Hence closely related to $\mathscr{G}_{gh}^{-}$ is the limiting operator $\mathscr{G}_{gh}^{-*}$ defined by

$$\mathscr{G}_{gh}^{-*} f(\mathbf{x}) := \lim_{\epsilon \searrow 0} \sup_{\mathbf{z}\in\mathbf{I}} h(\mathbf{z})\left(\frac{1}{\tilde{G}(\mathbf{z})}\int_{\mathbf{I}_{\mathbf{z}}} g(\mathbf{t}) f(\mathbf{t}\circ\mathbf{x})^{\epsilon}\, d\mathbf{t}\right)^{1/\epsilon}.$$

The purpose of this paper is to characterize the nonnegative function u so that the weighted inequality of the form

$$\left(\int_{\mathbf{E}} Tf(\mathbf{x})^q u(\mathbf{x})\, d\mathbf{x}\right)^{1/q} \le C\left(\int_{\mathbf{E}} f(\mathbf{x})^p\, d\mathbf{x}\right)^{1/p}, \quad f \ge 0, \tag{1.1}$$

where $0 < p, q < \infty$, holds for T to be any one of the operators given above. For $0 < p < \infty$ and $\eta\colon \mathbf{E} \mapsto [0, \infty]$, we denote

$$L^{+}_{p,\eta} = \left\{ f\colon \mathbf{E} \mapsto [0,\infty] : \|f\|_{p,\eta} := \left(\int_{\mathbf{E}} f(\mathbf{x})^p \eta(\mathbf{x})\, d\mathbf{x}\right)^{1/p} < \infty \right\}.$$

If $\eta \equiv 1$, we write L^{+}_{p} instead of $L^{+}_{p,\eta}$. If (1.1) holds with a finite constant C independent of f, we simply write $T\colon L^{+}_{p} \mapsto L^{+}_{q,u}$ and we denote the least constant C in (1.1) by $\|T\|_{L^{+}_{p}\to L^{+}_{q,u}}$.

There are a number of important works on the theory of weighted inequalities. We refer the reader to the books [8], [9], [10], [12], [13], [16], [17], [29], and the references given there. For the purpose of this paper, see also [2], [21], [24], [23], [27], [35], [36] for the Riemann-Liouville fractional integrals, [1], [3] for the one-dimensional integral operators with a homogeneous kernel, [2], [4], [22], [23] for the fractional one-sided maximal operators, [18], [27], [28], [32], [34] for the geometric mean operators, [5], [6], [7], [31], [30], [44] for the geometric maximal operators, [26], [33], [37], [42] for the multi-dimensional Hardy operator, [14], [15] for the fractional maximal functions and potentials with multiple kernels, and [11], [18], [38], [40], [41], [43] for the weighted Hardy-Littlewood averages and related operators.

By [2], [21], and [22] we see that, in general, the conditions for the boundedness of Riemann-Liouville fractional integrals and fractional one-sided maximal operators are quite different from the criterion for the boundedness of the Hardy operator $Hf(x) = \int_0^x f(t)\, dt$. On the other hand, it is interesting that for the Hardy averaging operator H_1, in some cases, $H_1\colon L^{+}_{p} \mapsto L^{+}_{q,u}$ is equivalent to $T\colon L^{+}_{p} \mapsto L^{+}_{q,u}$ for several types of averaging operators and related fractional maximal operators. In the case $m = 1$, $n_1 = 1$, and $\mathbf{E} = (0, \infty)$, Meskhi [24, Theorem 1 & Theorem 3] showed that (1.1) holds for $T = H_\alpha$,

$1 < p, q < \infty$, and $\alpha p > 1$ if and only if $\mathscr{B} < \infty$, where

$$\mathscr{B} = \begin{cases} \sup\limits_{t>0} \left(\int_t^\infty x^{-q} u(x)\, dx \right)^{1/q} t^{(p-1)/p}, & \text{if } p \le q, \\ \left\{ \int_0^\infty \left(\int_t^\infty x^{-q} u(x)\, dx \right)^{p/(p-q)} \cdot t^{p(q-1)/(p-q)}\, dt \right\}^{(p-q)/(pq)}, & \text{if } q < p. \end{cases}$$

Here $H_\alpha f(x) = \int_0^1 (1-t)^{\alpha-1} f(tx)\, dt$, $\alpha > 0$. We point out that the condition $\mathscr{B} < \infty$ is independent of α and it is also the well-known criterion for $H_1 \colon L_p^+ \mapsto L_{q,u}^+$. By using different methods, Prokhorov [35] proved (1.1) for $T = H_\alpha$, $0 < q < \infty$, and $\max(1, 1/\alpha) < p < \infty$. The criterions on u obtained in [35, Theorem 1 & Theorem 2] are equivalent to the condition $\mathscr{B} < \infty$. If $J_{\sigma\eta}^\alpha$ is the Erdelyi-Köber operator defined by $J_{\sigma\eta}^\alpha f(x) = \int_0^1 (1-t^\sigma)^{\alpha-1} t^{\sigma\eta+\sigma-1} f(tx)\, dt$, $\sigma > 0$, $0 < \alpha \le 1$, and $\eta > 1/\sigma - 1$, then by [25, Theorem 3.1 & Theorem 3.4], $J_{\sigma\eta}^\alpha \colon L_p^+ \mapsto L_{q,u}^+$ for $1/\alpha < p < \infty$ and $0 < q < \infty$ if and only if $\mathscr{B} < \infty$. Moreover, by [23, Theorem 3] we see that $\mathscr{B} < \infty$ is a necessary and sufficient condition for $\mathscr{M}_{\alpha\beta}^- \colon L_p^+ \mapsto L_{q,u}^+$, where $\mathscr{M}_{\alpha\beta}^- f(x) = \sup_{0<z<1} z^{-\beta} \int_{1-z}^1 (1-t)^{\alpha-1} f(tx)\, dt$, $1 < p \le q < \infty$, and $\beta + 1/p < \alpha \le 1$. In [27, Theorem 5.3 & Theorem 5.5] another criterion on u was obtained for (1.1) to hold with $T = H_\alpha$. The same condition also characterized $G_\alpha \colon L_p^+ \mapsto L_{q,u}^+$, where $0 < p, q < \infty$, $G_\alpha f(x) = \exp(\alpha \int_0^1 (1-t)^{\alpha-1} \log f(tx)\, dt)$, and $\alpha > 0$. See [27, Theorem 5.1]. In [18] the author showed that (1.1) holds for $T = G_\phi$ and $0 < p, q < \infty$ if and only if $A_\delta < \infty$ for all $\delta > 1$, where

$$A_\delta = \begin{cases} \sup\limits_{t>0} \left(\int_t^\infty x^{-\delta q/p} u(x)\, dx \right)^{1/q} t^{(\delta-1)/p}, & \text{if } p \le q, \\ \left\{ \int_0^\infty \left(\int_t^\infty x^{-\delta q/p} u(x)\, dx \right)^{p/(p-q)} \cdot t^{(\delta q-p)/(p-q)}\, dt \right\}^{(p-q)/(pq)}, & \text{if } q < p. \end{cases}$$

Note that $\mathscr{B} = A_p$. The function ϕ considered in [18] includes the cases $\phi(t) = \alpha t^{\alpha-1}$ and $\phi(t) = \alpha(1-t)^{\alpha-1}$, where $\alpha > 0$. These results point out an interesting fact that $T \colon L_p^+ \mapsto L_{q,u}^+$ for T to be operators H_α, $J_{\sigma\eta}^\alpha$, $\mathscr{M}_{\alpha\beta}^-$, and G_α are equivalent.

In general, the study of weighted estimates for multi-dimensional operators is more difficult. Many problems in the two-weight case remain open and the solutions for some solved problems are not easy to check. However, under suitable restrictions on weights, several weighted estimates in simple forms for some multi-dimensional operators have been obtained. In [37] Sawyer solved the two-weight problem for $I_2 \colon L^+_{p,v} \mapsto L^+_{q,u}$, where $1 < p \le q < \infty$, $I_2 f(\mathbf{x}) = \int_0^{x_1}\int_0^{x_2} f(t_1, t_2)\, dt_1\, dt_2$, and $\mathbf{E} = (0,\infty) \times (0,\infty)$. The criterion obtained in [37] consist of three weight conditions. If v is a multiple weight, that is, $v(x_1, x_2) = v_1(x_1)v_2(x_2)$, then a criterion under one condition was derived by Wedestig [42], and another Muckenhoupt-type condition was established by Meskhi [26]. Some results for a class of multi-dimensional integral operators and related operators that are similar to that given in the previous paragraph were established in [20]. In [14] and [15] Kokilashvili and Meskhi investigated criterions for $T \colon L^+_p \mapsto L^+_{q,u}$ in the multi-dimensional case and T are fractional maximal functions and potentials with multiple kernels. On the other hand, in the case $m = 1$ and $n_1 = n$, necessary and sufficient conditions on u for (1.1) to hold with $1 < q \le p < \infty$, $T = H_\phi$, where $\phi(t) = t^{n-1}$, can be obtained in [38]. Sinnamon [38] also showed that the one-dimensional results in the case $p < q$ do not carry over to dimension $n > 1$.

In this paper we obtain necessary and sufficient conditions on u so that (1.1) holds for T to be the multiple Hardy-Littlewood averaging operator H_ϕ and related operators G_ϕ, $\mathcal{M}^-_{gh}$, $\mathcal{G}^-_{gh}$, and $\mathcal{G}^{-*}_{gh}$. We show that, under suitable conditions on ϕ, g, and h, $T \colon L^+_p \mapsto L^+_{q,u}$ for T to be these operators are equivalent. Our theorems extend several one-dimensional results to multi-dimensional cases and to operators with multiple kernels. Moreover, we show that if $p < q$ and $n_i > 1$ for some $i = 1, \ldots, m$, inequality (1.1) holds for T to be any operators given above only if $u(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in \mathbf{E}$. A similar result for the case $m = 1$ and $\mathbf{E} = \mathbb{R}^n$ has been obtained by Sinnamon [38, Theorem 3.4].

For $i = 1, 2, \ldots, m$, if $n_i > 1$, we call a region $S_i \subseteq \mathbb{R}^{n_i}$ smoothly star-shaped if there exists a nonnegative, piecewise-C^1 function ψ_i defined on the unit sphere in $\mathbb{R}^{n_i}$ with $S_i = \{\, x \in \mathbb{R}^{n_i} \setminus \{0\} : |x| \le \psi_i(x/|x|)\,\}$. See also [39]. Define $E_i = \bigcup_{\alpha>0} \alpha S_i$, where $S_i \subseteq \mathbb{R}^{n_i}$ is a smoothly star-shaped region. For nonzero $x_i \in E_i$, there is a least positive dilation $\alpha_{x_i} S_i$ which contains x_i. Let $B_i = \{\, x \in \mathbb{R}^{n_i} \setminus \{0\} : |x| = \psi_i(x/|x|)\,\}$ and note that $x_i/\alpha_{x_i} \in B_i$ so that x_i is on the boundary of $\alpha_{x_i} S_i$. For any nonzero $x_i \in E_i$, we make the changes of variables $x_i = \xi_i \sigma_i$, where $\xi_i > 0$ and $\sigma_i \in B_i$. Then for any measurable function g defined on E_i, we have

$$\int_{E_i} g(x_i)\, dx_i = \int_{B_i} \int_0^\infty g(\xi_i \sigma_i)\xi_i^{n_i - 1}\, d\xi_i\, d\sigma_i. \tag{1.2}$$

If $n_i = 1$, we consider the case $E_i = (0,\infty)$ and $\int_{E_i} g(x_i)\, dx_i = \int_0^\infty g(x_i)\, dx_i$.

