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ARGUESIAN LATTICES OF DIMENSION $n \leqq 4$

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Introduction. It is known (cf. Jónsson [3]) that every lattice of commuting equivalence relations is Arguesian, but it is still an open question whether, conversely, every Arguesian lattice is isomorphic to a lattice of commuting equivalent relations. The principal purpose of this note is to establish the converse statement for lattices of dimension $n \leqq 4$. Actually we prove a stronger statement, namely that every Arguesian lattice of dimension $n \leqq 4$ is isomorphic to a sublattice of the lattice of all subspaces of an Arguesian projective geometry of dimension $n-1$. Thus it follows that a lattice of dimension $n \leqq 4$ is isomorphic to a lattice of commuting equivalence relations if and only if it is isomorphic to a sublattice of all subspaces of an Arguesian projective geometry. The corresponding statement for $n=5$ is false; in fact it is known (cf. [2]) that there exists a five dimensional lattice which is isomorphic to a lattice of commuting equivalence relations but cannot be embedded in a complemented modular lattice.

1. Preliminaries. Our notion of a projective geometry differs from the classical concept in that we do not exclude degenerate geometries where some or all of the lines pass through only two distinct points. Thus we only assume that each line passes through at least two distinct points, that any two distinct points determine a unique line that passes through both of them, and that any line which intersects two sides of a triangle in distinct points also meets the third side.

If a projective plane geometry S contains two distinct non-degenerate lines, then it is easy to see that every line of S is non-degenerate. From this it follows that degenerate projective plane geometries are of a rather trivial nature, in fact, all but one of the points lie on the same line, and all the remaining lines are degenerate and concurrent. From these observations we easily obtain:

Theorem 1.1. *Suppose S is a degenerate projective plane and S' is a (non-degenerate) projective plane with the property that there are at least as many points on each line of S' as there are points in S. Given a one-to-*

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one mapping $p \rightarrow p'$ of the points p on a line U of S onto points p' on a line U' of S', there exists an isomorphism f of S into S' such that $f(p) = p'$ for every point p on U.

A projective geometry S is said to be *Arguesian* if any two triangles which are centrally perspective are also axially perspective. As is well known, an n dimensional, non-degenerate Arguesian projective geometry S is isomorphic to the projective geometry $PG_n(D)$ whose points are the one-dimensional subspaces of an $n+1$ dimensional vector space over a suitable division ring D. For later reference we state two simple consequences of this fact:

THEOREM 1.2. *If $2 \leqq m \leqq n$, then every m dimensional non-degenerate Arguesian projective geometry is isomorphic to an m dimensional subspace of an n dimensional non-degenerate Arguesian geometry.*

THEOREM 1.3. *If m is any cardinal and S is an n dimensional non-degenerate Arguesian projective geometry, then there exists an isomorphism of S into an n dimensional non-degenerate, Arguesian, projective geometry S' with the property that there are at least m points on each line of S'.*

We shall later need the notion of a *quadrangular sextuplet* of points or of lines, used in the construction of a coordinate system for a non-degenerate projective Arguesian geometry. Roughly speaking, six points on a line are said to form a quadrangular sextuplet if they are the points of intersection of this line with the sides and the diagonals of a quadrangle. More precisely, an ordered sextuplet $\langle p_0, p_1, p_2, q_0, q_1, q_2 \rangle$ of points on a line U is said to be quadrangular if there exist four distinct points r_0, r_1, r_2, r_3, not on U, no three of which are collinear, such that p_i, r_i and r_3 are collinear for $i = 0, 1, 2$ and q_i, r_j and r_k are collinear for $i, j, k = 0, 1, 2$ with $i \neq j \neq k \neq i$. Dually, an ordered sextuplet $\langle U_0, U_1, U_2, V_0, V_1, V_2 \rangle$ of lines in a plane P and passing through a point p is said to be quadrangular if there exist lines W_0, W_1, W_2, W_3 in P which do not pass through p and no three of which are concurrent, such that U_i, W_i and W_3 are concurrent for $i = 0, 1, 2$ and V_i, W_j and W_k are concurrent for $i, j, k = 0, 1, 2$ with $i \neq j \neq k \neq i$.

