MATH. SCAND. 7 (1959), 146–156

FULL BANACH MEAN VALUES ON COUNTABLE GROUPS

HARRY KESTEN

1. Introduction. A functional L on the space of all bounded real functions on a group G is called a full Banach mean value if it satisfies the following conditions (f and g bounded real functions on G):

$$\inf_{x \in G} f(x) \leqq Lf \leqq \sup_{x \in G} f(x) , \tag{1.1}$$

$$L\{f(yxz)\} = L\{f(x)\} \qquad \text{for all} \quad y,z \in G , \tag{1.2}$$

$$L\{\lambda f\} = \lambda Lf \qquad \lambda \text{ real} , \tag{1.3}$$

$$L\{f+g\} = Lf+Lg \tag{1.4}$$

Recently E. Følner (Main theorem in [4]) has given necessary and sufficient conditions of a combinatorial character for the existence of such a mean value. Dixmier [2, p. 221] had already given slightly stronger sufficient conditions. It is the purpose of this note to derive from Følner's theorem another set of necessary and sufficient conditions for the existence of a mean value on countable groups. These conditions will be in terms of the spectral radii of matrices connected with random walks on the group and introduced by the author in his thesis [6]. It was shown that these spectral radii are determined by the probability of going from the unit element of G to the unit element in n steps ($n=0, 1, \ldots$) of such random walks (cf. [3] for the terminology on random walks and Markov chains). They are therefore connected with the number of ways in which the unit element can be written as the product of n generators of G.

Lemma 4, giving a bound for eigenvalues of symmetric matrices, may have some independent interest.

2. Some former results. When G is a countable (not necessarily infinite) group and $p(x)$ a symmetric probability distribution on G, that is

Received January 5, 1959.

Research sponsored by U. S. Army Ordnance, Office of Ordnance Research, Contract No. DA-36-034-ORD-2001.

(2.1) $$p(x) \geqq 0 \qquad (x \in G)\ ,$$

(2.2) $$p(x) = p(x^{-1})\ ,$$

(2.3) $$\sum_{x \in G} p(x) = 1\ ,$$

we associate a matrix[1] $M(G, p)$ with this group G and the probability distribution $p(x)$. The entries of M correspond to pairs of elements,

(2.4) $$M(G, p) = \|m_{x_i, x_j}\|$$

with

(2.5) $$m_{x_i, x_j} = p(x_i^{-1} x_j) \qquad (x_i, x_j \in G)\ .$$

By (2.1)–(2.3) M is a symmetric stochastic matrix and its dimension is equal to the order of the group. M was introduced in [6] as the matrix of transition probabilities corresponding to a random walk on G, in which every step consists of right multiplication by some element of G, $x \in G$, being chosen with probability $p(x)$. Thus when $y \in G$ was reached after n steps, one will reach yx with probability $p(x)$ after the $(n+1)^{\text{st}}$ step. One can consider M as a bounded linear operator on the Hilbert space $l^2(G)$ of functions $h = h(x)$ ($x \in G$; $h(x)$ complex) such that

$$\sum_{x \in G} |h(x)|^2 < \infty$$

by putting

(2.6) $$Mh(x) = \sum_{x \in G} m_{x,y} h(y)\ .$$

The spectrum of M is then defined as the set of all complex numbers λ such that $M - \lambda I$ does not have an inverse which is a bounded operator on $l^2(G)$. (I is the unit matrix of the same dimension as M.) The spectral radius of M is

$$\sup_{\lambda \in \text{spectrum of } M} |\lambda|\ .$$

The following results which were proved in [6] will be needed here.

LEMMA 1 (lemma 2.2 in [6]). *The spectrum of M is real and contained in* $[-1, +1]$. *Furthermore*

$$\lambda(G, p) \overset{\text{def}}{=} \max_{\lambda \in \text{spectrum of } M} \lambda$$

$$= \text{spectral radius of } M = \sup_{x \in G} \limsup_{n \to \infty} \{m_{x,x}^{(n)}\}^{1/n},$$

---

[1] The notation here differs slightly from the notation [6]. $M(G, p)$ would denoted by $M(G, G, P)$ where $P\{x|p\} = \frac{1}{2}p(x)$ (cf. p. 1 and (3.1) in [6]).

where $m^{(n)}_{x,x}$ *is a diagonal entry of the* nth *power of* M. *The entry* $m^{(n)}_{x,x}$ *equals the probability of going from* x *to* x *in* n *steps in the randomwalk defined on* G *by* $p(x)$.