Suppose that for some $1 \le \ell \le m$, $n_{k(s)} > 1$ for $s = 1, \ldots, \ell$. Then for $\mathbf{x} =$

$(x_1, \ldots, x_m) \in \mathbf{E}$, we write $x_i = \xi_i \sigma_i$ for $i = k(1), \ldots, k(\ell)$, where $\xi_i > 0$ and $\sigma_i \in B_i$. If $\ell < m$, we have $n_{j(s)} = 1$ and we consider the case $E_{j(s)} = (0, \infty)$ for $s = 1, \ldots, m - \ell$. Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m)$. In the case $\ell < m$, $\xi_i = x_i$ and we set $\sigma_i = 1$ for $i = j(1), \ldots, j(m - \ell)$. Then we write $\mathbf{x} = (\xi_1\sigma_1, \ldots, \xi_m\sigma_m) = \boldsymbol{\xi} \circ \boldsymbol{\sigma}$, where $\boldsymbol{\xi} \in \mathbf{R}_{\mathbf{0}}^{\infty} = (0, \infty) \times \cdots \times (0, \infty)$. For any nonnegative measurable function f defined on $\mathbf{E}$,

$$\int_{\mathbf{E}} f(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{B}} \int_{\mathbf{R}_{\mathbf{0}}^{\infty}} f(\boldsymbol{\xi} \circ \boldsymbol{\sigma})[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\, d\boldsymbol{\xi}\, d\boldsymbol{\sigma}, \tag{1.3}$$

where $d\mathbf{x} = dx_m \ldots dx_1$, $\mathbf{B} = B_{k(1)} \times \cdots \times B_{k(\ell)}$, $[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}] = \prod_{i=1}^{m} \xi_i^{n_i - 1}$, $d\boldsymbol{\xi} = d\xi_m \ldots d\xi_1$, and $d\boldsymbol{\sigma} = d\sigma_{k(\ell)} \ldots d\sigma_{k(1)}$. If each $n_i = 1$, we consider the case $E_1 = \cdots = E_m = (0, \infty)$ and (1.3) is reduced to $\int_{\mathbf{E}} f(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{R}_{\mathbf{0}}^{\infty}} f(\mathbf{x})\, d\mathbf{x}$.

Throughout this paper, we assume that all functions are measurable on their domains. For $0 < z < \infty$, we define z^* by $1/z + 1/z^* = 1$. The notation $A \ll D$ means that $A \le cD$ for some positive constant c depending at most on m, p, q, and a parameter τ. We also take $\exp(-\infty) = 0$, $\log 0 = -\infty$, $0^0 = \infty^0 = 1$, and $\infty/\infty = 0/0 = 0 \cdot \infty = 0$.

NOTATION. Let $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{0} = (0, \ldots, 0)$, and $\boldsymbol{\infty} = (\infty, \ldots, \infty)$. Let $\mathbf{n} = (n_1, \ldots, n_m)$. For $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and $\mathbf{b} = (b_1, \ldots, b_m)$, we write $\mathbf{a} \circ \mathbf{b} = (a_1b_1, \ldots, a_mb_m)$. In the case $\mathbf{b} \in \mathbb{R}^m$, we write $\mathbf{a} = \mathbf{b}$ if $a_i = b_i$ for all $i = 1, \ldots, m$, and $\mathbf{a} \ne \mathbf{b}$ if $\mathbf{a} = \mathbf{b}$ does not hold. We also write $\mathbf{a} > \mathbf{b}$ (or $\mathbf{a} \ge \mathbf{b}$) if $a_i > b_i$ (or $a_i \ge b_i$) for all $i = 1, \ldots, m$, and $\mathbf{a} < \mathbf{b}$ (or $\mathbf{a} \le \mathbf{b}$) if $\mathbf{b} > \mathbf{a}$ (or $\mathbf{b} \ge \mathbf{a}$). For $c \in \mathbb{R}$, let $c\mathbf{a} = (ca_1, \ldots, ca_m)$, $\mathbf{a}/c = c^{-1}\mathbf{a} = (a_1/c, \ldots, a_m/c)$ if $c \ne 0$. If $b_i \ne 0$ for all $i = 1, \ldots, m$, we write $c/\mathbf{b} = c\mathbf{b}^{-1} = (c/b_1, \ldots, c/b_m)$ and $\mathbf{a}/\mathbf{b} = \mathbf{a} \circ \mathbf{b}^{-1} = (a_1/b_1, \ldots, a_m/b_m)$. If $\mathbf{a} > \mathbf{0}$, we write $\mathbf{a}^c = (a_1^c, \ldots, a_m^c)$ and $\mathbf{a}^{\mathbf{b}} = (a_1^{b_1}, \ldots, a_m^{b_m})$. We also define $[\mathbf{a}] := \prod_{i=1}^{m} a_i$. If $\mathbf{0} \le \mathbf{a} \le \mathbf{b} \le \boldsymbol{\infty}$, let $\mathbf{R}_{\mathbf{a}}^{\mathbf{b}} = (a_1, b_1) \times \cdots \times (a_m, b_m)$.

2. Main Theorems

Let $0 < p, q < \infty$, $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{R}^m$, and $u\colon \mathbf{E} \mapsto [0, \infty]$. Define

$$A_{\mathbf{d}}^{pq}(u) = \begin{cases} \|A_{\mathbf{d}}^{pq}(u; \mathbf{x})\|_{p/(p-q)}, & \text{if } p > q,\ \mathbf{n} \ge \mathbf{1}, \\ \sup\limits_{\mathbf{x} \in \mathbf{E}} [\mathbf{x}]^{1-q/p} A_{\mathbf{d}}^{pq}(u; \mathbf{x}), & \text{if } p = q,\ \mathbf{n} \ge \mathbf{1}, \text{ or } p \le q,\ \mathbf{n} = \mathbf{1}, \end{cases}$$

where

$$A_{\mathbf{d}}^{pq}(u; \mathbf{x}) = \int_{\mathbf{R}_{\mathbf{1}}^{\infty}} u(\mathbf{t} \circ \mathbf{x}) \big[\mathbf{t}^{\mathbf{n}-(q/p)\mathbf{d}-\mathbf{1}}\big]\, d\mathbf{t}.$$

For $1 < \tau \le p$ we also define

$$\mathbb{K}^p_{\tau,\mathbf{d}} = \begin{cases} \left(\int_{\mathbf{I}} \phi(\mathbf{t})^{p/(p-\tau)} [\mathbf{t}^{(\tau\mathbf{1}-\mathbf{d})/(p-\tau)}]\, d\mathbf{t} \right)^{(p-\tau)/p}, & \text{if } 1 < \tau < p, \\ \sup_{\mathbf{t}\in\mathbf{I}} \phi(\mathbf{t})[\mathbf{t}^{\mathbf{1}-\mathbf{d}/p}], & \text{if } \tau = p. \end{cases}$$

THEOREM 2.1. *Let $T = H_\phi$. Let $1 < p = q < \infty$ and $\mathbf{n} \ge \mathbf{1}$, or $1 < p \le q < \infty$ and $\mathbf{n} = \mathbf{1}$. If* (1.1) *holds, then $A^{pq}_{\mathbf{d}}(u) < \infty$ for all $\mathbf{d} > \mathbf{n}$, and*

$$\left[\frac{\mathbf{n} \circ (\mathbf{d} - \mathbf{n})}{\mathbf{d}} \right]^{1/p} \left(\int_{\mathbf{I}} \phi(\mathbf{t})\, d\mathbf{t} \right) A^{pq}_{\mathbf{d}}(u)^{1/q} \le \|T\|_{L^+_p \to L^+_{q,u}}. \tag{2.1}$$

Conversely, if there exist $1 < \tau \le p$ and $\mathbf{d} > \mathbf{n}$ such that $\mathbb{K}^p_{\tau,\mathbf{d}} < \infty$ and $A^{pq}_{\mathbf{d}}(u) < \infty$, then (1.1) *holds and*

$$\|T\|_{L^+_p \to L^+_{q,u}} \ll \left[\frac{\tau - 1}{\mathbf{d} - \mathbf{n}} \right]^{(\tau-1)/p} \mathbb{K}^p_{\tau,\mathbf{d}} A^{pq}_{\mathbf{d}}(u)^{1/q}. \tag{2.2}$$

THEOREM 2.2. *Let $T = H_\phi$. Let $1 < q < p < \infty$ and $\mathbf{n} \ge \mathbf{1}$. If there exist $p/q < \tau \le p$ and $\mathbf{d} > \mathbf{n}$ such that $\mathbb{K}^p_{\tau,\mathbf{d}} < \infty$ and $A^{pq}_{\mathbf{d}}(u) < \infty$, then we have* (1.1) *and*

$$\|T\|_{L^+_p \to L^+_{q,u}} \ll \left[\frac{\tau - 1}{\mathbf{d} - \mathbf{n}} \right]^{(\tau q-p)/(pq)} \mathbb{K}^p_{\tau,\mathbf{d}} A^{pq}_{\mathbf{d}}(u)^{1/q}. \tag{2.3}$$

If $u = \prod_{i=1}^m u_i$, where $u_i \colon E_i \mapsto [0, \infty]$, then $A^{pq}_{\mathbf{d}}(u) < \infty$ for all $\mathbf{d} > \mathbf{n}$ is necessary for (1.1) *to hold and*

$$\left[\frac{q(\mathbf{d} - \mathbf{n})}{p} \right]^{1/q} \left(\int_{\mathbf{I}} \phi(\mathbf{t}) \left[\mathbf{t}^{(q\mathbf{d}-p\mathbf{n})/(p^2-pq)} \right] d\mathbf{t} \right) A^{pq}_{\mathbf{d}}(u)^{1/q} \le \|T\|_{L^+_p \to L^+_{q,u}}. \tag{2.4}$$

THEOREM 2.3. *Let $1 < p < q < \infty$, $\mathbf{n} \ge \mathbf{1}$, and $\mathbf{n} \ne \mathbf{1}$. Then* (1.1) *holds for $T = H_\phi$ if and only if $u(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in \mathbf{E}$.*

We note that Theorem 2.3 still holds for any operator T that maps functions in L^+_p into $L^+_{q,u}$ and satisfies the inequality $\|H_\phi f\|_{q,u} \le c\|Tf\|_{q,u}$ for all $f \in L^+_p$.

As an application, we consider the case $\phi(\mathbf{t}) = [(\mathbf{1} - \mathbf{t}^{\mathbf{b}})^{\mathbf{a}-\mathbf{1}} \circ \mathbf{t}^{\mathbf{c}\circ\mathbf{b}+\mathbf{b}-\mathbf{1}}]$, where $\mathbf{a} = (a_1, \dots, a_m) > \mathbf{0}$, $\mathbf{b} = (b_1, \dots, b_m) > \mathbf{0}$, and $\mathbf{c} = (c_1, \dots, c_m) >$

$-\mathbf{1}$. In this case, $H_\phi f(\mathbf{x})$ is reduced to

$$J_{\mathbf{abc}} f(\mathbf{x}) = \int_{\mathbf{I}} [(\mathbf{1} - \mathbf{t}^{\mathbf{b}})^{\mathbf{a}-\mathbf{1}} \circ \mathbf{t}^{\mathbf{c}\circ\mathbf{b}+\mathbf{b}-\mathbf{1}}] f(\mathbf{t} \circ \mathbf{x})\, d\mathbf{t}$$

and $\Phi = \int_{\mathbf{I}} \phi(\mathbf{t})\, d\mathbf{t} = \prod_{i=1}^m B(c_i+1, a_i)/b_i$, where $B(\cdot,\cdot)$ is the beta function.

COROLLARY 2.4. *Let* $T = J_{\mathbf{abc}}$.

(1) *Let* $1 < p = q < \infty$ *and* $\mathbf{n} \ge \mathbf{1}$, *or* $1 < p \le q < \infty$ *and* $\mathbf{n} = \mathbf{1}$. *Suppose that* $p\mathbf{a} > \mathbf{1}$ *and* $p(\mathbf{c}+\mathbf{1}) \circ \mathbf{b} > \mathbf{n}$. *Then* (1.1) *holds if and only if* $A_{\mathbf{d}}^{pq}(u) < \infty$ *for all* $\mathbf{d} > \mathbf{n}$.