THEOREM 1.4. *Suppose S' and S are non-degenerate n dimensional Arguesian projective geometries, and let there be given a one-to-one correspondence $p \rightarrow p'$ between the points on a line U of S and the points on a line U' of S'. If, for any quadrangular sextuplet $\langle p_0, p_1, p_2, q_0, q_1, q_2 \rangle$ of points on U, the sextuplet $\langle p_0', p_1', p_2', q_0', q_1', q_2' \rangle$ is also quadrangular, then there exists an isomorphism f of S onto S' such that $f(p) = p'$ for every point p on U.*

For basic notions and results from lattice theory we refer the reader to Birkhoff [1]. We shall use $\leqq$ for the relation of lattice inclusion and write $x+y$ and xy for the sum, or least upper bound, and the product, or greatest lower bound, of two elements x and y. If $b \leqq a$, then a/b is the sublattice (quotient) consisting of all elements x with $b \leqq x \leqq a$. If a covers b, that is, if $b < a$, and if there is no element x with $b < x < a$, then we write $b \ll a$. The dimension of an element a of a finite dimensional modular lattice will be denoted by $\delta(a)$.

The family $L(S)$ of all subspaces of an $n-1$ dimensional projective geometry S is an n dimensional complemented modular lattice under set-inclusion. Conversely, any n dimensional complemented modular lattice A is isomorphic to the lattice of all subspaces of an $n-1$ dimensional projective geometry S. In fact, we may take for S the set consisting of all the atoms of A and define the line through two distinct atoms p and q to be the set of all those atoms which are contained in the lattice sum $p+q$. The isomorphism is then established by associating with each element x of A the set U_x consisting of all those atoms of A which are contained in x.

In view of these facts we shall sometimes adopt a geometric language when speaking of an n dimensional complemented modular lattice A. Thus we refer to the one and two dimensional elements of A as points and lines, respectively, and say that A is degenerate or non-degenerate according to whether the projective geometry associated with A is degenerate or non-degenerate.

A lattice A is said to be *Arguesian*[1] if it satisfies the following condition: *For any* $a_0,a_1,a_2,b_0,b_1,b_2 \in A$, *if*

$$y = (a_1+a_2)(b_1+b_2)[(a_0+a_1)(b_0+b_1) + (a_0+a_2)(b_0+b_2)],$$

then

$$(a_0+b_0)(a_1+b_1)(a_2+b_2) \leqq a_1(a_2+y) + b_1(b_2+y).$$

It is not hard to show that every Arguesian lattice is modular (cf. Jónsson [2]).

THEOREM 1.5. *In order for a projective geometry S to be Arguesian it is necessary and sufficient that the lattice $L(S)$ of all subspaces of S be Arguesian.*

The following theorem (cf. Jónsson [2]) will be used several times in Section 3:

THEOREM 1.6. *Suppose I and D are, respectively, an ideal and a dual ideal of a lattice A, such that*

[1] A lattice theoretic analogue of Desargues' Law was first given in Schutzenberger [4], the particular form of this condition that is used here can be found in Jónsson [3].

$$A = I \cup D \quad \textit{and} \quad I \cap D \neq \emptyset .$$

If f is an isomorphism of I into a lattice B, if g is an isomorphism of D into B, and if f and g agree on $I \cap D$, then f and g possess a common extension which maps A isomorphically into B.

2. Modular lattices of dimension $n \leqq 4$. A zero dimensional lattice consists of just one element, $0 = 1$, and a one dimensional lattice consists of exactly two elements 0 and 1. A two dimensional lattice consists of 0 and 1, and of one or more atoms. Since the sum of two distinct atoms is always 1, and their product is always 0, it is clear that such a lattice is completely determined up to isomorphism if we know the number of its atoms. The following trivial observation will be used several times in the next section:

THEOREM 2.1. *If A and A' are two dimensional lattices and A' has at least as many elements as A, then A is isomorphic to a sublattice of A'. In fact, given an atom p of A and an atom p' of A', there exists an isomorphism f of A into A' such that $f(p) = p'$.*