LEMMA 2. (Cor. 1 in [6]) *If the set* H *of elements* x *for which* $p(x)$ *is positive, that is*

$$H = \{x \in G \mid p(x) > 0\}$$

generates G *and*

$$\lambda(G, p) = 1 ,$$

then

$$\lambda(G', q) = 1$$

for any subgroup $G' \subseteq G$ *and any symmetric probability distribution* $q(x)$ *on* G'.

In addition we need the following lemma:

LEMMA 3. *If there exists for every finitely generated subgroup* $G' \subseteq G$ *a symmetric probability distribution* $q(x)$ *on* G' *such that the set*

$$H' = \{x \in G' \mid q(x) > 0\}$$

generates G' *and*

$$\lambda(G', q) = 1 ,$$

then

$$\lambda(G, p) = 1$$

for any probability distribution $p(x)$ *on* G.

PROOF. Let $p(x)$ be a symmetric probability distribution on G. Choose an ε $(0 < \varepsilon < 1)$ and a finite subset S of G such that

$$\sum_{x \in S \cup S^{-1}} p(x) \geqq 1 - \varepsilon$$

where $S^{-1} = \{x \mid x^{-1} \in S\}$ (obviously such a set S exists for every $\varepsilon > 0$). Take for G' the subgroup generated by $S \cup S^{-1}$ and put

$$r(x) = p^{-1} p(x) \qquad (x \in G') , \tag{2.7}$$

where

$$p = \sum_{x \in G'} p(x) .$$

By our assumptions and by lemma 2,

$$\lambda(G', r) = 1 . \tag{2.8}$$

But by lemma 1, $\lambda(G', r)$ will be equal to the upper bound of the spectrum of the matrix

$$\tilde{M} = \|\tilde{m}_{x_i, x_j}\| \qquad (x_i, x_j \in G)$$

with

$$\tilde{m}_{x_i,x_j} = \begin{cases} r(x_i^{-1}x_j) & \text{if} \quad x_i^{-1}x_j \in G' , \\ 0 & \text{otherwise} . \end{cases}$$

If we denote the entries of $M(G, p)$ by m_{x_i,x_j}, then

$$\sum_{x_j \in G} |m_{x_i,x_j} - \tilde{m}_{x_i,x_j}| = \sum_{x \in G'} p(x)(p^{-1} - 1) + \sum_{x \in G - G'} p(x) \tag{2.9}$$
$$= (1-p) + (1-p) \leqq 2\varepsilon .$$

Therefore (cf. [7])

$$|\lambda(G, p) - \lambda(G', r)| \leqq 2\varepsilon \tag{2.10}$$

or

$$\lambda(G', r) - 2\varepsilon = 1 - 2\varepsilon \leqq \lambda(G, p) \leqq 1 . \tag{2.11}$$

Since ε can be chosen arbitrarily small, the lemma follows from (2.11).

For completeness we quote Følner's theorem [4]:

"A necessary condition that a group G have a full Banach mean value is that for every k in the interval $0 < k < 1$, and arbitrary, finitely many, elements $a_1, \ldots, a_n$ from G, there exists a finite subset E of G such that

$$N(E \cap Ea_i) \geqq kN(E) \qquad \text{for} \quad i = 1, \ldots, n , \tag{2.12}$$

where $N(\cdot)$ denotes the number of elements in the set between the brackets.

A sufficient condition that a group G have a full Banach mean value is that there exists a k_0 in the interval $0 < k_0 < 1$ such that for arbitrary, finitely many, not necessarily different, elements $a_1, \ldots, a_n$ from G there exists a finite subset E of G such that

$$n^{-1} \sum_{i=1}^{n} N(E \cap Ea_i) \geqq k_0 N(E) . \tag{2.13}$$

It follows that either of the two conditions are necessary and sufficient."