(2) *Let* $1 < q < p < \infty$ *and* $\mathbf{n} \ge \mathbf{1}$. *Let* $q\mathbf{a} > \mathbf{1}$ *and* $p(\mathbf{c}+\mathbf{1}) \circ \mathbf{b} > \mathbf{n}$. *If* $A_{\mathbf{d}}^{pq}(u) < \infty$ *for some* $\mathbf{n} < \mathbf{d} < p(\mathbf{c}+\mathbf{1}) \circ \mathbf{b}$, *then* (1.1) *holds. Conversely, if* (1.1) *holds and* $u = \prod_{i=1}^m u_i$, *where* $u_i \colon E_i \mapsto [0,\infty]$, *then* $A_{\mathbf{d}}^{pq}(u) < \infty$ *for all* $\mathbf{d} > \mathbf{n}$.

(3) *Let* $1 < p < q < \infty$, $\mathbf{n} \ge \mathbf{1}$, *and* $\mathbf{n} \ne \mathbf{1}$. *Then* (1.1) *holds if and only if* $u(\mathbf{x}) = 0$ *for almost all* $\mathbf{x} \in \mathbf{E}$.

Corollary 2.4 can be obtained by Theorems 2.1–2.3 and the fact that $\mathbb{K}_{\tau,\mathbf{d}}^p < \infty$ for any $1 < \tau < \min\{p, pa_1, \dots, pa_m\}$ and $\mathbf{n} < \mathbf{d} < p(\mathbf{c}+\mathbf{1}) \circ \mathbf{b}$.

THEOREM 2.5. *Let* $T = \mathscr{M}_{gh}^-$ *and* $\phi(\mathbf{t}) = g(\mathbf{t})h(\mathbf{1}-\mathbf{t})$. *Suppose that h is nonincreasing on each variable and* $h(\mathbf{1}^-) > 0$.

(1) *Let* $1 < p = q < \infty$ *and* $\mathbf{n} \ge \mathbf{1}$, *or* $1 < p \le q < \infty$ *and* $\mathbf{n} = \mathbf{1}$. *If* (1.1) *holds, then* $A_{\mathbf{d}}^{pq}(u) < \infty$ *for all* $\mathbf{d} > \mathbf{n}$. *Conversely, if there exist* $1 < \tau \le p$ *and* $\mathbf{d} > \mathbf{n}$ *such that* $\mathbb{K}_{\tau,\mathbf{d}}^p < \infty$ *and* $A_{\mathbf{d}}^{pq}(u) < \infty$, *then we have* (1.1).

(2) *Let* $1 < q < p < \infty$ *and* $\mathbf{n} \ge \mathbf{1}$. *If there exist* $p/q < \tau \le p$ *and* $\mathbf{d} > \mathbf{n}$ *such that* $\mathbb{K}_{\tau,\mathbf{d}}^p < \infty$ *and* $A_{\mathbf{d}}^{pq}(u) < \infty$, *then we have* (1.1). *Conversely, if* (1.1) *holds and* $u = \prod_{i=1}^m u_i$, *where* $u_i \colon E_i \mapsto [0,\infty]$, *then* $A_{\mathbf{d}}^{pq}(u) < \infty$ *for all* $\mathbf{d} > \mathbf{n}$.

(3) *Let* $1 < p < q < \infty$, $\mathbf{n} \ge \mathbf{1}$, *and* $\mathbf{n} \ne \mathbf{1}$. *Then* (1.1) *holds if and only if* $u(\mathbf{x}) = 0$ *for almost all* $\mathbf{x} \in \mathbf{E}$.

Since $h(\mathbf{1}^-) H_g f(\mathbf{x}) \le \mathscr{M}_{gh}^- f(\mathbf{x}) \le H_\phi f(\mathbf{x})$ for $f \in L_p^+$ and $\mathbf{x} \in E$, we have

$$h(\mathbf{1}^-) \|H_g\|_{L_p^+ \to L_{q,u}^+} \le \|\mathscr{M}_{gh}^-\|_{L_p^+ \to L_{q,u}^+} \le \|H_\phi\|_{L_p^+ \to L_{q,u}^+}. \tag{2.5}$$

Then Theorem 2.5 can be obtained by Theorems 2.1–2.3 and (2.5). It is interesting to note that Theorem 2.5(3) still holds for any $T = \mathscr{M}_{gh}^-$ such that $h(\mathbf{1}^-) > 0$.

If we choose $g(\mathbf{t}) = [(\mathbf{1}-\mathbf{t})^{\mathbf{a}-\mathbf{1}} \circ \mathbf{t}^{\mathbf{c}}]$, $h(\mathbf{z}) = [\mathbf{z}^{-\mathbf{b}}]$, where $\mathbf{a} > \mathbf{0}$, $\mathbf{b} \geq \mathbf{0}$, and $\mathbf{c} > -\mathbf{1}$, then $\mathscr{M}_{gh}^{-}$ is reduced to

$$\mathscr{M}_{\mathbf{abc}} f(\mathbf{x}) = \sup_{\mathbf{z}\in\mathbf{I}} [\mathbf{z}^{-\mathbf{b}}] \int_{\mathbf{I}_{\mathbf{z}}} \big[(\mathbf{1}-\mathbf{t})^{\mathbf{a}-\mathbf{1}} \circ \mathbf{t}^{\mathbf{c}}\big] f(\mathbf{t}\circ\mathbf{x})\, d\mathbf{t}.$$

By Corollary 2.4 and Theorem 2.5 we have Corollary 2.6.

COROLLARY 2.6. *Let $T = \mathscr{M}_{\mathbf{abc}}$.*

(1) *Let $1 < p = q < \infty$ and $\mathbf{n} \geq \mathbf{1}$, or $1 < p \leq q < \infty$ and $\mathbf{n} = \mathbf{1}$. Suppose that $p(\mathbf{a}-\mathbf{b}) > \mathbf{1}$ and $p(\mathbf{c}+\mathbf{1}) > \mathbf{n}$. Then* (1.1) *holds if and only if $A_{\mathbf{d}}^{pq}(u) < \infty$ for all $\mathbf{d} > \mathbf{n}$.*

(2) *Let $1 < q < p < \infty$ and $\mathbf{n} \geq \mathbf{1}$. Let $q(\mathbf{a}-\mathbf{b}) > \mathbf{1}$ and $p(\mathbf{c}+\mathbf{1}) > \mathbf{n}$. If $A_{\mathbf{d}}^{pq}(u) < \infty$ for some $\mathbf{n} < \mathbf{d} < p(\mathbf{c}+\mathbf{1})$, we have* (1.1). *Conversely, if* (1.1) *holds and $u = \prod_{i=1}^{m} u_i$, where $u_i \colon E_i \mapsto [0,\infty]$, then $A_{\mathbf{d}}^{pq}(u) < \infty$ for all $\mathbf{d} > \mathbf{n}$.*

(3) *Let $1 < p < q < \infty$, $\mathbf{n} \geq \mathbf{1}$, and $\mathbf{n} \neq \mathbf{1}$. Then* (1.1) *holds if and only if $u(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in \mathbf{E}$.*

In the following we investigate (1.1) for $T = G_\phi$, $\mathscr{G}_{gh}^{-}$, and $\mathscr{G}_{gh}^{-*}$. First we consider the case $T = G_\phi$. Suppose that $\phi\colon \mathbf{I} \mapsto (0,\infty)$ satisfies the following conditions:

(K1) $\Phi = \int_{\mathbf{I}} \phi(\mathbf{t})\, d\mathbf{t} = 1$,

(K2) $M_1 = \exp \int_{\mathbf{I}} \phi(\mathbf{t}) \log \phi(\mathbf{t})\, d\mathbf{t} < \infty$,

(K3) $M_2 = \exp\big(\int_{\mathbf{I}} \phi(\mathbf{t}) \log[\mathbf{t}^{\mathbf{1}-\mathbf{n}}]\, d\mathbf{t}\big) < \infty$.

We first deal with the existence of G_ϕ.

LEMMA 2.7. *Let $0 < p < \infty$ and let $\phi\colon \mathbf{I} \mapsto (0,\infty)$ satisfy* (K1)–(K3). *Then for all $f \in L_p^{+}$, $G_\phi f(\mathbf{x})$ exists and is finite for almost all $\mathbf{x} \in \mathbf{E}$.*

PROOF. Let $f \in L_p^{+}$. If $\mathbf{n} \neq \mathbf{1}$, by (1.3) we have $\int_{\mathbf{R}_0^\infty} f(\boldsymbol{\xi}\circ\boldsymbol{\sigma})^p [\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\, d\boldsymbol{\xi} < \infty$ for almost all $\boldsymbol{\sigma} \in \mathbf{B}$. Therefore $\int_{\mathbf{I}} f((\mathbf{t}\circ\boldsymbol{\eta})\circ\boldsymbol{\sigma})^p [\mathbf{t}^{\mathbf{n}-\mathbf{1}}]\, d\mathbf{t}$ is finite for all $\boldsymbol{\eta} > \mathbf{0}$ and for almost all $\boldsymbol{\sigma} \in \mathbf{B}$. This implies that $\int_{\mathbf{I}} f(\mathbf{t}\circ\mathbf{x})^p [\mathbf{t}^{\mathbf{n}-\mathbf{1}}]\, d\mathbf{t}$ is finite for almost all $\mathbf{x} \in \mathbf{E}$.

Let $\mathbf{x}$ be such an element in $\mathbf{E}$. Then

$$\exp\bigg(\int_{\mathbf{I}} \phi(\mathbf{t}) \log\big(\phi(\mathbf{t})^{-1} f(\mathbf{t}\circ\mathbf{x})^p [\mathbf{t}^{\mathbf{n}-\mathbf{1}}]\big)\, d\mathbf{t}\bigg)$$

exists and is finite. Since

$$\begin{aligned}(G_\phi f(\mathbf{x}))^p &= G_\phi(f^p)(\mathbf{x})\\ &= M_1 M_2 \exp\bigg(\int_{\mathbf{I}} \phi(\mathbf{t}) \log\big(\phi(\mathbf{t})^{-1}[\mathbf{t}^{\mathbf{n}-\mathbf{1}}] f(\mathbf{t}\circ\mathbf{x})^p\big)\, d\mathbf{t}\bigg),\end{aligned}$$

where M_1 and M_2 are defined by (K2) and (K3), respectively, we see that $G_\phi f(\mathbf{x})$ exists and is finite.

The proof of the case $\mathbf{n} = \mathbf{1}$ is similar and we omit the details.

THEOREM 2.8. *Let $T = G_\phi$, where $\phi\colon \mathbf{I} \mapsto (0,\infty)$ satisfies* (K1)–(K3).