We shall henceforth assume that A is an n dimensional modular lattice, let a be the sum of all the atoms of A, and let b be the product of all the dual atoms of A. The conditions $\delta(a) = n$ and $\delta(b) = 0$ are equivalent and imply that A is complemented. If $\delta(a) = 1$, then a is an atom of A, and in fact a is the only atom of A. In this case A is completely determined by its $n-1$ dimensional sublattice $1/a$. Similarly, if $\delta(b) = n-1$, then the study of A reduces to the study of its $n-1$ dimensional sublattice $b/0$. We shall therefore be concerned here with the cases in which

$$1 < \delta(a) < n \quad \text{and} \quad 0 < \delta(b) < n-1 .$$

THEOREM 2.2. *For $n = 3$, 4, if*

$$0 < \delta(b) < \delta(a) < n ,$$

then

$$b < a \quad \textit{and} \quad A = a/0 \cup 1/b .$$

PROOF. Our hypothesis implies that either b is an atom or else a is a dual atom, for otherwise we would have $1 < \delta(b) < \delta(a) < 3$, which is impossible. If b is an atom, then $b \leqq a$ because a is the sum of all the atoms. Similarly, if a is a dual atom, then $b \leqq a$ because b is the product of all the dual atoms. Thus in either case, $b \leqq a$, and since $\delta(b) < \delta(a)$, it follows that $b < a$.

By the definitions of a and b, 0 and all the atoms of A belong to $a/0$, and 1 and all the dual atoms belong to $1/b$. It follows that if $n=3$, then every element x of A belongs to either $a/0$ or to $1/b$, and in order to prove this for $n=4$ we need only consider the case when $\delta(x)=2$.

Assume that b is an atom. If x does not belong to $a/0$, then there exists a unique atom p such that $p \leqq x$, for if q were another such atom, then we could have $x=p+q \leqq a$. For each dual atom y we have $\delta(x+y) \leqq 4$, and therefore

$$\delta(xy) = \delta(x)+\delta(y)-\delta(x+y) \geqq 2+3-4 = 1 ,$$

which shows that $xy \neq 0$. Therefore xy contains an atom, and since p is the only atom contained in x we infer that $p \leqq xy \leqq y$. Thus p is contained in all the dual atoms, and is therefore contained in their product b. Inasmuch as b was assumed to be an atom, it follows that $b=p \leqq x$, and therefore $x \in 1/b$.

The case in which a is a dual atom can be treated similarly.

THEOREM 2.3. *If* $n=4$, $\delta(a)=2$ *and* $\delta(b)=2$, *then*

$$a/0 \cup 1/b = A-X$$

where X *is the set of all irreducible elements* $x \in A$ *with* $\delta(x)=2$. *Furthermore, each element of* X *covers a unique atom and is covered by a unique dual atom, and two elements of* X *cover the same atom if and only if they are covered by the same dual atom. Finally, if* $a \neq b$, *then* ab *is an atom and is covered by* b, $a+b$ *is a dual atom and covers* a, *and* $ab \ll x \ll a+b$ *for every element* $x \in X$.

PROOF. Observe that all the two dimensional elements of A except a are additively irreducible, and that all the two dimensional elements except b are multiplicatively irreducible. Hence the only two dimensional elements which belong to $A-X$ are a and b. Since $a/0$ consists of 0 and a and of all the atoms of A, while $1/b$ consists of 1 and b and of all the dual atoms of A, the first part of the theorem follows.

Being irreducible, each element x of X covers a unique atom and is covered by a unique dual atom. If two distinct elements x and y of X cover the same atom, then this atom must be equal to xy. But $xy \ll x$ implies that $y \ll x+y$, and $xy \ll y$ implies that $x \ll x+y$. Therefore x and y are covered by the same dual atom, namely $x+y$. Similarly, if x and y are covered by the same dual atom, then they cover the same atom.

Now suppose $a \neq b$. Observe that $ab \neq 0$ because b contains an atom and all the atoms are contained in a. Therefore ab must be an atom, and inasmuch as b is additively irreducible, ab must be the only atom covered by b. Similarly $a+b$ is a dual atom and is the only dual atom covering a.

If $x \in X$, then x covers a unique atom p, and since $p < a$ it follows that $p = ax$. From $ax \ll x$ we infer that $a \ll a + x$, and hence that $a + x = a + b$. Therefore $a + b$ is the unique dual atom which covers x. Similarly ab is the unique atom which is covered by x.