3. Statement and proof of the theorem for countable groups. We assume everywhere in this section that G is a countable group and $p(x)$ an arbitrary but fixed symmetric probability distribution on G such that

$$H = \{x \in G \mid p(x) > 0\}$$

generates G.

For any symmetric matrix M (not necessarily of the type $M(G, p)$) we define its spectrum as in section 2, and we shall write

$$\lambda(M) = \sup_{\lambda \in \text{spectrum of } M} \lambda . \tag{3.1}$$

When M is finite, $\lambda(M)$ is its largest eigenvalue; when $M=M(G, p)$, then $\lambda(M)=\lambda(G, p)$.

THEOREM. *When H generates G, a necessary and sufficient condition for the existence of a full Banach mean value on G is that*

$$\lambda(G, p) = 1 .$$

PROOF OF NECESSITY. Let $a_1, \ldots, a_n$ be any finite set of elements of G and let G' be the subgroup of G generated by

$$A = \{a_1, a_2, \ldots, a_n, a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}\} .$$

Define

$$q(x) = (2n)^{-1}\cdot\{\text{number of times } x \text{ occurs in } A\} \qquad (x \in G') . \tag{3.2}$$

Note that it is possible to have $q(x) > (2n)^{-1}$ if not all the $2n$ elements of A are different.

By lemma 3 it suffices to have $\lambda(G', q)=1$ for any such set $a_1, \ldots, a_n$. However if G has a full Banach mean value, so has its subgroup G' ([4, theorem 2] or [1, 4D]), and by Følner's theorem there exists for any k $(0<k<1)$ a finite set $E \subseteq G'$ such that

$$(2n)^{-1}\sum_{i=1}^{n}\{N(E \cap Ea_i)+N(E \cap Ea_i^{-1})\} \geqq kN(E) . \tag{3.3}$$

Let k be fixed $(0<k<1)$ and E a finite set $\subseteq G'$ such that (3.3) is satisfied. Put

$$h_E(x) = \begin{cases} N(E)^{-\frac{1}{2}} & \text{if } \ x \in E, \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

and

$$M(G', q) = \|m'_{x_i,x_j}\| \qquad (x_i, x_j \in G') . \tag{3.5}$$

Then

$$\begin{aligned} \lambda(G', q) &= \sup_{h \in l^2(G')} \left\{\sum_{x\in G'} |h(x)|^2\right\}^{-1} \sum_{x,y\in G'} h(x)\, m'_{x,y}\overline{h(y)} \\ &\geqq \sum_{x,y\in G'} h_E(x)\, m'_{x,y} h_E(y) . \end{aligned} \tag{3.6}$$

However, for $x \in E$

$$N(E)^{\frac{1}{2}}\, 2n \sum_{y\in G'} m'_{x,y}\, h_E(y) \tag{3.7}$$

is exactly the number of sets among $E \cap Ea_1, E \cap Ea_2, \ldots, E \cap Ea_n$, $E \cap Ea_1^{-1}, E \cap Ea_2^{-1}, E \cap Ea_n^{-1}$ which contain x.

Therefore

$$\lambda(G', q) \geqq N(E)^{-1}(2n)^{-1}\sum_{i=1}^{n}\{N(E \cap Ea_i)+N(E \cap Ea_i^{-1})\} \geqq k . \tag{3.8}$$

Since k is arbitrary between 0 and 1, $\lambda(G', q) = 1$ for every finitely generated subgroup G'. An application of lemma 3 completes the proof of the first part of our theorem.