(1) *Let $0 < p = q < \infty$ and $\mathbf{n} \geq \mathbf{1}$, or $0 < p \leq q < \infty$ and $\mathbf{n} = \mathbf{1}$. If* (1.1) *holds, then $A^{pq}_{\mathbf{d}}(u) < \infty$ for all $\mathbf{d} > \mathbf{n}$ and we have* (2.1). *Conversely, if there exist $\tau > 1$ and $\mathbf{d} > \mathbf{n}$ such that $M_{\tau,\mathbf{d}} = \exp\big(\int_{\mathbf{I}} \phi(\mathbf{t}) \log(\phi(\mathbf{t})[\mathbf{t}^{\mathbf{1}-\mathbf{d}/\tau}])\, d\mathbf{t}\big) < \infty$ and $A^{pq}_{\mathbf{d}}(u) < \infty$, then* (1.1) *holds and we have* (2.2) *with $\mathbb{K}^p_{\tau,\mathbf{d}}$ being replaced by $M^{\tau/p}_{\tau,\mathbf{d}}$.*

(2) *Let $0 < q < p < \infty$ and $\mathbf{n} \geq \mathbf{1}$. If there exist $\tau > p/q$ and $\mathbf{d} > \mathbf{n}$ such that $M_{\tau,\mathbf{d}} < \infty$ and $A^{pq}_{\mathbf{d}}(u) < \infty$, then* (1.1) *holds and we have* (2.3) *with $\mathbb{K}^p_{\tau,\mathbf{d}}$ being replaced by $M^{\tau/p}_{\tau,\mathbf{d}}$. Conversely, if* (1.1) *holds and $u = \prod_{i=1}^m u_i$, where $u_i\colon E_i \mapsto [0,\infty]$, then $A^{pq}_{\mathbf{d}}(u) < \infty$ for all $\mathbf{d} > \mathbf{n}$ and we have* (2.4) *with the integral $\int_{\mathbf{I}} \phi(\mathbf{t})[\mathbf{t}^{(q\mathbf{d}-p\mathbf{n})/(p^2-pq)}]\, d\mathbf{t}$ being replaced by $\exp\big(\int_{\mathbf{I}} \phi(\mathbf{t}) \log[\mathbf{t}^{(q\mathbf{d}-p\mathbf{n})/(p^2-pq)}]\, d\mathbf{t}\big)$.*

(3) *Let $0 < p < q < \infty$, $\mathbf{n} \geq \mathbf{1}$, and $\mathbf{n} \neq \mathbf{1}$. Then* (1.1) *holds if and only if $u(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in \mathbf{E}$.*

As an application, we consider the case $\phi(\mathbf{t}) = [\mathbf{a} \circ (\mathbf{1}-\mathbf{t})^{\mathbf{a}-\mathbf{1}}]$, $\mathbf{a} = (a_1,\dots,a_m) > \mathbf{0}$. In this case, ϕ satisfies (K1)–(K3) and $G_\phi f(x)$ can be reduced to

$$\mathscr{J}_{\mathbf{a}} f(\mathbf{x}) = \exp\bigg(\int_{\mathbf{I}} \big[\mathbf{a}\circ(\mathbf{1}-\mathbf{t})^{\mathbf{a}-\mathbf{1}}\big] \log f(\mathbf{t}\circ\mathbf{x})\, d\mathbf{t}\bigg).$$

Moreover,

$$M_1 = [\mathbf{a}\circ \mathbf{e}^{\frac{\mathbf{1}}{\mathbf{a}}-\mathbf{1}}] \quad \text{and} \quad M_2 = \Big[\mathbf{e}^{\left(\gamma\mathbf{1}+\frac{\Gamma'(\mathbf{a}+\mathbf{1})}{\Gamma(\mathbf{a}+\mathbf{1})}\right)\circ(\mathbf{n}-\mathbf{1})}\Big],$$

where γ is the Euler constant, $\mathbf{e} = (e,\dots,e)$, $\Gamma(\mathbf{a}+\mathbf{1}) = (\Gamma(a_1+1),\dots,\Gamma(a_m+1))$, and $\Gamma'(\mathbf{a}+\mathbf{1}) = (\Gamma'(a_1+1),\dots,\Gamma'(a_m+1))$. We also have

$$M_{\tau,\mathbf{d}} = \Big[\mathbf{a}\circ \mathbf{e}^{\frac{\mathbf{1}}{\mathbf{a}}-\mathbf{1}+\left(\gamma\mathbf{1}+\frac{\Gamma'(\mathbf{a}+\mathbf{1})}{\Gamma(\mathbf{a}+\mathbf{1})}\right)\circ(\mathbf{d}/\tau-\mathbf{1})}\Big].$$

COROLLARY 2.9. *Let* $T = \mathscr{J}_{\mathbf{a}}$.

(1) *Let* $0 < p = q < \infty$ *and* $\mathbf{n} \geq \mathbf{1}$, *or* $0 < p \leq q < \infty$ *and* $\mathbf{n} = \mathbf{1}$. *Then* (1.1) *holds if and only if* $A_{\mathbf{d}}^{pq}(u) < \infty$ *for all* $\mathbf{d} > \mathbf{n}$.

(2) *Let* $0 < q < p < \infty$ *and* $\mathbf{n} \geq \mathbf{1}$. *If* $A_{\mathbf{d}}^{pq}(u) < \infty$ *for some* $\mathbf{d} > \mathbf{n}$, *we have* (1.1). *Conversely, if* (1.1) *holds and* $u = \prod_{i=1}^{m} u_i$, *where* $u_i\colon E_i \mapsto [0, \infty]$, *then* $A_{\mathbf{d}}^{pq}(u) < \infty$ *for all* $\mathbf{d} > \mathbf{n}$.

(3) *Let* $0 < p < q < \infty$, $\mathbf{n} \geq \mathbf{1}$, *and* $\mathbf{n} \neq \mathbf{1}$. *Then* (1.1) *holds if and only if* $u(\mathbf{x}) = 0$ *for almost all* $\mathbf{x} \in \mathbf{E}$.

The results of the case $m = 1$, $n_1 = 1$, $\mathbf{E} = (0, \infty)$, and $0 < p, q < \infty$ of Corollary 2.9(1) and (2) have been obtained by [18, Corollary 3.4] and [27, Theorem 5.1].

THEOREM 2.10. *Let* $T = \mathscr{G}_{gh}^{-}$ *or* $\mathscr{G}_{gh}^{-*}$. *Suppose that* $h(\mathbf{1}^{-}) > 0$, *conditions* (K1)–(K3) *are satisfied with* ϕ *replaced by* g, *and* $h(\mathbf{z})^{p/\lambda}/\tilde{G}(\mathbf{z})$ *is nonincreasing on each variable for some* $\lambda > 1$. *Let* $\phi(\mathbf{t}) = g(\mathbf{t})h(\mathbf{1} - \mathbf{t})^{p/\lambda}/\tilde{G}(\mathbf{1} - \mathbf{t})$.

(1) *Let* $0 < p = q < \infty$ *and* $\mathbf{n} \geq \mathbf{1}$, *or* $0 < p \leq q < \infty$ *and* $\mathbf{n} = \mathbf{1}$. *If* (1.1) *holds, then* $A_{\mathbf{d}}^{pq}(u) < \infty$ *for all* $\mathbf{d} > \mathbf{n}$. *Conversely, if there exist* $1 < \tau \leq \lambda$ *and* $\mathbf{d} > \mathbf{n}$ *such that* $\mathbb{K}_{\tau,\mathbf{d}}^{\lambda} < \infty$ *and* $A_{\mathbf{d}}^{pq}(u) < \infty$, *then we have* (1.1).

(2) *Let* $0 < q < p < \infty$ *and* $\mathbf{n} \geq \mathbf{1}$. *If there exist* $p/q < \tau \leq \lambda$ *and* $\mathbf{d} > \mathbf{n}$ *such that* $\mathbb{K}_{\tau,\mathbf{d}}^{\lambda} < \infty$ *and* $A_{\mathbf{d}}^{pq}(u) < \infty$, *then we have* (1.1). *Conversely, if* (1.1) *holds and* $u = \prod_{i=1}^{m} u_i$, *where* $u_i\colon E_i \mapsto [0, \infty]$, *then* $A_{\mathbf{d}}^{pq}(u) < \infty$ *for all* $\mathbf{d} > \mathbf{n}$.

(3) *Let* $0 < p < q < \infty$, $\mathbf{n} \geq \mathbf{1}$, *and* $\mathbf{n} \neq \mathbf{1}$. *Then* (1.1) *holds if and only if* $u(\mathbf{x}) = 0$ *for almost all* $\mathbf{x} \in \mathbf{E}$.

PROOF. For $f \in L_p^+$ and $\mathbf{x} \in \mathbf{E}$, we define $k(\mathbf{x})^\lambda = f(\mathbf{x})^p$. Then

$$h(\mathbf{1}^-)(G_g k(\mathbf{x}))^{\lambda/p} \leq Tf(\mathbf{x}) \leq (\mathscr{M}_{gh^*}^{-} k(\mathbf{x}))^{\lambda/p},$$

where $T = \mathscr{G}_{gh}^{-}$ or $\mathscr{G}_{gh}^{-*}$, and $h^*(\mathbf{z}) = h(\mathbf{z})^{p/\lambda}/\tilde{G}(\mathbf{z})$. This implies that

$$h(\mathbf{1}^-)\|G_g\|_{L_\lambda^+ \to L_{\lambda q/p,u}^+}^{\lambda/p} \leq \|T\|_{L_p^+ \to L_{q,u}^+} \leq \|\mathscr{M}_{gh^*}^{-}\|_{L_\lambda^+ \to L_{\lambda q/p,u}^+}^{\lambda/p}.$$

Then Theorem 2.10 can be proved by Theorem 2.5 and Theorem 2.8.

It is interesting to note that Theorem 2.10(3) still holds for any $T = \mathscr{G}_{gh}^{-}$ and $\mathscr{G}_{gh}^{-*}$ such that $h(\mathbf{1}^-) > 0$ and conditions (K1)–(K3) are satisfied with ϕ being replaced by g.

Let $g(\mathbf{t}) = [\mathbf{a} \circ (\mathbf{1} - \mathbf{t})^{\mathbf{a}-\mathbf{1}}]$ and $h(\mathbf{z}) = [\mathbf{z}^{\mathbf{b}}]$, where $\mathbf{a} = (a_1, \ldots, a_m) > \mathbf{0}$, $\mathbf{b} = (b_1, \ldots, b_m) > \mathbf{0}$. Then for $\mathbf{x} \in \mathbf{E}$,

$$\mathscr{G}^-_{gh} f(\mathbf{x}) = \mathscr{G}^-_{\mathbf{ab}} f(\mathbf{x}) = \sup_{\mathbf{z}\in\mathbf{I}} [\mathbf{z}^{\mathbf{b}}] \exp\left(\left[\frac{\mathbf{a}}{\mathbf{z}^{\mathbf{a}}}\right] \int_{\mathbf{I}_{\mathbf{z}}} [(\mathbf{1} - \mathbf{t})^{\mathbf{a}-\mathbf{1}}] \log f(\mathbf{t} \circ \mathbf{x})\, d\mathbf{t}\right),$$

$$\mathscr{G}^{-*}_{gh} f(\mathbf{x}) = \mathscr{G}^{-*}_{\mathbf{ab}} f(\mathbf{x}) = \lim_{\epsilon \searrow 0} \sup_{\mathbf{z}\in\mathbf{I}} [\mathbf{z}^{\mathbf{b}}] \left(\left[\frac{\mathbf{a}}{\mathbf{z}^{\mathbf{a}}}\right] \int_{\mathbf{I}_{\mathbf{z}}} [(\mathbf{1} - \mathbf{t})^{\mathbf{a}-\mathbf{1}}] f(\mathbf{t} \circ \mathbf{x})^{\epsilon}\, d\mathbf{t}\right)^{1/\epsilon}.$$

Choose λ such that $\lambda \mathbf{1} > \frac{p\mathbf{b}}{\mathbf{a}}$. Then $h(\mathbf{z})^{p/\lambda}/\tilde{G}(\mathbf{z})$ is nonincreasing in each variable. Let $\phi(\mathbf{t}) = [\mathbf{a} \circ (\mathbf{1} - \mathbf{t})^{(p/\lambda)\mathbf{b}-\mathbf{1}}]$. We have

$$\mathbb{K}^{\lambda}_{\tau,\mathbf{d}} = \prod_{i=1}^{m} a_i B\left(\frac{pb_i - \tau}{\lambda - \tau}, \frac{\lambda - d_i}{\lambda - \tau}\right)^{(\lambda-\tau)/\lambda} < \infty$$

for $\tau < \lambda$, $\tau \mathbf{1} < p\mathbf{b}$, and $\mathbf{d} = (d_1, \ldots, d_m) < \lambda \mathbf{1}$.

COROLLARY 2.11. *Let* $T = \mathscr{G}^-_{\mathbf{ab}}$ *or* $\mathscr{G}^{-*}_{\mathbf{ab}}$, *and* $p\mathbf{b} > (\max\{1, p/q\})\mathbf{1}$.

(1) *Let* $0 < p = q < \infty$ *and* $\mathbf{n} \geq \mathbf{1}$, *or* $0 < p \leq q < \infty$ *and* $\mathbf{n} = \mathbf{1}$. *Then* (1.1) *holds if and only if* $A^{pq}_{\mathbf{d}}(u) < \infty$ *for all* $\mathbf{d} > \mathbf{n}$.