These two theorems give a reasonably complete picture of all modular lattices of dimensions 3 and 4. The complemented case is regarded for this purpose as being known, and the case in which either a is an atom or b is a dual atom reduces trivially to a lower dimensional case. The remaining possibilities are illustrated in Fig. 1.

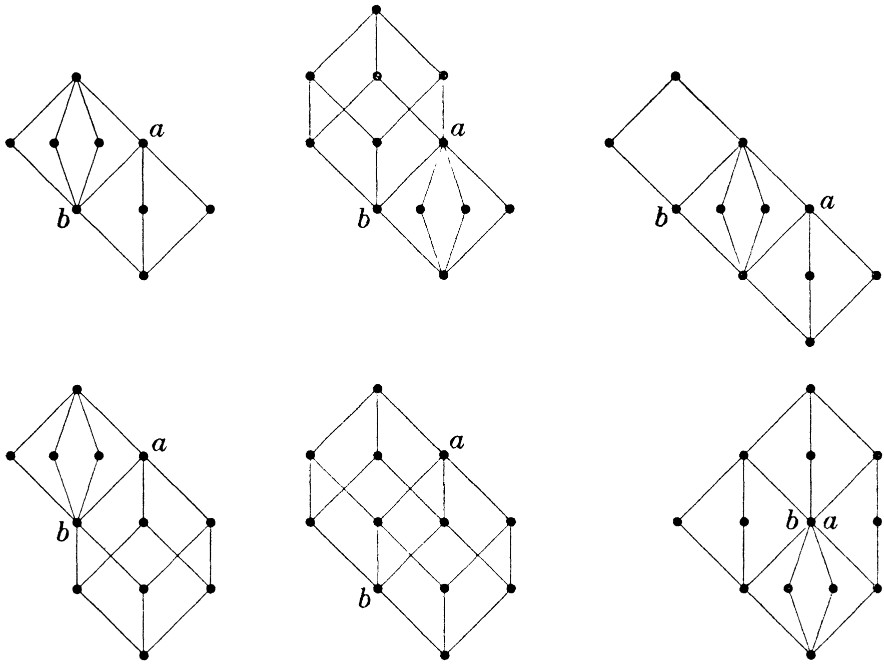

Fig. 1.

3. Arguesian lattices of dimension $n \leqq 4$. Our principal result is

THEOREM 3.1. *In order for a lattice A of dimension $n \leqq 4$ to be isomorphic to a sublattice of the lattice of all subspaces of an $n-1$ dimensional Arguesian projective geometry it is necessary and sufficient that A be Arguesian.*

PROOF. Since every sublattice of an Arguesian lattice is Arguesian, the necessity of this condition follows from Theorem 1.5.

For $n \leqq 2$ it follows from Theorem 2.1 and the remark preceding it that every n dimensional lattice is isomorphic to a sublattice of the lattice of all subspaces of an $n-1$ dimensional Arguesian projective geometry. We may therefore assume that $n=3$ or $n=4$.

Let a be the sum of all the atoms of A and let b be the product of all the dual atoms of A. In view of Theorem 1.5 we may assume that A is not complemented, and hence that $a \neq 1$ and $b \neq 0$. Also, the cases in which a is an atom or b is a dual atom reduce trivially to lower dimensional cases. We therefore assume that

$$1 < \delta(a) < n \qquad \text{and} \qquad 0 < \delta(b) < n-1 .$$

If $n=3$, then this implies that $\delta(a)=2$ and $\delta(b)=1$. For later use (Case 3 below) we shall prove for this situation a somewhat stronger conclusion than is called for in the theorem.

LEMMA A. *Suppose $n=3$, $\delta(a)=2$ and $\delta(b)=1$. If S is any plane projective geometry with the property that there are at least as many points on each line of S as there are elements in A, then A is isomorphic to a sublattice of $L(S)$.*

PROOF OF LEMMA A. By Theorem 2.2, we have $A=a/0 \cup 1/b$. Our assumptions regarding S imply that there are at least as many lines through each point of S as there are elements in A. By Theorem 2.1 it follows that if p is any point of S and U is any line through p, then $a/0$ and $1/b$ are isomorphic to sublattices of the quotients $U/\emptyset$ and P/p of $L(S)$. The isomorphism of $a/0$ into $U/\emptyset$ necessarily maps a onto U, and it can be so chosen that it maps b onto p. Similarly, the isomorphism of $1/b$ into P/p necessarily maps b onto p, and it can be so chosen that it maps a onto U. Then the two isomorphisms agree on the common part of $a/0$ and $1/b$, and it follows by Theorem 1.6 that they have a common extension which maps A isomorphically into $L(S)$.

