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SUMMABILITY METHODS AND UNBOUNDED SEQUENCES

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We wish to investigate the summability properties of regular matrices for unbounded sequences. The properties for bounded sequences have been described by Brudno [2] (see also [4] and [5]). The problem for unbounded sequences turns out to be markedly different. If a matrix $B=(b_{mn})$ sums a bounded sequence $\{s'_n\}$ that is not summable by $A=(a_{mn})$, then B sums a non-enumerable set of independent bounded sequences that are not A summable. We shall see that if B sums an unbounded sequence $\{s'_n\}$ that is not A summable, the sequences that are B summable may all be of the form $\{Cs'_n+\sigma_n\}$ where C is a constant and $\{\sigma_n\}$ is A summable.

A matrix $A=(a_{mn})$; $m, n=1, 2, \ldots$ is regular if the following conditions are fulfilled:

1° $\sum_n |a_{mn}| \leqq H$ for every m;
2° $\lim_{m\to\infty} a_{mn}=0$ for every n;
3° $\alpha_m=\sum_n a_{mn} \to 1$ as $m \to \infty$.

We shall consider regular matrices with finite rows, i.e. satisfying the following additional condition

$$4^\circ \begin{cases} a_{mn}=0 & \text{when } n>\lambda(m), \\ a_{m,\lambda(m)} \neq 0. \end{cases}$$

Lemma 1. *If A is a regular matrix satisfying condition 4° and*

$$5^\circ \begin{cases} \lambda(m)=m, \\ a_{mn}=0 \quad \textit{for} \quad n<m-1, \\ |a_{m,m-1}a_{mm}{}^{-1}| \geqq K>1 \quad \textit{for} \quad m \geqq 2, \end{cases}$$

then every A summable sequence has the form

$$\{s_m\} = \{Cs'_m+\sigma_m\},$$

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where C is a constant, $\{s'_m\}$ a certain unbounded sequence and $\{\sigma_n\}$ a convergent sequence.

PROOF. It will be convenient to use the notations

$$a_m = a_{mm}, \qquad A_m = a_1 a_2 \dots a_m, \qquad m = 1, 2, \dots$$
$$b_m = a_{m,m-1}, \qquad B_m = b_2 b_3 \dots b_m, \qquad m = 2, 3, \dots$$

Note that conditions 3° and 5° imply that b_m is bounded away from 0; actually $b_m > \frac{1}{2}$ from a certain m.

That the sequence $s_1, s_2, \dots$ is A summable means that the sequence

(1) $$t_1 = a_1 s_1, \quad t_2 = b_2 s_1 + a_2 s_2, \quad t_3 = b_3 s_2 + a_3 s_3, \quad \dots$$

converges to a limit s. If we replace s_m by $s_m - s$, t_m will be replaced by $t_m - \alpha_m s$; hence, we may assume that $t_m \to 0$. From (1) follows

$$s_1 = A_1^{-1} t_1,$$
$$s_2 = -A_2^{-1} B_2 (t_1 - A_1 B_2^{-1} t_2),$$
$$s_3 = A_3^{-1} B_3 (t_1 - A_1 B_2^{-1} t_2 + A_2 B_3^{-1} t_3),$$

and generally

$$\begin{aligned} s_m &= (-1)^{m-1} A_m^{-1} B_m \big(t_1 - A_1 B_2^{-1} t_2 + \dots + (-1)^{m-1} A_{m-1} B_m^{-1} t_m\big) \\ &= (-1)^{m-1} A_m^{-1} B_m \big(t_1 - A_1 B_2^{-1} t_2 + A_2 B_3^{-1} t_3 - \dots\big) - \\ &\quad - \big(b_{m+1}^{-1} t_{m+1} - (a_{m+1} b_{m+1}^{-1}) b_{m+2}^{-1} t_{m+2} + \\ &\quad + (a_{m+1} b_{m+1}^{-1})(a_{m+2} b_{m+2}^{-1}) b_{m+3}^{-1} t_{m+3} - \dots\big). \end{aligned}$$

For the absolute value of the last term we have the upper bound

$$(1 + K^{-1} + K^{-2} + \dots) \max_{\mu > m} |b_\mu^{-1} t_\mu|,$$

and since b_μ^{-1} is bounded, this tends to zero. We have thus finished the proof of Lemma 1 with

$$s'_m = (-1)^{m-1} A_m^{-1} B_m,$$
$$C = t_1 - B_2^{-1} t_2 + A_2 B_3^{-1} t_3 - A_3 B_4^{-1} t_4 + \dots.$$

We remark that every sequence $\{s_m\}$ which satisfies the condition

$$|s_{m-1}^{-1} s_m| \geqq K > 1$$

is A summable for some matrix A satisfying the conditions of Lemma 1. In fact, if we choose

$$a_{11} = 1; \quad a_{mm} = -s_{m-1}(s_m - s_{m-1})^{-1}, \quad a_{m,m-1} = s_m (s_m - s_{m-1})^{-1},$$

the conditions of Lemma 1 are satisfied, and we get $t_m = 0$, $m = 2, 3, \dots$.