(2) *Let* $0 < q < p < \infty$ *and* $\mathbf{n} \geq \mathbf{1}$. *If there exists* $\mathbf{d} > \mathbf{n}$ *such that* $A^{pq}_{\mathbf{d}}(u) < \infty$, *then* (1.1) *holds. Conversely, if* (1.1) *holds and* $u = \prod_{i=1}^{m} u_i$, *where* $u_i \colon E_i \mapsto [0, \infty]$, *then* $A^{pq}_{\mathbf{d}}(u) < \infty$ *for all* $\mathbf{d} > \mathbf{n}$.

(3) *Let* $0 < p < q < \infty$, $\mathbf{n} \geq \mathbf{1}$, *and* $\mathbf{n} \neq \mathbf{1}$. *Then* (1.1) *holds if and only if* $u(\mathbf{x}) = 0$ *for almost all* $\mathbf{x} \in \mathbf{E}$.

In the following we consider a simple case as an example of our main results. Let $m = n = 2$, $n_1 = n_2 = 1$, $E_1 = E_2 = (0, \infty)$, $\mathbf{E} = (0, \infty) \times (0, \infty)$, and $\mathbf{I} = (0, 1) \times (0, 1)$. The operators considered in this paper are reduced to the following forms.

$$H_{\phi} f(x_1, x_2) := \int_0^1 \int_0^1 \phi(t_1, t_2) f(t_1 x_1, t_2 x_2)\, dt_2\, dt_1,$$

$$G_{\phi} f(x_1, x_2) := \exp\left(\int_0^1 \int_0^1 \phi(t_1, t_2) \log f(t_1 x_1, t_2 x_2)\, dt_2\, dt_1\right),$$

$$\mathscr{M}^-_{gh} f(x_1, x_2) := \sup_{0<z_1,z_2<1} h(z_1, z_2) \int_{1-z_1}^1 \int_{1-z_2}^1 g(t_1, t_2) f(t_1 x_1, t_2 x_2)\, dt_2\, dt_1,$$

$$\mathcal{G}_{gh}^{-}f(x_1,x_2) := \sup_{0<z_1,z_2<1} h(z_1,z_2)\exp\biggl(\frac{1}{\tilde{G}(z_1,z_2)}\int_{1-z_1}^{1}\int_{1-z_2}^{1} g(t_1,t_2)\log f(t_1x_1,t_2x_2)\,dt_2\,dt_1\biggr),$$

$$\mathcal{G}_{gh}^{-*}f(x_1,x_2) := \lim_{\epsilon\searrow 0}\sup_{0<z_1,z_2<1} h(z_1,z_2)\biggl(\frac{1}{\tilde{G}(z_1,z_2)}\int_{1-z_1}^{1}\int_{1-z_2}^{1} g(t_1,t_2)f(t_1x_1,t_2x_2)^{\epsilon}\,dt_2\,dt_1\biggr)^{1/\epsilon},$$

where $\tilde{G}(z_1,z_2)=\int_{1-z_1}^{1}\int_{1-z_2}^{1} g(t_1,t_2)\,dt_2\,dt_1$. Inequality (1.1) is then reduced to

$$\biggl(\int_0^\infty\int_0^\infty Tf(x_1,x_2)^q u(x_1,x_2)\,dx_2\,dx_1\biggr)^{1/q} \le C\biggl(\int_0^\infty\int_0^\infty f(x_1,x_2)^p\,dx_2\,dx_1\biggr)^{1/p}. \quad (2.6)$$

Here $f\ge 0$ and T is any one of the operators given above. For $0<p,q<\infty$, $d_1,d_2\in\mathbb{R}$, and $u\colon(0,\infty)\times(0,\infty)\mapsto[0,\infty]$,

$$A_{d_1,d_2}^{pq}(u) = \begin{cases} \bigl\|A_{d_1,d_2}^{pq}(u;x_1,x_2)\bigr\|_{p/(p-q)}, & \text{if } p>q,\\ \sup\limits_{0<x_1,x_2<\infty}(x_1x_2)^{1-q/p}A_{d_1,d_2}^{pq}(u;x_1,x_2), & \text{if } p\le q,\end{cases}$$

where

$$A_{d_1,d_2}^{pq}(u;x_1,x_2)=\int_1^\infty\int_1^\infty u(t_1x_1,t_2x_2)t_1^{-qd_1/p}t_2^{-qd_2/p}\,dt_2\,dt_1.$$

For $1<\tau\le p$ we also have

$$\mathbb{K}_{\tau,d_1,d_2}^{p} = \begin{cases} \biggl\{\displaystyle\int_0^1\int_0^1 \phi(t_1,t_2)^{\frac{p}{p-\tau}}t_1^{\frac{\tau-d_1}{p-\tau}}t_2^{\frac{\tau-d_2}{p-\tau}}\,dt_2\,dt_1\biggr\}^{(p-\tau)/p}, & \text{if } 1<\tau<p,\\ \sup\limits_{0<t_1,t_2<1}\phi(t_1,t_2)t_1^{1-d_1/p}t_2^{1-d_2/p}, & \text{if } \tau=p.\end{cases}$$

Theorem 2.1 shows that for $T=H_\phi$ and $1<p\le q<\infty$, if (2.6) holds, then $A_{d_1,d_2}^{pq}(u)<\infty$ for all $d_1>1$ and $d_2>1$. Conversely, if there exist $1<\tau\le p$ and $d_1>1$, $d_2>1$ such that $\mathbb{K}_{\tau,d_1,d_2}^{p}<\infty$ and $A_{d_1,d_2}^{pq}(u)<\infty$, then (2.6) holds. Theorem 2.2 gives results for the case $1<q<p<\infty$. Similarly, Theorems 2.5, 2.8, and 2.10 establish (2.6) for $T = \mathscr{M}^{-}_{gh}$, G_ϕ, and $\mathscr{G}^{-}_{gh}$, $\mathscr{G}^{-*}_{gh}$, respectively, under the condition $A^{pq}_{d_1,d_2}(u) < \infty$.

3. Proof of Theorem 2.1

We first consider the sufficiency part. Let $h^\tau = f^p$, $1 < \tau \le p$, and $\mathbf{d} > \mathbf{n}$. Then

$$H_\phi f(\mathbf{x}) \le \mathbb{K}^p_{\tau,\mathbf{d}} \left(\int_{\mathbf{I}} \left[\mathbf{t}^{\mathbf{d}-\tau\mathbf{1}}\right]^{1/\tau} h(\mathbf{t}\circ\mathbf{x})\, d\mathbf{t} \right)^{\tau/p}. \tag{3.1}$$

Consider the case $1 < p = q < \infty$, $\mathbf{n} \ge \mathbf{1}$, and $\mathbf{n} \ne \mathbf{1}$. By (3.1) and (1.3) we have

$$\begin{aligned}\int_{\mathbf{E}} & H_\phi f(\mathbf{x})^p u(\mathbf{x})\, d\mathbf{x} \\ &\le (\mathbb{K}^p_{\tau,\mathbf{d}})^p \int_{\mathbf{B}} \int_{\mathbf{R}_0^\infty} \left(\int_{\mathbf{I}} \left[\mathbf{t}^{\mathbf{d}-\tau\mathbf{1}}\right]^{1/\tau} h(\mathbf{t}\circ(\boldsymbol{\xi}\circ\boldsymbol{\sigma}))\, d\mathbf{t} \right)^\tau u(\boldsymbol{\xi}\circ\boldsymbol{\sigma}) \left[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}\right] d\boldsymbol{\xi}\, d\boldsymbol{\sigma}.\end{aligned}$$

Let $g(\mathbf{s}) = [\mathbf{s}^{\mathbf{d}-\tau\mathbf{1}}]^{1/\tau} h(\mathbf{s}\circ\boldsymbol{\sigma})$. Then

$$\int_{\mathbf{I}} \left[\mathbf{t}^{\mathbf{d}-\tau\mathbf{1}}\right]^{1/\tau} h(\mathbf{t}\circ(\boldsymbol{\xi}\circ\boldsymbol{\sigma}))\, d\mathbf{t} = \left[\boldsymbol{\xi}^{-\mathbf{d}}\right]^{1/\tau} \int_{\mathbf{R}_0^{\boldsymbol{\xi}}} g(\mathbf{s})\, d\mathbf{s}$$

and by [33, Theorem 2.1], we have

$$\begin{aligned}\int_{\mathbf{R}_0^\infty} & \left(\int_{\mathbf{R}_0^{\boldsymbol{\xi}}} g(\mathbf{s})\, d\mathbf{s} \right)^\tau u(\boldsymbol{\xi}\circ\boldsymbol{\sigma}) \left[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}-\mathbf{d}}\right] d\boldsymbol{\xi} \\ &\ll \left[\frac{\tau-1}{\mathbf{d}-\mathbf{n}}\right]^{\tau-1} C(\boldsymbol{\sigma}) \int_{\mathbf{R}_0^\infty} g(\boldsymbol{\xi})^\tau \left[\boldsymbol{\xi}^{\tau\mathbf{1}+\mathbf{n}-\mathbf{1}-\mathbf{d}}\right] d\boldsymbol{\xi},\end{aligned}$$

where

$$C(\boldsymbol{\sigma}) = \sup_{\mathbf{s}>\mathbf{0}} \left[\mathbf{s}^{\mathbf{d}-\mathbf{n}}\right] \int_{\mathbf{R}_\mathbf{s}^\infty} u(\boldsymbol{\xi}\circ\boldsymbol{\sigma}) \left[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{d}-\mathbf{1}}\right] d\boldsymbol{\xi} = \sup_{\mathbf{s}>\mathbf{0}} A^{pp}_{\mathbf{d}}(u; \mathbf{s}\circ\boldsymbol{\sigma}).$$

By $\int_{\mathbf{R}_0^\infty} g(\boldsymbol{\xi})^\tau [\boldsymbol{\xi}^{\tau\mathbf{1}+\mathbf{n}-\mathbf{1}-\mathbf{d}}]\, d\boldsymbol{\xi} = \int_{\mathbf{R}_0^\infty} f(\boldsymbol{\xi}\circ\boldsymbol{\sigma})^p [\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\, d\boldsymbol{\xi}$ we obtain

$$\begin{aligned}\|H_\phi f\|_{p,u} &\ll \mathbb{K}^p_{\tau,\mathbf{d}} \left[\frac{\tau-1}{\mathbf{d}-\mathbf{n}}\right]^{(\tau-1)/p} \left(\int_{\mathbf{B}} C(\boldsymbol{\sigma}) \int_{\mathbf{R}_0^\infty} f(\boldsymbol{\xi}\circ\boldsymbol{\sigma})^p \left[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}\right] d\boldsymbol{\xi}\, d\boldsymbol{\sigma} \right)^{1/p} \\ &\le \left[\frac{\tau-1}{\mathbf{d}-\mathbf{n}}\right]^{(\tau-1)/p} \mathbb{K}^p_{\tau,\mathbf{d}} A^{pp}_{\mathbf{d}}(u)^{1/p} \|f\|_p.\end{aligned}$$

Consider the case $1 < p \le q < \infty$, $\mathbf{n} = \mathbf{1}$, and $E_1 = \cdots = E_m = (0, \infty)$. By (3.1), we have

$$\int_{\mathbf{E}} H_\phi f(\mathbf{x})^q u(\mathbf{x})\, d\mathbf{x} \le (\mathbb{K}^p_{\tau,\mathbf{d}})^q \int_{\mathbf{R}_0^\infty} \left(\int_{\mathbf{I}} \big[\mathbf{t}^{\mathbf{d}-\tau\mathbf{1}}\big]^{1/\tau} h(\mathbf{t} \circ \mathbf{x})\, d\mathbf{t} \right)^{\tau q/p} u(\mathbf{x})\, d\mathbf{x}.$$