We henceforth assume that $n=4$ and therefore $\delta(a)=2, 3$ and $\delta(b)=1, 2$. The proof will be divided into six cases.

Case 1. $\delta(a)=2$ and $\delta(b)=1$.

By Theorem 2.2 we have $b<a$ and $A=a/0 \cup 1/b$. Let S be a three dimensional Arguesian projective geometry with the properties that there are at least as many points on each line of S as there are atoms in A, and that $1/b$ is isomorphic to a sublattice of $L(P)=P/\emptyset$ where P is a plane in S. The existence of S with these properties is guaranteed by Theorems 1.2 and 1.3 in case the projective geometry associated with $1/b$ is non-degenerate, and by Theorem 1.1 in the alternative case. If p is a point in S which is not in P, then

$$P+p = S \qquad \text{and} \qquad Pp=\emptyset .$$

The quotients S/p and $P/\emptyset$ are therefore isomorphic, and consequently $1/b$ is isomorphic to a sublattice of S/p. This isomorphism g maps b onto p and maps a onto a line U through p. By Theorem 2.1 there exists an isomorphism f of $a/0$ into $U/\emptyset$ which maps b onto p, and since f and g agree on the common part of the quotients $a/0$ and $1/b$, they have a common extension which maps A isomorphically into $L(S)$.

Case 2. $\delta(a)=3$ and $\delta(b)=2$.
This is the dual of Case 1 and can be treated similarly.

Case 3. $\delta(a)=2$, $\delta(b)=2$ and $a \neq b$.
Let X be the set of all irreducible elements $x \in A$ with $\delta(x)=2$. By Theorem 2.3, the sublattice $B=A-X$ of A is the union of the quotients $a/0$ and $1/b$, and every element of X belongs to the quotient $(a+b)/ab$. Hence

$$A = (a+b)/0 \cup 1/b .$$

The atoms of the three dimensional lattice $(a+b)/0$ are precisely the atoms of A, and their sum is the dual atom a of $(a+b)/0$. The dual atoms of $(a+b)/0$ are a, b, and the elements of X, and their product is ab. We can therefore apply Lemma A with the lattice A replaced by $(a+b)/0$. Let S be a three dimensional Arguesian projective geometry with the property that there are at least as many points on each line of S as there are elements in A, and let P be a plane in S. By Lemma A, the quotient $(a+b)/0$ is isomorphic to a sublattice of $L(P)=P/\emptyset$. This isomorphism f maps $a+b$ onto P and maps b onto a line U in P. By Theorem 2.1 there exists an isomorphism g of $1/b$ into S/U which maps $a+b$ onto P, and since f and g agree on the common part of the quotients $(a+b)/0$ and $1/b$, they have a common extension which maps A isomorphically into $L(S)$.

Case 4. $\delta(a)=2$, $\delta(b)=2$ and $a=b$.
Let X and B be as in Case 3. By Theorem 2.3 B is the union of the quotients $a/0$ and $1/a$. Furthermore, if P is the set of all those atoms of A which are covered by some member of X, and if P' is the set of all those dual atoms of A which cover some member of X, then there exists a one-to-one correspondence $p \to p'$ between P and P' such that, for each $x \in X$ and $p \in P$, x covers p if and only if x is covered by p'. Each quotient p'/p then consists of p, p' and a, and of all those elements x of X which cover p, each element $x \in X$ belongs to exactly one such quotient, and if $p \neq q$, then p'/p and q'/q have only the element a in common.

Let S be a three dimensional Arguesian lattice with the property that there are at least as many points on each line of S as there are elements in A, and let U be a line in S. By Theorem 2.1 there exist isomorphisms f and g of $a/0$ and $1/a$ into $U/\emptyset$ and S/U, respectively. For any $p \in P$, $f(p)$ is a point of U and $g(p')$ is a plane through U. Applying Theorem 2.1 again we infer that there exists an isomorphism h_p of p'/p into $g(p')/f(p)$ which maps a into U. Inasmuch as any two of the mappings f, g, and h_p with $p \in P$ agree on their common domain, we see that they all have a common extension h which maps A into $L(S)$. We complete the proof by showing h is an isomorphism.