DEFINITION. Let A be a matrix. A matrix B is called *stronger than* A if it sums all A summable sequences, and it is called *strictly stronger than* A if it is stronger than A and sums a sequence which is not A summable.

THEOREM 1. *To a regular matrix A satisfying* 4° *corresponds a regular matrix B satisfying* 4°, *strictly stronger than A, so that every matrix which is stronger than A and sums a sequence which is B summable but not A summable, is stronger than B.*

PROOF. The matrix A transforms a sequence $\{s_n\}$ into $\{t_m\}$ where

$$t_m = a_{m1}s_1 + \ldots + a_{m,\lambda(m)} s_{\lambda(m)} .$$

The effect of a permutation of the rows of A will be that the terms of $\{t_m\}$ are permuted in the same manner and this will not change the convergence properties of $\{t_m\}$. Therefore, we can assume that $\lambda(m)$ is increasing, i.e. that

$$\lambda(1) = \ldots = \lambda(m_1) < \lambda(m_1+1) = \ldots = \lambda(m_2) < \lambda(m_2+1) = \ldots .$$

For convenience, we put $m_0 = 0$. Correspondingly, we have a division of every transformed sequence $\{t_m\}$ in sections so that the terms

$$t_{m_{\nu-1}+1}, \ldots, t_{m_\nu}$$

constitute the νth section.

We shall now construct a sequence $s'_1, s'_2, \ldots$ so that the transformed sequence $t'_1, t'_2, \ldots$ has the following property:

$$|t'_m t'^{-1}_n| \geqq K > 1 \tag{2}$$

when

$$m_{\nu-1} < n \leqq m_\nu, \quad m_\nu < m \leqq m_{\nu+1}, \quad \nu \geqq 1 .$$

In order to do this, we choose $\{s'_n\}$ so that all terms are 0 except the terms $s'_{\lambda(m)}$. We first choose $s'_{\lambda(m_1)} \neq 0$. Next, we choose $s'_{\lambda(m_2)}$ so that the terms of the second section of $\{t'_m\}$ satisfy (2), and the construction proceeds by induction.

The next step of the proof is the construction of a matrix D with the property that the set of D summable sequences is identical with the set of all sequences $\{Ct'_m + u_m\}$ where C is a constant whereas $\{t'_m\}$ is the sequence introduced above and u_m is a convergent sequence.

The rows of the matrix D will be indexed by pairs (p,q) of numbers so that p and q correspond to adjacent sections, i.e.

$$m_{\nu-1} < p \leqq m_\nu, \quad m_\nu < q \leqq m_{\nu+1}, \quad \nu \geqq 1 .$$

Thus, the rows of D fall in sections so that the ν'th section contains

$(m_\nu - m_{\nu-1})(m_{\nu+1} - m_\nu)$ rows. The arrangement of the rows within the sections being of no importance for the summability properties, we may assume that the rows are arranged lexicographically with respect to p and q.

Let $(d_{mn}) = (d_{(p,q),\,n})$ be chosen as follows:

$$d_{(p,q),\,n} = \begin{cases} t'_q\,(t'_q - t'_p)^{-1} & \text{for } n = p\,, \\ -t'_p\,(t'_q - t'_p)^{-1} & \text{for } n = q\,, \\ 0 & \text{for } n \neq p,q\,. \end{cases}$$

It follows from (2) that D satisfies 1°, and it is obvious that D satisfies the conditions 2°, 3° and 4°. In particular D is regular so that every convergent sequence is D summable. It is clear that also $\{t'_m\}$ is D summable, and hence that every sequence $\{Ct'_m + u_m\}$, where C is constant and $\{u_m\}$ convergent, is D summable.