Let $g(\mathbf{s}) = [\mathbf{s}^{\mathbf{d}-\tau\mathbf{1}}]^{1/\tau} h(\mathbf{s})$. By [33, Theorem 2.1] and similar calculation to that given in the previous case we have

$$\begin{aligned} \|H_\phi f\|_{q,u} &\le \mathbb{K}^p_{\tau,\mathbf{d}} \left\{ \int_{\mathbf{R}_0^\infty} \left(\int_{\mathbf{R}_0^{\mathbf{x}}} g(\mathbf{s})\, d\mathbf{s} \right)^{\tau q/p} u(\mathbf{x}) [\mathbf{x}^{-\mathbf{d}}]^{q/p}\, d\mathbf{x} \right\}^{1/q} \\ &\ll \left[\frac{\tau - 1}{\mathbf{d} - \mathbf{1}} \right]^{(\tau-1)/p} \mathbb{K}^p_{\tau,\mathbf{d}} A^{pq}_{\mathbf{d}}(u)^{1/q} \|f\|_p. \end{aligned}$$

In the following we consider the necessity part. Suppose that (1.1) holds for $T = H_\phi$. Let $1 < p = q < \infty$, $\mathbf{n} = (n_1, \dots, n_m) \ge \mathbf{1}$, $\mathbf{n} \ne \mathbf{1}$, $\mathbf{d} = (d_1, \dots, d_m) > \mathbf{n}$, and $\mathbf{a} = (a_1, \dots, a_m) > \mathbf{0}$. Suppose that for some $1 \le \ell \le m$, $n_{k(s)} > 1$ for $s = 1, \dots, \ell$. Then for $\mathbf{x} = (x_1, \dots, x_m) \in \mathbf{E}$, we write $x_i = \xi_i \sigma_i$ for $i = k(1), \dots, k(\ell)$, where $\xi_i > 0$ and $\sigma_i \in B_i$. If $\ell < m$, we have $n_{j(s)} = 1$ and we consider the case $E_{j(s)} = (0, \infty)$ for $s = 1, \dots, m - \ell$. Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)$ and $\boldsymbol{\sigma} = (\sigma_{k(1)}, \dots, \sigma_{k(\ell)})$. In the case $\ell < m$, $\xi_i = x_i$ for $i = j(1), \dots, j(m - \ell)$. Let $\mathbf{B} = B_{k(1)} \times \cdots \times B_{k(\ell)}$ and let ψ be a nonnegative function defined on $\mathbf{B}$ such that ψ^p is integrable on $\mathbf{B}$. We choose $f(\mathbf{x}) = g_{\mathbf{a}}(\boldsymbol{\xi}) \psi(\boldsymbol{\sigma})$, where

$$g_{\mathbf{a}}(\boldsymbol{\xi}) = \sum_{j_1=1}^{2} \cdots \sum_{j_m=1}^{2} \prod_{i=1}^{m} \chi_{I_{j_i}}(\xi_i) a_i^{-n_i/p} \left(\frac{a_i}{\xi_i} \right)^{d_i(j_i-1)/p}. \tag{3.2}$$

Here $I_{j_i} = (0, a_i)$ if $j_i = 1$ and $I_{j_i} = (a_i, \infty)$ if $j_i = 2$. We first show that $\int_{\mathbf{E}} f(\mathbf{x})^p\, d\mathbf{x} < \infty$. Since

$$\begin{aligned} \int_{\mathbf{R}_0^\infty} g_{\mathbf{a}}(\boldsymbol{\xi})^p [\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\, d\boldsymbol{\xi} &= \sum_{j_1=1}^{2} \cdots \sum_{j_m=1}^{2} \prod_{i=1}^{m} \int_{I_{j_i}} a_i^{-n_i} \left(\frac{a_i}{\xi_i} \right)^{d_i(j_i-1)} \xi_i^{n_i-1}\, d\xi_i \\ &= \sum_{j_1=1}^{2} \cdots \sum_{j_m=1}^{2} \prod_{i=1}^{m} \frac{(-1)^{j_i}}{d_i(j_i - 1) - n_i} = \left[\frac{\mathbf{d}}{\mathbf{n}(\mathbf{d} - \mathbf{n})} \right], \end{aligned}$$

we see that

$$\int_{\mathbf{E}} f(\mathbf{x})^p \, d\mathbf{x} = \int_{\mathbf{B}} \left(\int_{\mathbf{R}_0^\infty} g_{\mathbf{a}}(\boldsymbol{\xi})^p [\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}] \, d\boldsymbol{\xi} \right) \psi(\boldsymbol{\sigma})^p \, d\boldsymbol{\sigma}$$
$$= \left[\frac{\mathbf{d}}{\mathbf{n}(\mathbf{d}-\mathbf{n})} \right] \int_{\mathbf{B}} \psi(\boldsymbol{\sigma})^p \, d\boldsymbol{\sigma} < \infty.$$

Here $d\boldsymbol{\sigma} = d\sigma_{k(\ell)} \ldots d\sigma_{k(1)}$. On the other hand, for $\boldsymbol{\xi} > \mathbf{a}$, we have

$$\begin{aligned} H_\phi f(\mathbf{x}) &= \int_{\mathbf{I}} \phi(\mathbf{t}) g_{\mathbf{a}}(\mathbf{t} \circ \boldsymbol{\xi}) \psi(\boldsymbol{\sigma}) \, d\mathbf{t} \\ &= \sum_{j_1=1}^{2} \cdots \sum_{j_m=1}^{2} \psi(\boldsymbol{\sigma}) \Bigg\{ \left(\prod_{i=1}^{m} a_i^{-n_i/p} \left(\frac{a_i}{\xi_i} \right)^{d_i(j_i-1)/p} \right) \\ &\quad \times \int_0^1 \cdots \int_0^1 \phi(t_1, \ldots, t_m) \left(\prod_{i=1}^{m} \chi_{I_{j_i}}(t_i \xi_i) t_i^{d_i(1-j_i)/p} \right) dt_m \ldots dt_1 \Bigg\} \\ &\geq \left[\mathbf{a}^{\mathbf{d}-\mathbf{n}} \boldsymbol{\xi}^{-\mathbf{d}} \right]^{1/p} \psi(\boldsymbol{\sigma}) \int_{\mathbf{I}} \phi(\mathbf{t}) \, d\mathbf{t}. \end{aligned}$$

The last inequality is based on the facts that $a_i^{-n_i} > a_i^{d_i - n_i} \xi_i^{-d_i}$ and $t_i^{-d_i} > 1$ for $0 < t_i < 1$. Let $\Phi = \int_{\mathbf{I}} \phi(\mathbf{t}) \, d\mathbf{t}$. Then

$$\int_{\mathbf{E}} H_\phi f(\mathbf{x})^p u(\mathbf{x}) \, d\mathbf{x} \geq \Phi^p [\mathbf{a}^{\mathbf{d}-\mathbf{n}}] \int_{\mathbf{B}} \int_{\mathbf{R}_{\mathbf{a}}^\infty} u(\boldsymbol{\xi} \circ \boldsymbol{\sigma}) \left[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}-\mathbf{d}} \right] \psi(\boldsymbol{\sigma})^p \, d\boldsymbol{\xi} \, d\boldsymbol{\sigma}.$$

Inequality (1.1) with $T = H_\phi$ implies

$$\int_{\mathbf{B}} C^p \left[\frac{\mathbf{d}}{\mathbf{n}(\mathbf{d}-\mathbf{n})} \right] \psi(\boldsymbol{\sigma})^p \, d\boldsymbol{\sigma}$$
$$\geq \Phi^p \int_{\mathbf{B}} \left([\mathbf{a}^{\mathbf{d}-\mathbf{n}}] \int_{\mathbf{R}_{\mathbf{a}}^\infty} u(\boldsymbol{\xi} \circ \boldsymbol{\sigma}) \left[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}-\mathbf{d}} \right] d\boldsymbol{\xi} \right) \psi(\boldsymbol{\sigma})^p \, d\boldsymbol{\sigma}.$$

Since this inequality holds for all nonnegative functions ψ defined on $\mathbf{B}$ such that ψ^p is integrable on $\mathbf{B}$, we obtain

$$C^p \left[\frac{\mathbf{d}}{\mathbf{n}(\mathbf{d}-\mathbf{n})} \right] \geq \Phi^p [\mathbf{a}^{\mathbf{d}-\mathbf{n}}] \int_{\mathbf{R}_{\mathbf{a}}^\infty} u(\boldsymbol{\xi} \circ \boldsymbol{\sigma}) \left[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}-\mathbf{d}} \right] d\boldsymbol{\xi} = \Phi^p A_{\mathbf{d}}^{pp}(u; \mathbf{a} \circ \boldsymbol{\sigma})$$

for almost all $\boldsymbol{\sigma} \in \mathbf{B}$. Moreover, this inequality holds for all $\mathbf{a} > \mathbf{0}$ and hence we have (2.1).

Consider the case $1 < p \le q < \infty$, $\mathbf{n} = \mathbf{1}$, and $E_1 = \cdots = E_m = (0, \infty)$. Let $\mathbf{d} > \mathbf{1}$ and $f(\mathbf{x}) = g_{\mathbf{a}}(\mathbf{x})$, where $\mathbf{a} > \mathbf{0}$ and $g_{\mathbf{a}}$ is defined by (3.2). Then $\int_{\mathbf{E}} f(\mathbf{x})^p \, d\mathbf{x} = \left[\frac{\mathbf{d}}{\mathbf{d}-\mathbf{1}}\right]$. For $\mathbf{x} > \mathbf{a}$, we have $H_\phi f(\mathbf{x}) \ge [\mathbf{a}^{\mathbf{d}-\mathbf{1}}\mathbf{x}^{-\mathbf{d}}]^{1/p}\Phi$. Therefore

$$\int_{\mathbf{E}} H_\phi f(\mathbf{x})^q u(\mathbf{x}) \, d\mathbf{x} \ge \Phi^q [\mathbf{a}^{\mathbf{d}-\mathbf{1}}]^{q/p} \int_{\mathbf{R}_{\mathbf{a}}^\infty} u(\mathbf{x})[\mathbf{x}^{-\mathbf{d}}]^{q/p} \, d\mathbf{x}.$$

If (1.1) holds for $T = H_\phi$, then

$$\begin{aligned} C\left[\frac{\mathbf{d}}{\mathbf{d}-\mathbf{1}}\right]^{1/p} &\ge \Phi[\mathbf{a}^{\mathbf{d}-\mathbf{1}}]^{1/p}\left(\int_{\mathbf{R}_{\mathbf{a}}^\infty} u(\mathbf{x})[\mathbf{x}^{-\mathbf{d}}]^{q/p} \, d\mathbf{x}\right)^{1/q} \\ &= \Phi[\mathbf{a}]^{1/q-1/p} A_{\mathbf{d}}^{pq}(u; \mathbf{a})^{1/q}. \end{aligned}$$

Since this inequality holds for all $\mathbf{a} > \mathbf{0}$, we have (2.1).