First observe that if $p \in P$, then the quotient $1/p$ is the union of its ideal p'/p and of its dual ideal $1/a$. From this it follows by Theorem 1.6 that h maps $1/p$ isomorphically into $L(S)$. Similarly, h maps $p'/0$ isomorphically into $L(S)$. If p, $q \in P$, then the set

$$C_{p,q} = p'/0 \cup 1/q$$

is a sublattice of A, and since

$$p'/0 \cap 1/q \neq \emptyset ,$$

we may apply Theorem 1.6 again to infer that h maps $C_{p,q}$ isomorphically into $L(S)$. Also by Theorem 1.6, h maps $a/0 \cup 1/a$ isomorphically into $L(S)$. Finally, for any two elements x, $y \in A$, either x and y both belong to $a/0 \cup 1/a$, or else they both belong to $C_{p,q}$ for some p, $q \in X$. In either case it follows that

$$h(x+y) = h(x)+h(y) \qquad \text{and} \qquad h(xy) = h(x)h(y) ,$$

and that $h(x)=h(y)$ if and only if $x=y$. This shows that h is in fact an isomorphism.

Case 5. $\delta(a)=3$, $\delta(b)=1$, and the projective geometry associated with one of the quotients $a/0$ and $1/b$ is degenerate.

We shall assume that the quotient associated with $1/b$ is degenerate; the proof under the alternative assumption is analogous. By Theorem 2.2 we have

$$b < a \qquad \text{and} \qquad A = a/0 \cup 1/b .$$

Let S be a three dimensional non-degenerate (hence necessarily Arguesian) projective geometry with the properties that there are at least as many points on each line in S as there are dual atoms in A, and that $a/0$ is isomorphic to a sublattice of $L(P)=P/\emptyset$ where P is a plane in S. The existence of S with these properties follows from Theorems 1.2 and 1.3 in case the projective geometry associated with $a/0$ is non-degenerate, and from Theorem 1.1 in the alternative case.

The isomorphism f of $a/0$ into $P/\emptyset$ maps b onto a point p in P, and it maps the ideal a/b of $1/b$ isomorphically into the ideal P/p of S/p. Our hypothesis regarding S implies that in the projective geometry associated with S/p there are at least as many points on each line as there are atoms in $1/b$. From this it follows by Theorem 1.1 that there exists an isomorphism g of $1/b$ into S/p which agrees with f on a/b. Therefore f and g have a common extension which maps A isomorphically into $L(S)$.

Case 6. $\delta(a)=3$, $\delta(b)=1$, and the projective geometries associated with the quotients $a/0$ and $1/b$ are non-degenerate.

By Theorem 2.2 we have

$$b < a \qquad \text{and} \qquad A = a/0 \cup 1/b \, .$$

The proof in this case will be based on the following

LEMMA B. *Suppose the hypothesis of Case* 6 *is satisfied, and assume that*

$$b \leqslant x_i \leqslant a \qquad \textit{and} \qquad b \leqslant y_i \leqslant a \qquad \textit{for} \quad i = 0, 1, 2 \, .$$

Then $\langle x_0,x_1,x_2,y_0,y_1,y_2\rangle$ *is a quadrangular sextuplet of points in* $1/b$ *if and only if* $\langle y_0,y_1,y_2,x_0,x_1,x_2\rangle$ *is a quadrangular sextuplet of lines in* $a/0$.

REMARK. This is closely related to the fact that a quadrangular sextuplet of lines in an Arguesian projective geometry meet a line (in their plane but not passing through their point of intersection) in a quadrangular sextuplet of points. Of course, if we wanted to make use of this result in the present situation, we would have to assume the theorem that we are trying to prove. However, our reasoning is motivated by the proof of the classical theorem, as illlustrated in Fig. 2.

PROOF OF LEMMA B. The elements x_i, y_i are points on the line a in $1/b$, and they are also lines through the point b in $a/0$.