Let $\{t_m\}$ denote an arbitrary D summable sequence. We shall prove that $\{t_m\}$ has the form $\{Ct'_m + u_m\}$. We consider all possible sequences $p_1 < p_2 < \ldots$ of integers so that $\{t'_{p_\mu}\}$ contains exactly one term from each section of $\{t'_m\}$. Let $D_{p_1 p_2} \ldots$ denote the matrix consisting of the rows of D with indices $(p_1, p_2), (p_2, p_3), \ldots$. The sequences $\{t'_{p_\mu}\}$ and $\{t_{p_\mu}\}$ are $D_{p_1 p_2} \ldots$ summable. Since this matrix satisfies the conditions of Lemma 1, all $D_{p_1 p_2} \ldots$ summable sequences have the form $\{Ct''_{p_\mu} + u'_{p_\mu}\}$ where C is constant and $\{u'_{p_\mu}\}$ converges. It follows in particular that

$$t'_{p_\mu} = C^* t''_{p_\mu} + u''_{p_\mu}\,,$$

where $C^* \neq 0$, as $\{t'_{p_\mu}\}$ is unbounded. Hence

$$t''_{p_\mu} = C^{*-1} t'_{p_\mu} - C^{*-1} u''_{p_\mu}\,,$$

and we have proved that all $D_{p_1 p_2} \ldots$ summable sequences have the form

$$\{CC^{*-1} t'_{p_\mu} + (u'_{p_\mu} - C^{*-1} u''_{p_\mu})\}\,.$$

We have thus proved that each of the subsequences $\{t_{p_\mu}\}$ has the form $\{Ct'_{p_\mu} + u'_{p_\mu}\}$. The constant C is uniquely determined by the condition that $\{t_{p_\mu} - Ct'_{p_\mu}\}$ is a bounded sequence. This implies that C is independent of the choice of $p_1, p_2 \ldots$, since two subsequences with an infinity of common terms must correspond to the same value of C. We can then write $t_m = Ct'_m + u_m$ and $\{u_m\}$ has the property that each of the subsequences $\{u_{p_\mu}\}$ converges, but this implies that $\{u_m\}$ converges.

We can now prove Theorem 1 with $B = DA$. Every sequence $\{Cs'_n + v_n\}$ where $\{v_n\}$ is A summable is by A transformed into $\{Ct'_m + u_m\}$ where $\{u_m\} = A\{v_n\}$ is convergent. The sequence $\{Ct'_m + u_m\}$ is D summable, hence $\{Cs'_n + v_n\}$ is B summable. It follows that B is strictly stronger than A. On the other hand, let $\{s_n\}$ be a B summable sequence. Then $A\{s_n\}$ is

D summable, hence $A\{s_n\} = \{Ct'_m + u_m\}$ where u_m converges. We put $s_n = Cs'_n + v_n$, and it follows that

$$A\{v_n\} = A\{s_n\} - CA\{s'_n\} = A\{s_n\} - C\{t'_m\} = \{u_m\},$$

hence $\{v_n\}$ is A summable. Thus, if a matrix B' sums every A summable sequence and one sequence $\{Cs'_n + v_n\}$ with $C \neq 0$, then B' sums $\{s'_n\}$ and, hence, every B summable sequence.

Definition. Two sequences $\{s'_n\}$, $\{s''_n\}$ are called *independent with respect to a matrix A* if no linear combination

$$\{C's'_n + C''s''_n\} \quad \text{with} \quad (C', C'') \neq (0, 0)$$

is A summable.

Let A be a given matrix. Theorem 1 states that there exists a matrix B strictly stronger than A, so that a maximal system of B summable sequences independent with respect to A contains only one sequence. In this case the sequences which are B summable but not A summable are unbounded. In fact, if a matrix B sums a *bounded* sequence which is not A summable, there exists, according to Brudno ([2], see also [5]), a matrix C strictly stronger than A so that B is strictly stronger than C. The following theorem is interesting in this connection:

Theorem 2. *Let A be a regular matrix satisfying* 4°, *and let B denote a regular matrix stronger than A, so that there exist two B summable sequences independent with respect to A. Then there exists a matrix C strictly stronger than A so that B is strictly stronger than C.*

Proof. According to the conditions of the theorem there exist two B summable sequences $\{s'_n\}$ and $\{s''_n\}$ independent with respect to A, and we may even suppose that both sequences are B summable with sum 0. The matrix A transforms $\{s'_n\}$ into $\{t'_m\}$ and $\{s''_n\}$ into $\{t''_m\}$. No sequence $\{C't'_m + C''t''_m\}$, $(C', C'') \neq (0, 0)$, is convergent. Our proof will depend on the nature of the sequences $\{t'_n\}$, $\{t''_n\}$, but we shall start with some remarks which will be useful in all the particular cases.

We are going to choose certain subsequences $\{t'_{\mu_m}\}$ and $\{t''_{\mu_m}\}$ of $\{t'_m\}$ and $\{t''_m\}$. These subsequences are the transforms of $\{s'_n\}$ and $\{s''_n\}$ by the matrix $A^* = \{a_{\mu_m n}\}$ consisting of some of the rows of A.