In order to prove the sufficiency part we first prove[2]

LEMMA 4. *If $B = \|b_{ij}\|$ is a symmetric substochastic $N \times N$ matrix, that is*

$$b_{ij} = b_{ji} \geqq 0 \,, \tag{3.9}$$

$$\sum_{j=1}^{N} b_{ij} \leqq 1 \,, \tag{3.10}$$

such that for any set $S \subseteq \{1, 2, \ldots, N\}$ of s indices $(1 \leqq s \leqq N)$

$$s^{-1} \sum_{i,j \in S} b_{ij} \leqq k < 1 \qquad (k > 0) \,, \tag{3.11}$$

then

$$\lambda(B) \leqq 4k(1 + 2k^{-1})^{\frac{2}{3}} = O(k^{\frac{1}{3}}) \qquad (k \to 0) \,. \tag{3.12}$$

REMARK. Actually it follows from the proof that

$$\lambda(B) = O(k^{\frac{1}{2}-\varepsilon}) \qquad \text{for every} \quad \varepsilon > 0 \,.$$

PROOF. Since B is symmetric and real

$$\lambda(B) = \sup_{z} \left\{ \sum_{i=1}^{N} z_i^2 \right\}^{-1} \sum_{i,j=1}^{N} z_i b_{ij} z_j \qquad (z_i \text{ real}) \,.$$

Let $y = (y_1, \ldots, y_N)$ be an eigenvector for the eigenvalue $\lambda(B)$, satisfying therefore,

$$\lambda(B) = \left\{ \sum_{i=1}^{N} y_i^2 \right\}^{-1} \sum_{i,j} y_i b_{ij} y_j \,. \tag{3.13}$$

Because of (3.9) we can choose $y_i \geqq 0$ ([5]). In addition we may assume

$$\sum_{i=1}^{N} y_i^2 = 1$$

and

$$y_1 \geqq y_2 \geqq \cdots \geqq y_N \,. \tag{3.14}$$

Let $m (\geqq 2)$ be that integer which satisfies

$$(m-1)\, \tfrac{1}{2}k < 1, \qquad m\, \tfrac{1}{2}k \geqq 1 \,. \tag{3.15}$$

---

[2] The author is indebted to Dr. H. Furstenberg for an inspiring discussion regarding the proof of this lemma. Although it seems likely that similar estimates are known, the author was unable to derive the lemma from other bounds for eigenvalues, given in the literature.

For convenience we define $b_{ij}=0$ when $i>N$ or $j>N$, and $y_i=0$ when $i>N$. The relations (3.9)–(3.11) remain valid and one has, using (3.14)

$$\begin{aligned}(3.16)\quad \lambda(B) &= \sum_{i,j=1}^{\infty} y_i b_{ij} y_j \\ &= 2\sum_{i<j} y_i b_{ij} y_j + \sum_{j=1}^{\infty} y_j b_{jj} y_j \\ &= 2\sum_{p=0}^{\infty}\sum_{j=pm+1}^{(p+1)m}\sum_{i<j} y_i b_{ij} y_j + \sum_{p=0}^{\infty}\sum_{j=pm+1}^{(p+1)m} y_j b_{jj} y_j \\ &\leqq 2\sum_{p=0}^{\infty}\sum_{j=pm+1}^{(p+1)m}\left\{\sum_{i=1}^{p} y_i b_{ij} y_j + \sum_{i=p+1}^{j-1} y_{p+1} b_{ij} y_j\right\} + \sum_{p=0}^{\infty}\sum_{j=pm+1}^{(p+1)m} y_{p+1} b_{jj} y_j \,.\end{aligned}$$

Also

$$(3.17)\quad \sum_{p=0}^{\infty}\sum_{j=pm+1}^{(p+1)m}\sum_{i=1}^{p} y_i b_{ij} y_j \leqq \sum_{i=1}^{\infty}\sum_{j\geqq im+1} y_i b_{ij} y_j \leqq \sum_{i=1}^{\infty}\sum_{j=(i-1)m+1}^{im} y_i \tfrac{1}{2}k y_j$$

because y_j is non-increasing in j and

$$\sum_{j\geqq im+1} b_{ij} \leqq 1 \leqq m\tfrac{1}{2}k \leqq \sum_{j=(i-1)m+1}^{im} \tfrac{1}{2}k \,.$$