4. Proof of Theorem 2.2

First we prove the sufficiency part. Consider the case $\mathbf{n} \ge \mathbf{1}$ and $\mathbf{n} \ne \mathbf{1}$. Suppose that $p/q < \tau \le p$, $\mathbf{d} > \mathbf{n}$, and $\mathbb{K}_{\tau,\mathbf{d}}^p < \infty$ and $A_{\mathbf{d}}^{pq}(u) < \infty$. Let $h^\tau = f^p$. Inequalities (3.1) and (1.3) imply

$$\begin{aligned} &\int_{\mathbf{E}} H_\phi f(\mathbf{x})^q u(\mathbf{x}) \, d\mathbf{x} \\ &\le (\mathbb{K}_{\tau,\mathbf{d}}^p)^q \int_{\mathbf{B}} \int_{\mathbf{R}_{\mathbf{0}}^\infty} \left(\int_{\mathbf{I}} [\mathbf{t}^{\mathbf{d}-\tau\mathbf{1}}]^{1/\tau} h(\mathbf{t} \circ (\boldsymbol{\xi} \circ \boldsymbol{\sigma})) \, d\mathbf{t}\right)^{\tau q/p} u(\boldsymbol{\xi} \circ \boldsymbol{\sigma})[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}] \, d\boldsymbol{\xi} \, d\boldsymbol{\sigma} \end{aligned}$$

and then by [33, Theorem 3.1] and by letting $g(\mathbf{s}) = [\mathbf{s}^{\mathbf{d}-\tau\mathbf{1}}]^{1/\tau} h(\mathbf{s} \circ \boldsymbol{\sigma})$, we have

$$\begin{aligned} &\left\{\int_{\mathbf{R}_{\mathbf{0}}^\infty} \left(\int_{\mathbf{I}} [\mathbf{t}^{\mathbf{d}-\tau\mathbf{1}}]^{1/\tau} h(\mathbf{t} \circ (\boldsymbol{\xi} \circ \boldsymbol{\sigma})) \, d\mathbf{t}\right)^{\tau q/p} u(\boldsymbol{\xi} \circ \boldsymbol{\sigma})[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}] \, d\boldsymbol{\xi}\right\}^{p/(\tau q)} \\ &\qquad = \left\{\int_{\mathbf{R}_{\mathbf{0}}^\infty} \left(\int_{\mathbf{R}_{\mathbf{0}}^{\xi}} g(\mathbf{s}) \, d\mathbf{s}\right)^{\tau q/p} u(\boldsymbol{\xi} \circ \boldsymbol{\sigma})\left[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}-(q/p)\mathbf{d}}\right] d\boldsymbol{\xi}\right\}^{p/(\tau q)} \\ &\qquad \ll \left[\frac{\tau-1}{\mathbf{d}-\mathbf{n}}\right]^{(\tau q-p)/(\tau q)} C_1(\boldsymbol{\sigma})^{p/\tau} \left\{\int_{\mathbf{R}_{\mathbf{0}}^\infty} g(\boldsymbol{\xi})^\tau \left[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{d}+\tau\mathbf{1}-\mathbf{1}}\right] d\boldsymbol{\xi}\right\}^{1/\tau}, \end{aligned}$$

where $\int_{\mathbf{R}_0^\infty} g(\boldsymbol{\xi})^\tau[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{d}+\tau\mathbf{1}-\mathbf{1}}]\,d\boldsymbol{\xi} = \int_{\mathbf{R}_0^\infty} h(\boldsymbol{\xi}\circ\boldsymbol{\sigma})^\tau[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\,d\boldsymbol{\xi}$ and

$$C_1(\boldsymbol{\sigma}) = \left\{\int_{\mathbf{R}_0^\infty}\left(\int_{\mathbf{R}_\mathbf{s}^\infty} u(\boldsymbol{\xi}\circ\boldsymbol{\sigma})\big[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}-(q/p)\mathbf{d}}\big]\,d\boldsymbol{\xi}\right)^{p/(p-q)} \times \left[\mathbf{s}^{((\mathbf{d}-\mathbf{n}+\mathbf{1})q-p\mathbf{1})/(p-q)}\right]d\mathbf{s}\right\}^{(p-q)/pq}.$$

Hence

$$\begin{aligned}
&\|H_\phi f\|_{q,u}\\
&\ll \mathbb{K}_{\tau,\mathbf{d}}^p\left[\frac{\tau-1}{\mathbf{d}-\mathbf{n}}\right]^{(\tau q-p)/(pq)}\left\{\int_\mathbf{B} C_1(\boldsymbol{\sigma})^q\left(\int_{\mathbf{R}_0^\infty} h(\boldsymbol{\xi}\circ\boldsymbol{\sigma})^\tau[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\,d\boldsymbol{\xi}\right)^{q/p} d\boldsymbol{\sigma}\right\}^{1/q}\\
&\le \mathbb{K}_{\tau,\mathbf{d}}^p\left[\frac{\tau-1}{\mathbf{d}-\mathbf{n}}\right]^{(\tau q-p)/(pq)}\left\{\int_\mathbf{B} C_1(\boldsymbol{\sigma})^{pq/(p-q)}\,d\boldsymbol{\sigma}\right\}^{(p-q)/pq}\|f\|_p.
\end{aligned}$$

Here $\int_\mathbf{B} C_1(\boldsymbol{\sigma})^{pq/(p-q)}\,d\boldsymbol{\sigma} = A_\mathbf{d}^{pq}(u)^{p/(p-q)}$. Therefore we have (1.1) and (2.3) with $T = H_\phi$.

If $\mathbf{n} = \mathbf{1}$ and $E_1 = \cdots = E_m = (0,\infty)$, we have

$$\int_\mathbf{E} H_\phi f(\mathbf{x})^q u(\mathbf{x})\,d\mathbf{x} \le (\mathbb{K}_{\tau,\mathbf{d}}^p)^q\int_{\mathbf{R}_0^\infty}\left(\int_\mathbf{I}[\mathbf{t}^{\mathbf{d}-\tau\mathbf{1}}]^{1/\tau}h(\mathbf{t}\circ\mathbf{x})\,d\mathbf{t}\right)^{\tau q/p}u(\mathbf{x})\,d\mathbf{x}.$$

Then a similar proof can also be applied to obtain (2.3).

In the following we suppose that (1.1) holds and $u = \prod_{i=1}^m u_i$, where $u_i\colon E_i \mapsto [0,\infty]$. For $\mathbf{x} = (x_1,\dots,x_m) \in \mathbf{E}$, we write $x_i = \xi_i\sigma_i$ if $n_i > 1$ and $x_i = \xi_i$ if $n_i = 1$, where $\xi_i > 0$ and $\sigma_i \in B_i$. For a positive integer M we define

$$u_{i,M}(x_i) = \min\{u_i(x_i), M\}\chi_{(0,M)}(\xi_i) + \min\big\{u_i(x_i), \xi_i^{-q(n_i+1)/r}\big\}\chi_{(M,\infty)}(\xi_i),$$

where $1/r = 1/q - 1/p$. Let $u_M(\mathbf{x}) = \prod_{i=1}^M u_{i,M}(x_i)$. In the case $\mathbf{n} \ge \mathbf{1}$ and $\mathbf{n} \ne \mathbf{1}$, we write $\mathbf{x} = \boldsymbol{\xi}\circ\boldsymbol{\sigma}$, where $\boldsymbol{\xi} = (\xi_1,\dots,\xi_m)$, $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_m)$, and $\sigma_i = 1$ if $n_i = 1$. For $\mathbf{d} > \mathbf{n}$, let

$$f_M(\mathbf{x}) = \left[\boldsymbol{\xi}^{(q\mathbf{d}-p\mathbf{n})/(p^2-pq)}\right]U_M(\boldsymbol{\sigma};\boldsymbol{\xi})^{1/(p-q)}, \tag{4.1}$$

where $U_M(\boldsymbol{\sigma};\boldsymbol{\xi}) = [\boldsymbol{\xi}^{\mathbf{n}-(q/p)\mathbf{d}}]A_\mathbf{d}^{pq}(u_M;\mathbf{x})$. By (1.3) and the dual result of [33, Theorem 2.1],

$$\int_{\mathbf{E}} f_M(\mathbf{x})^p \, d\mathbf{x} = \int_{\mathbf{B}} \int_{\mathbf{R}_0^\infty} U_M(\boldsymbol{\sigma}; \boldsymbol{\xi})^{p/(p-q)} \big[\boldsymbol{\xi}^{(q\mathbf{d}-p\mathbf{n})/(p-q)+\mathbf{n}-\mathbf{1}}\big] \, d\boldsymbol{\xi} \, d\boldsymbol{\sigma}$$

$$\leq C^* \int_{\mathbf{B}} \int_{\mathbf{R}_0^\infty} u_M(\boldsymbol{\xi} \circ \boldsymbol{\sigma})^{p/(p-q)} [\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}] \, d\boldsymbol{\xi} \, d\boldsymbol{\sigma} < \infty,$$

where C^* is some finite constant. On the other hand, since $\| H_\phi f_M \|_{q,u_M} \leq C \| f_M \|_p$ and

$$H_\phi f_M(\mathbf{x}) \geq \mathbb{K}\big[\boldsymbol{\xi}^{(q\mathbf{d}-p\mathbf{n})/(p^2-pq)}\big] U_M(\boldsymbol{\sigma}; \boldsymbol{\xi})^{1/(p-q)},$$

where $\mathbb{K} = \int_{\mathbf{I}} \phi(\mathbf{t}) \big[\mathbf{t}^{(q\mathbf{d}-p\mathbf{n})/(p^2-pq)}\big] \, d\mathbf{t}$, we have

$$\mathbb{K}\left\{ \int_{\mathbf{B}} \int_{\mathbf{R}_0^\infty} [\boldsymbol{\xi}^{(q^2\mathbf{d}-pq\mathbf{n})/(p^2-pq)+\mathbf{n}-\mathbf{1}}] U_M(\boldsymbol{\sigma}; \boldsymbol{\xi})^{q/(p-q)} u_M(\boldsymbol{\xi} \circ \boldsymbol{\sigma}) \, d\boldsymbol{\xi} \, d\boldsymbol{\sigma} \right\}^{1/q}$$

$$\leq C \left\{ \int_{\mathbf{B}} \int_{\mathbf{R}_0^\infty} [\boldsymbol{\xi}^{(q\mathbf{d}-p\mathbf{n})/(p-q)+\mathbf{n}-\mathbf{1}}] U_M(\boldsymbol{\sigma}; \boldsymbol{\xi})^{p/(p-q)} \, d\boldsymbol{\xi} \, d\boldsymbol{\sigma} \right\}^{1/p} < \infty.$$

Since

$$\int_0^\infty \xi_i^{(qd_i-pn_i)/(p-q)+n_i-1} U_{i,M}(\sigma_i; \xi_i)^{p/(p-q)} \, d\xi_i$$

$$\leq \frac{p}{q(d_i - n_i)} \int_0^\infty \xi_i^{(q^2 d_i-pqn_i)/(p^2-pq)+n_i-1} U_{i,M}(\sigma_i; \xi_i)^{q/(p-q)} u_{i,M}(\xi_i \sigma_i) \, d\xi_i,$$

where $U_{i,M}(\sigma_i; \xi_i) = \int_{\xi_i}^\infty u_{i,M}(y_i \sigma_i) y_i^{n_i - qd_i/p-1} \, dy_i$, we have

$$\int_{\mathbf{R}_0^\infty} \big[\boldsymbol{\xi}^{(q\mathbf{d}-p\mathbf{n})/(p-q)+\mathbf{n}-\mathbf{1}}\big] U_M(\boldsymbol{\sigma}; \boldsymbol{\xi})^{p/(p-q)} \, d\boldsymbol{\xi}$$

$$\leq \left[\frac{p}{q(\mathbf{d} - \mathbf{n})}\right] \int_{\mathbf{R}_0^\infty} \big[\boldsymbol{\xi}^{(q^2\mathbf{d}-pq\mathbf{n})/(p^2-pq)+\mathbf{n}-\mathbf{1}}\big] U_M(\boldsymbol{\sigma}; \boldsymbol{\xi})^{q/(p-q)} u_M(\boldsymbol{\xi} \circ \boldsymbol{\sigma}) \, d\boldsymbol{\xi}.$$

Then

$$C \geq \mathbb{K}\left[\frac{q(\mathbf{d} - \mathbf{n})}{p}\right]^{1/q}$$

$$\cdot \left\{ \int_{\mathbf{B}} \int_{\mathbf{R}_0^\infty} \big[\boldsymbol{\xi}^{(q\mathbf{d}-p\mathbf{n})/(p-q)+\mathbf{n}-\mathbf{1}}\big] U_M(\boldsymbol{\sigma}; \boldsymbol{\xi})^{p/(p-q)} \, d\boldsymbol{\xi} \, d\boldsymbol{\sigma} \right\}^{1/q-1/p}.$$

By letting $M \to \infty$, $u_M \uparrow u$ and we have (2.4).

The proof of the case $\mathbf{n} = \mathbf{1}$ and $E_1 = \cdots = E_m = (0, \infty)$ is similar. We omit the details.