Next, we choose a regular matrix D, which transforms one of the sequences $\{t'_{\mu_m}\}$, $\{t''_{\mu_m}\}$ into a sequence which does not converge to zero, while D transforms a certain linear combination $\{C't'_{\mu_m} + C''t''_{\mu_m}\}$, $(C', C'') \neq (0, 0)$, into a sequence converging to zero. Then $\{C's'_n + C''s''_n\}$ is DA^* summable with sum zero while $\{s'_n\}$ or $\{s''_n\}$ lacks this property. Since the matrix D is regular every A summable sequence is DA^* summable. Finally we form a matrix C, which consists of all rows of B and all rows of DA^*. All summable sequences and the sequence $\{C's'_n + C''s''_n\}$ are C summable, hence C is strictly stronger than A. On the other hand, every C summable sequence is B summable, and one of the sequences $\{s'_n\}$, $\{s''_n\}$ is B summable, but not C summable, hence B is strictly stronger than C.

The proof of Theorem 2 will be finished when we have chosen A^* and D with the properties stated in the preceding section. We shall first assume that one of the sequences $\{t'_m\}$, $\{t''_m\}$, say $\{t'_m\}$ contains a subsequence $\{t'_{\mu_m}\}$ convergent to a limit $\neq 0$. We can choose this subsequence so that t''_{μ_m} tends to a finite limit or to infinity. We shall consider the two cases separately.

(i) If $t'_{\mu_m} \to u \neq 0$ and $t''_{\mu_m} \to v$, the sequence $\{us''_n - vs'_n\}$ is A^* summable with sum 0, while s'_n is A^* summable with sum $\neq 0$. We may then choose D as the unit matrix and the conditions will be satisfied.

(ii) If $t'_{\mu_m} \to u \neq 0$ and $|t''_{\mu_m}| \to \infty$, we may assume that

$$|t''_{\mu_{m+1}}| > 2|t''_{\mu_m}| .$$

According to the remark following Lemma 1, we can choose D so that $\{t''_{\mu_m}\}$ is D summable with sum 0 while $\{t'_{\mu_m}\}$ is D summable with sum $u \neq 0$. With this choice the conditions will be satisfied.

In the remaining case we know that every convergent subsequence of $\{t'_m\}$ and $\{t''_m\}$ has limit 0. Assume first that we can choose $\{\mu_m\}$ so that one subsequence, say $\{t'_{\mu_m}\}$ converges to 0, while $\{t''_{\mu_m}\}$ diverges. We may then take D as the unit matrix and the conditions will be satisfied. We shall now assume that it is impossible to choose $\{\mu_m\}$ in this way. We know that no sequence $\{C't'_m + C''t''_m\}$ converges. This property will be preserved if we delete all terms with $|t'_m| < 1$, $|t''_m| < 1$. When these terms are deleted, the absolute values of the terms of both sequences will tend to ∞. We can then choose $\{\mu_m\}$ so that $\{t'_{\mu_m}\}$ satisfies the condition

$$|t'_{\mu_{m+1}}| > 2|t_{\mu_m}| . \tag{3}$$

If $\{C't'_{\mu_m} + t''_{\mu_m}\}$ does not converge for any C', we can construct D so that $\{t'_{\mu_m}\}$ is D summable to 0 while $\{t''_{\mu_m}\}$ is not D summable. If $\{C't'_{\mu_m} + t''_{\mu_m}\}$ converges to t for some C', then we consider the sequence $\{C't'_m + t''_m\}$. By the independence of $\{s'_m\}$ and $\{s''_m\}$ with respect to A, there exists an $\varepsilon > 0$ and a sequence $\{\nu_m\}$ such that

$$|C't'_{\nu_m} + t''_{\nu_m} - t| > \varepsilon, \qquad \text{for all} \quad m .$$

We then choose a subsequence $\{t_{\varkappa_m}\}$ of $\{t_m\}$ satisfying (3) and containing an infinite number of terms from each of the sequences $\{t'_{\mu m}\}$ and $\{t'_{\nu m}\}$. Then, finally, we can construct a matrix D which sums $\{t'_{\varkappa m}\}$ to 0 but does not sum $\{t''_{\varkappa m}\}$.

This completes the proof of the theorem.

We remark that any sequence $\{s_n\}$, for which $s_n = O(n^{\frac{1}{2}})$ and is zero everywhere save for a subsequence $\{n_k\}$ so that the counting function of $\{n_k\}$ is $o(n^{\frac{1}{2}})$, is $(C, 1)$ summable (see Lorentz [3]). It follows that the iteration of $(C, 1)$ with itself sums a non-enumerable set of unbounded sequences that are not $(C, 1)$ summable though the set of bounded sequences is the same. In this case there exists a non-enumerable set of matrices of the type described in Theorem 2.

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