Assume that $\langle x_0,x_1,x_2,y_0,y_1,y_2\rangle$ is a quadrangular sextuplet of points in $1/b$. Then there exist distinct points c_0,c_1,c_2,c_3 in $1/b$, none of which lies on a and no three of which are collinear, such that

$$x_0 = a(c_0+c_3), \qquad x_1 = a(c_1+c_3), \qquad x_2 = a(c_2+c_3) \, ,$$
$$y_0 = a(c_1+c_2), \qquad y_1 = a(c_2+c_0), \qquad y_2 = a(c_0+c_1) \, .$$

It readily follows that

(1) $\quad x_0 \neq x_1, \quad y_0 \neq y_1, \quad x_0 \neq y_1, \quad x_1 \neq y_0, \quad y_2 \neq x_0,x_1,y_0,y_1 \, .$

For instance, since the lines c_0+c_3 and c_1+c_3 are distinct and meet in the point c_3, which is not on the line a, the points x_0 and x_1 in which these two lines cut the line a must be distinct. From (1) we easily see that there exists a line z of $a/0$ passing through the point b, such that $\langle y_0,y_1,y_2,x_0,x_1,z\rangle$

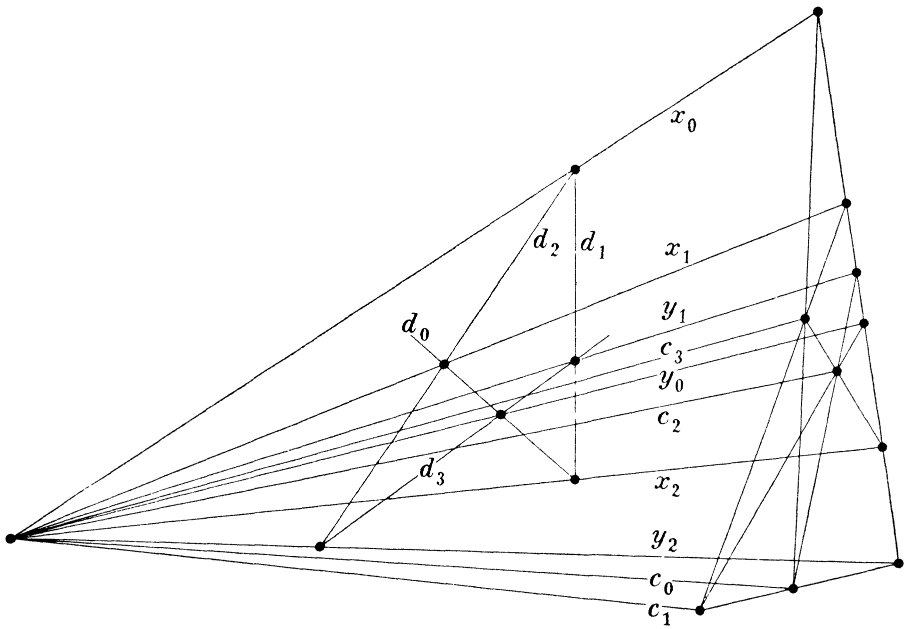

Fig. 2.

is a quadrangular sextuplet of lines in $a/0$. Consequently we can find four distinct lines d_0,d_1,d_2,d_3 in $a/0$ such that none of them passes through b, no three of them are concurrent, and

$$y_0 = b+d_0d_3, \quad y_1 = b+d_1d_3, \quad y_2 = b+d_2d_3\,,$$
$$x_0 = b+d_1d_2, \quad x_1 = b+d_2d_0, \quad z = b+d_0d_1\,.$$

Applying the definition of an Arguesian lattice with

$$a_0 = c_3, \quad a_1 = c_0, \quad a_2 = c_2\,,$$
$$b_0 = d_0d_2 \quad b_1 = d_2d_3, \quad b_2 = d_3d_0\,,$$

and therefore

(2) $$y = (c_0+c_2)(d_2d_3+d_3d_0)\,[(c_3+c_0)(d_0d_2+d_2d_3)+(c_3+c_2)(d_0d_2+d_3d_0)]\,,$$

we obtain

(3) $$(c_3+d_0d_2)(c_2+d_3d_0)(c_0+d_2d_3) \leqq c_0(c_2+y)+d_2d_3(d_3d_0+y)\,.$$