Similarly one obtains

$$(3.18)\quad \sum_{p=0}^{\infty}\sum_{j=pm+1}^{(p+1)m}\left\{\sum_{i=p+1}^{j-1} y_{p+1} 2b_{ij} y_j + y_{p+1} b_{jj} y_j\right\} \leqq \sum_{p=0}^{\infty}\sum_{j=pm+1}^{(p+1)m} y_{p+1} k y_j$$

because

$$y_{p+1} y_{j_1} \geqq y_{p+1} y_{j_2}$$

when

$$j_1 \leqq j_2$$

and

$$y_{p_1+1} y_{j_1} \geqq y_{p_2+1} y_{j_2}$$

when

$$p_1 < p_2, \quad p_1 m+1 \leqq j_1 \leqq (p_1+1)m < p_2 m+1 \leqq j_2 \leqq (p_2+1)m,$$

and in addition for $1 \leqq r' \leqq m$, $-1 \leqq r < \infty$

$$\begin{aligned}\sum_{p=0}^{r}\sum_{j=pm+1}^{(p+1)m}\left\{\sum_{i=p+1}^{j-1} 2b_{ij} + b_{jj}\right\} + \sum_{j=(r+1)m+1}^{(r+1)m+r'}\left\{\sum_{i=r+2}^{j-1} 2b_{ij} + b_{jj}\right\} &\leqq \sum_{j=1}^{(r+1)m+r'}\left\{\sum_{i<j} 2b_{ij} + b_{jj}\right\} \\ &= \sum_{i,j=1}^{(r+1)m+r'} b_{ij} \\ &\leqq \big((r+1)m+r'\big)k \\ &= \sum_{p=0}^{r}\sum_{j=pm+1}^{(p+1)m} k + \sum_{j=(r+1)m+1}^{(r+1)m+r'} k \,.\end{aligned}$$

Combining (3.16)–(3.18) gives

$$(3.19)\qquad \begin{aligned}\lambda(B) &\leqq \sum_{i=1}^{\infty}\sum_{j=(i-1)m+1}^{im} y_i k y_j + \sum_{p=0}^{\infty}\sum_{j=pm+1}^{(p+1)m} y_{p+1} k y_j \\ &= 2\sum_{i=1}^{\infty}\sum_{j=(i-1)m+1}^{im} y_i k y_j \\ &\leqq \lambda(C)\end{aligned}$$

where $C=\|c_{ij}\|$ is the symmetric infinite matrix with entries

$$(3.20)\qquad \begin{aligned} c_{11} &= 2k\,, \\ c_{i,(i-1)m+r'} &= c_{(i-1)m+r',i} = k \\ &\qquad (1 \leqq r' \leqq m \text{ but not } i = r' = 1)\,, \\ c_{ij} &= 0 \quad \text{otherwise}\,. \end{aligned}$$

It therefore suffices to find an upper bound for $\lambda(C)$.

Denote by $C^{(1)}=\|c_{ij}^{(1)}\|$ the matrix which has the same first row and column as C and zero entries everywhere else, i.e.

$$\begin{aligned} c_{11}^{(1)} &= 2k\,, \\ c_{r'1}^{(1)} &= c_{1r'}^{(1)} = k \qquad (r' = 2, \ldots, m)\,, \\ c_{ij}^{(1)} &= 0 \qquad \text{otherwise}\,. \end{aligned}$$

A trivial computation shows that

$$(3.21)\qquad \begin{aligned}\lambda(C^{(1)}) &= k(1+m^{\frac{1}{2}}) \\ &\leqq k\{1+(1+2k^{-1})^{\frac{1}{2}}\} \\ &= O(k^{\frac{1}{2}}) \qquad (k \to 0)\,.\end{aligned}$$

Thus

$$(3.22)\qquad \begin{aligned}\lambda(B) &\leqq \lambda(C) \\ &\leqq k\{1 + (1+2k^{-1})^{\frac{1}{2}}\} + \lambda(C-C^{(1)}) \\ &= k\{1 + (1+2k^{-1})^{\frac{1}{2}}\} + \lambda(C^{(2)})\,,\end{aligned}$$

where $C^{(2)}$ is the matrix formed by deleting the first row and column of C. We notice that the matrix

$$C^{(3)} = (m+1)^{-1}k^{-1}C^{(2)}$$

is a symmetric substochastic matrix. Adding $(m+1)^{-1}$ to the entries $C^{(3)}_{i,im}$ for $i=1,\ldots,m-1$ one obtains a stochastic matrix D with entries