5. Proof of Theorem 2.3

Suppose that (1.1) holds for $T = H_\phi$. Assume $n_{k(s)} > 1$ for $s = 1, \ldots, \ell$. If $\ell = m$, then $n_i > 1$ for each $i = 1, \ldots, m$. If $1 \le \ell < m$, we set $n_{j(s)} = 1$ and $E_{j(s)} = (0, \infty)$ for $s = 1, \ldots, m - \ell$. For $\epsilon > 0$ and $\rho_{k(s)} \in B_{k(s)}$, we define $D_\epsilon(\rho_{k(s)}) = \{\sigma \in B_{k(s)} : \|\sigma - \rho_{k(s)}\| < \epsilon\}$, $s = 1, \ldots, \ell$. For each s, it is easy to see that for almost all $\rho_{k(s)} \in B_{k(s)}$, $|D_\epsilon(\rho_{k(s)})| > 0$ for all $\epsilon > 0$, where $|D_\epsilon(\rho_{k(s)})|$ is the measure of $D_\epsilon(\rho_{k(s)})$ as a subset of $B_{k(s)}$. Now fix such a $\rho_{k(s)} \in B_{k(s)}$ and let $\boldsymbol{\rho}^* = (\rho_{k(1)}, \ldots, \rho_{k(\ell)})$. Let $\epsilon > 0$ and $\mathbf{D}_\epsilon(\boldsymbol{\rho}^*) = D_\epsilon(\rho_{k(1)}) \times \cdots \times D_\epsilon(\rho_{k(\ell)})$. For $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbf{E}$, we write $x_i = \xi_i \sigma_i$ and let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$. Here if $\ell = m$ then $\sigma_i \in B_i$ for $i = 1, \ldots, m$, and if $1 \le \ell < m$ then $\xi_i = x_i$ and we set $\sigma_i = 1$ for $i = j(1), \ldots, j(m - \ell)$. Let $\boldsymbol{\sigma}^* = (\sigma_{k(1)}, \ldots, \sigma_{k(\ell)})$. Let $\mathbf{a} = (a_1, \ldots, a_m) > \mathbf{0}$ and define

$$f_\epsilon(\mathbf{x}) = \chi_{\mathbf{R}_0^{\mathbf{a}}}(\boldsymbol{\xi}) \chi_{\mathbf{D}_\epsilon(\boldsymbol{\rho}^*)}(\boldsymbol{\sigma}^*).$$

Then

$$\|f_\epsilon\|_p^p = \left[\frac{\mathbf{a}^{\mathbf{n}}}{\mathbf{n}}\right] \prod_{s=1}^{\ell} |D_\epsilon(\rho_{k(s)})|$$

and

$$\|H_\phi f_\epsilon\|_{q,u}^q \ge \Phi^q \int_{\mathbf{D}_\epsilon(\boldsymbol{\rho}^*)} \int_{\mathbf{R}_0^{\mathbf{a}}} u(\boldsymbol{\xi} \circ \boldsymbol{\sigma})[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\, d\boldsymbol{\xi}\, d\boldsymbol{\sigma}^*.$$

Here $\Phi = \int_{\mathbf{I}} \phi(\mathbf{t})\, d\mathbf{t}$ and $d\boldsymbol{\sigma}^* = d\sigma_{k(\ell)} \ldots d\sigma_{k(1)}$. Since $\|H_\phi f_\epsilon\|_{q,u} \le C \|f_\epsilon\|_p$, this implies

$$\Phi^q \int_{\mathbf{D}_\epsilon(\boldsymbol{\rho}^*)} \int_{\mathbf{R}_0^{\mathbf{a}}} u(\boldsymbol{\xi} \circ \boldsymbol{\sigma})[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\, d\boldsymbol{\xi}\, d\boldsymbol{\sigma}^* \le C^q \left[\frac{\mathbf{a}^{\mathbf{n}}}{\mathbf{n}}\right]^{q/p} \left(\prod_{s=1}^{\ell} |D_\epsilon(\rho_{k(s)})|\right)^{q/p}.$$

Hence

$$\frac{1}{\prod_{s=1}^{\ell} |D_\epsilon(\rho_{k(s)})|} \int_{\mathbf{D}_\epsilon(\boldsymbol{\rho}^*)} \int_{\mathbf{R}_0^{\mathbf{a}}} u(\boldsymbol{\xi} \circ \boldsymbol{\sigma})[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\, d\boldsymbol{\xi}\, d\boldsymbol{\sigma}^*$$
$$\le C^q \Phi^{-q} \left[\frac{\mathbf{a}^{\mathbf{n}}}{\mathbf{n}}\right]^{q/p} \left(\prod_{s=1}^{\ell} |D_\epsilon(\rho_{k(s)})|\right)^{q/p-1}. \quad (5.1)$$

Inequality (5.1) holds for almost all $\rho_{k(s)} \in B_{k(s)}$, $s = 1, \ldots, \ell$, and for all $\epsilon > 0$. By taking limits as $\epsilon \to 0$, $|D_\epsilon(\rho_{k(s)})| \to 0$. Since $p < q$, the right-hand side of (5.1) approaches zero and the left-hand side of (5.1) approaches $\int_{\mathbf{R}_0^{\mathbf{a}}} u(\boldsymbol{\xi} \circ \boldsymbol{\rho})[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\, d\boldsymbol{\xi}$ for almost all $\boldsymbol{\rho}^* \in B_{k(1)} \times \cdots \times B_{k(\ell)}$. Here if $\ell = m$ then $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_m) = \boldsymbol{\rho}^*$ and if $\ell < m$ then we set $\rho_i = 1$ for $i = j(1), \ldots, j(m-\ell)$. This implies $\int_{\mathbf{R}_0^{\mathbf{a}}} u(\boldsymbol{\xi} \circ \boldsymbol{\rho})[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\, d\boldsymbol{\xi} = 0$ for almost all $\boldsymbol{\rho}^* \in B_{k(1)} \times \cdots \times B_{k(\ell)}$. Since $\mathbf{a} > \mathbf{0}$ is arbitrary, we have $u(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in \mathbf{E}$.

6. Proof of Theorem 2.8

The proof is similar to that given in Theorem 2.1–2.3. For any $\tau > 1$, let $h^\tau = f^p$. Then $G_\phi f(\mathbf{x}) = G_\phi h(\mathbf{x})^{\tau/p}$ and (1.1) for $T = G_\phi$ holds if and only if

$$\left(\int_{\mathbf{E}} G_\phi h(\mathbf{x})^{\tau q/p} u(\mathbf{x})\, d\mathbf{x} \right)^{p/(\tau q)} \leq C^{p/\tau} \left(\int_{\mathbf{E}} h(\mathbf{x})^\tau\, d\mathbf{x} \right)^{1/\tau} \tag{6.1}$$

holds with the same constant C. For $\mathbf{d} > \mathbf{n}$ and $\tau > \max\{1, p/q\}$ such that $M_{\tau,\mathbf{d}} < \infty$, we write $\log h(\mathbf{t}\circ\mathbf{x}) = \log(\phi(\mathbf{t})[\mathbf{t}^{\mathbf{1}-\mathbf{d}/\tau}]) + \log(\phi(\mathbf{t})^{-1}[\mathbf{t}^{\mathbf{d}/\tau-\mathbf{1}}]h(\mathbf{t}\circ\mathbf{x}))$. Then $G_\phi h(\mathbf{x}) \leq M_{\tau,\mathbf{d}} \int_{\mathbf{I}} [\mathbf{t}^{\mathbf{d}/\tau-\mathbf{1}}] h(\mathbf{t} \circ \mathbf{x})\, d\mathbf{t}$ and therefore

$$\int_{\mathbf{E}} G_\phi h(\mathbf{x})^{\tau q/p} u(\mathbf{x})\, d\mathbf{x} \leq M_{\tau,\mathbf{d}}^{\tau q/p} \int_{\mathbf{E}} \left(\int_{\mathbf{I}} [\mathbf{t}^{\mathbf{d}/\tau-\mathbf{1}}] h(\mathbf{t} \circ \mathbf{x})\, d\mathbf{t} \right)^{\tau q/p} u(\mathbf{x})\, d\mathbf{x}.$$

By similar proofs to that given in Theorem 2.1–2.2, the sufficiency part of (1)–(2) can be obtained.

In the following, we suppose that (1.1) holds for $T = G_\phi$. Then (6.1) holds for $\tau = 1$. Consider the case $0 < p \leq q < \infty$. Let $\mathbf{n} = (n_1, \ldots, n_m) \geq \mathbf{1}$, $\mathbf{d} = (d_1, \ldots, d_m) > \mathbf{n}$, and $\mathbf{a} = (a_1, \ldots, a_m) > \mathbf{0}$. The result of the necessary part of the case $p = q$, $\mathbf{n} \geq \mathbf{1}$, and $\mathbf{n} \neq \mathbf{1}$ can be obtained by a similar proof to that given in Theorem 2.1 with $h(\mathbf{x}) = g_{\mathbf{a}}(\boldsymbol{\xi})\psi(\boldsymbol{\sigma})$, where $g_{\mathbf{a}}(\boldsymbol{\xi})$ is given in (3.2) with $p = 1$ and the inequality $G_\phi h(\mathbf{x}) \geq [\mathbf{a}^{\mathbf{d}-\mathbf{n}}\boldsymbol{\xi}^{-\mathbf{d}}]\psi(\boldsymbol{\sigma})$ for $\boldsymbol{\xi} > \mathbf{a}$. If $p \leq q$, $\mathbf{n} = \mathbf{1}$, and $E_1 = \cdots = E_m = (0, \infty)$, let $h(\mathbf{x}) = g_{\mathbf{a}}(\mathbf{x})$ with $p = 1$. Then $\int_{\mathbf{E}} h(\mathbf{x})\, d\mathbf{x} = \left[\frac{\mathbf{d}}{\mathbf{d}-\mathbf{1}}\right]$ and $G_\phi h(\mathbf{x}) \geq [\mathbf{a}^{\mathbf{d}-\mathbf{1}}\mathbf{x}^{-\mathbf{d}}]$ for $\mathbf{x} > \mathbf{a}$. Then the necessity part of this case can be easily derived by a similar argument to that given in Theorem 2.1. The result of the case $q < p$ can also be obtained by a similar argument to that given in Theorem 2.2 with $h_M(\mathbf{x}) = f_M(\mathbf{x})^p$ for any positive integer M and the inequality

$$G_\phi h_M(\mathbf{x}) \geq M_{\mathbf{d}} \big[\boldsymbol{\xi}^{(q\mathbf{d}-p\mathbf{n})/(p-q)}\big] U_M(\boldsymbol{\sigma}; \boldsymbol{\xi})^{p/(p-q)},$$

where f_M is given by (4.1) and $M_{\mathbf{d}} = \exp\big(\int_{\mathbf{I}} \phi(\mathbf{t}) \log[\mathbf{t}^{(q\mathbf{d}-p\mathbf{n})/(p-q)}]\, d\mathbf{t}\big)$.

In the case $0 < p < q < \infty$, $\mathbf{n} \geq \mathbf{1}$, and $\mathbf{n} \neq \mathbf{1}$, if (6.1) holds for $\tau = 1$, by choosing $h = f_\epsilon$, we have

$$\int_{\mathbf{E}} G_\phi h(\mathbf{x})^{q/p} u(\mathbf{x})\, d\mathbf{x} \geq \int_{\mathbf{D}_\epsilon(\boldsymbol{\rho}^*)} \int_{\mathbf{R}_0^{\mathbf{a}}} u(\boldsymbol{\xi} \circ \boldsymbol{\sigma})[\boldsymbol{\xi}^{\mathbf{n}-\mathbf{1}}]\, d\boldsymbol{\xi}\, d\boldsymbol{\sigma}^*,$$

where f_ϵ, $\mathbf{D}_\epsilon(\boldsymbol{\rho}^*)$, and $d\boldsymbol{\sigma}^*$ are given in Theorem 2.3. Therefore by a similar proof to that given in Theorem 2.3 we have $u(\mathbf{x}) = 0$ for almost all $\mathbf{x} \in \mathbf{E}$.

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