Since $b \leqq c_0$, we have

$$c_0+d_2d_3 = c_0+b+d_2d_3 = c_0+y_2\,,$$

and inasmuch as c_0, c_1, and y_2 are distinct but collinear points in $1/b$, it follows that

$$c_1 \leqq c_0 + d_2 d_3 .$$

For similar reasons we have

$$c_1 \leqq c_2 + d_3 d_0 \quad \text{and} \quad c_1 \leqq c_3 + d_0 d_2 .$$

It follows by (3) that

$$(4) \qquad c_1 \leqq c_0(c_2 + y) + d_2 d_3 (d_3 d_0 + y) .$$

We have

$$(c_0 + c_2)(d_2 d_3 + d_3 d_0) \leqq (c_0 + c_2) d_3 = (c_0 + c_2) a d_3$$
$$= y_1 d_3 = (b + d_1 d_3) d_3 = b d_3 + d_1 d_3 = d_1 d_3$$

and, similarly,

$$(c_3 + c_0)(d_0 d_2 + d_2 d_3) \leqq d_1 d_2, \qquad (c_3 + c_2)(d_0 d_2 + d_3 d_0) \leqq x_2 d_0$$

Consequently, by (2),

$$y \leqq d_1 d_3 (d_1 d_2 + x_2 d_0) .$$

Now if x_2 and z were distinct, then the three points $d_1 d_3$, $d_1 d_2$, $x_2 d_0$ would not be collinear. We would then have $y = 0$ and consequently, by (4),

$$c_1 \leqq c_0 c_2 + d_0 d_2 d_3 .$$

But $c_0 c_2 = b$ because c_0 and c_2 are distinct points in $1/b$, and $d_0 d_2 d_3 = 0$ because the lines d_0, d_2 and d_3 in $a/0$ are not concurrent. We would therefore have $c_1 \leqq b$, which is impossible because c_1 is a point in $1/b$. We therefore conclude that $z = x_2$, and the proof of the forward implication is complete.

Now suppose $\langle y_0, y_1, y_2, x_0, x_1, x_2 \rangle$ is a quadrangular sextuplet of lines in $a/0$. Then (1) holds and, regarding x_0, x_1, y_0, y_1, y_2 as points on the line a of $1/b$, we infer that there exists a point z on a such that $\langle x_0, x_1, z, y_0, y_1, y_2 \rangle$ is a quadrangular sextuplet of points in $1/b$. By the first part of the proof it follows that $\langle y_0, y_1, y_2, x_0, x_1, z \rangle$ is a quadrangular sextuplet of lines in $a/0$. Consequently, $z = x_2$. This proves the backward implication.

We now return to the proof of the main theorem for the case under consideration, Case 6. By Theorem 1.2 there exists a non-degenerate, three dimensional projective geometry S such that $a/0$ is isomorphic to $P/\emptyset$ where P is a plane in S. Let f be such an isomorphism, and let p be the point onto which f maps the atom b of $a/0$. The function f maps the ideal a/b of $1/b$ isomorphically onto the ideal P/p of S/p. Observe that if $\langle x_0,x_1,x_2,y_0,y_1,y_2\rangle$ is a quandrangular sextuplet of points on the line a of $1/b$, and if f maps x_i onto X_i and y_i onto Y_i then $\langle X_0,X_1,X_2,Y_0,Y_1,Y_2\rangle$ is a quadrangular sextuplet of points on the line P of S/p. In fact, by Lemma B, $\langle y_0,y_1,y_2,x_0,x_1,x_2\rangle$ is a quadrangular sextuplet of lines in $a/0$, whence it follows that $\langle Y_0,Y_1,Y_2,X_0,X_1,X_2\rangle$ is a quadrangular sextuplet of lines in $P/\emptyset$. Applying Lemma B again, this time with A replaced by the sublattice $P/\emptyset \cup S/p$ of $L(S)$, we conclude that $\langle X_0,X_1,X_2,Y_0,Y_1,Y_2\rangle$ is a quadrangular sextuplet of points in S/p.

Using Theorem 1.4 we see that there exists an isomorphism g of $1/b$ onto S/p which agrees with f on a/b. Therefore f and g have a common extension which maps A isomorpically into $L(S)$.

Since Cases 1–6 exhaust all the possible situations which were not disposed of in the preliminary discussion, this completes the proof of Theorem 3.1.

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