$$(3.23)\qquad \begin{aligned} &d_{i,im} = 2(m+1)^{-1} \\ &d_{i,im+r'} = (m+1)^{-1} \end{aligned}\Bigg\}\quad \begin{aligned} i &= 1, \ldots, m-1\,, \\ r' &= 1, \ldots, m-1\,, \end{aligned}$$

$$\begin{aligned} &d_{i,[i/m]} = (m+1)^{-1} \\ &d_{i,im+r'} = (m+1)^{-1} \end{aligned}\Bigg\}\quad \begin{aligned} i &\geqq m\,, \\ r' &= 0, \ldots, m-1\,, \end{aligned}$$

$$d_{ij} = 0 \quad \text{otherwise}$$

(when $[\alpha]$ = the largest integer $\leqq \alpha$). D can be considered as the matrix of transition probabilities of a Markov-chain with the positive integers as possible states. From the state i one can go to each one of the states $im+r'$ ($r'=0, \ldots, m-1$) with a probability of at least $(m+1)^{-1}$. We shall call such a transition a step to the right. A step to the right multiplies the state number at least by a factor m $\big((im+r')i^{-1} \geqq m\big)$. When $i \geqq m$ one can also go from i to the state $[i/m]$ with a probability $(m+1)^{-1}$. Such a transition will be called a step to the left. A step to the left multiplies the state number at least by a factor $(2m-1)^{-1}$, since $[i/m]i^{-1} \geqq (2m-1)^{-1}$ when $i \geqq m$. Starting in any state i_0, after n steps, r to the right and $(n-r)$ to the left, one ends up in a state with a number at least equal to $m^r(2m-1)^{r-n}i_0$. Thus

$$(3.24)\qquad \begin{aligned} d^{(n)}_{i_0,i_0} &= \text{probability of going from } i_0 \text{ to } i_0 \text{ in } n \text{ steps} \\ &\leqq P\{m^r(2m-1)^{r-n} \leqq 1\} \\ &= P\{r \leqq n\log(2m-1)\big(\log m(2m-1)\big)^{-1}\} \\ &\leqq \sum_{u=1}^{\alpha n}\binom{n}{u}(m+1)^{u-n}\big(m(m+1)^{-1}\big)^u\,, \end{aligned}$$

where

$$\alpha n = \big[n\log(2m-1)\big(\log m(2m-1)\big)^{-1}\big] \leqq 2\cdot\tfrac{1}{3}n\,.$$

It follows that

$$(3.25)\qquad \begin{aligned} d^{(n)}_{i_0,i_0} &\leqq \{2(m+1)^{\alpha-1}\big(m(m+1)^{-1}\big)^{\alpha}\}^n \sum_{u=1}^{\alpha n}\binom{n}{u}2^{-n} \\ &\leqq \{2(m+1)^{-1}m^{\alpha}\}^n\,. \end{aligned}$$

Since $0 \leqq c^{(3)}_{ij} \leqq d_{ij}$,

$$(3.26)\qquad \begin{aligned} \lambda(C^{(3)}) &\leqq \sup_{i_0}\limsup_{n\to\infty}\{c^{(3)}_{i_0,i_0}{}^{(n)}\}^{1/n} \\ &\leqq \sup_{i_0}\limsup_{n\to\infty}\{d^{(n)}_{i_0,i_0}\}^{1/n} \\ &\leqq 2(m+1)^{-1}m^{\frac{2}{3}}\,. \end{aligned}$$

From the definition of $C^{(3)}$ it follows then, that

$$(3.27)\qquad \lambda(C^{(2)}) \leqq 2km^{\frac{2}{3}} \leqq 2k(1+2k^{-1})^{\frac{2}{3}}\,.$$

The lemma follows now from (3.22) and (3.27).

PROOF OF SUFFICIENCY. By Følner's theorem it suffices to show that there exists a $k_0 > 0$ such that for every finite set of elements $a_1, \ldots, a_n$ there exists a finite set E with

$$n^{-1} \sum_{i=1}^{n} N(E \cap Ea_i) \geqq k_0 N(E) . \tag{3.28}$$

Let $a_1, \ldots, a_n$ be any fixed set of elements from G. Let k_0 be any positive number such that

$$4k_0(1+2k_0^{-1})^{\frac{2}{3}} < 1 . \tag{3.29}$$

We shall prove that there exists a set E, satisfying (3.28) for this k_0. As before denote the group generated by

$$A = \{a_1, a_2, \ldots, a_n, a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}\}$$

by G' and define for any $x \in G'$

$$q(x) = (2n)^{-1} \cdot \{\text{number of times } x \text{ occurs in } A\} . \tag{3.30}$$

Since $\lambda(G', q) = 1$ there exists (cf. [9, p. 218]) a finite diagonal submatrix[3] B of $M(G', q)$ such that

$$\lambda(B) > 4k_0(1+2k_0^{-1})^{\frac{2}{3}} \tag{3.31}$$

because of (3.29). By lemma 4 this implies that B has a diagonal submatrix, say

$$B^{(1)} = \|b_{ij}^{(1)}\| \qquad (1 \leqq i, j \leqq N)$$

such that

$$\sum_{i,j}^{N} b_{ij}^{(1)} \geqq k_0 N .$$

Clearly $B^{(1)}$ is also a diagonal submatrix of

$$M(G', q) = \|m_{x_i, x_j}\| \qquad (x_i, x_j \in G') .$$

Therefore it is possible to find elements $y_1, \ldots, y_N$ in G' such that $b_{ij}^{(1)} = m_{y_i, y_j}$. The set $E = \{y_1, \ldots, y_N) \subseteq G'$ satisfies (3.28). In fact, as in (3.7),

$$\begin{aligned} (2n)^{-1} \sum_{i=1}^{n} \{N(E \cap Ea_i) + N(E \cap Ea_i^{-1})\} &= \sum_{i,j=1}^{N} b_{ij}^{(1)} \\ &\geqq k_0 N(E) . \end{aligned} \tag{3.32}$$

Since

$$N(E \cap Ea_i^{-1}) = N\big((E \cap Ea_i^{-1})a_i\big) = N(Ea_i \cap E)$$

this completes the proof of the theorem.

---

[3] By a diagonal submatrix we mean a submatrix whose rows and columns correspond to the same set of elements $x \in G'$.

The theorem explains the strong agreement between several of the theorems in [6] (about $\lambda(G, P) = 1$ or $\lambda(G, P) < 1$) and known theorems about the existence of a full Banach mean value. One might want a more direct construction of the mean value when $\lambda(G, P) = 1$ (not using Følner's theorem) but the author has been unable to find one.

Since [6] only deals with countable groups it seems desirable to find extensions of the results in [6] for more general groups. The question of the existence of mean values on semigroups has also been treated in literature (e.g. [1] [2]) whereas random walks on semigroups have been discussed by Schützenberger [8]. It seems harder to extend the results of [6] to random walks on semigroups because the corresponding matrices of transition probabilities cannot, in general, be chosen symmetric in such a case.

REFERENCES

1. Mahlon M. Day, *Amenable semigroups*, Illinois J. Math. 1 (1957), 509–544.
2. J. Dixmier, *Les moyennes invariantes dans les semi-groups et leurs applications*, Acta Sci. Math. Szeged 12 (1950), Pars A, 213–227.
3. W. Feller, *An introduction to probability theory and its applications*, New York, 1950.
4. Erling Følner, *On groups with full Banach mean value*, Math. Scand. 3 (1955), 243–254.
5. G. Frobenius, *Über Matrizen aus nicht negativen Elementen*, S.-B. Preuss. Akad. Berlin, 1912, 456–477.
6. Harry Kesten, *Symmetric random walks on groups*, Thesis, Cornell University, Ithaca, N. Y. (1958). To appear in Trans. Amer. Math. Soc.
7. J. Schur, *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen*, J. Reine Angew. Math. 140 (1911), 1–28.
8. M. P. Schützenberger, *Random walks on monoids*, Abstract 544–45, Notices Amer. Math. Soc. 6 (1958), 341.
9. A. Wintner, *Spektraltheorie der unendlichen Matrizen*, Leipzig, 1929.

PRINCETON UNIVERSITY, N. J., U. S. A.