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ON THE UNIQUENESS OF THE CAUCHY PROBLEM II

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1. Introduction. In this paper we shall again apply the methods used in [2], this time to elliptic differential equations with *variable* coefficients. For elliptic equations we shall thus obtain a new proof and a slight extension of the results proved by Calderón [1] using singular integral operators. A simplified version of Calderón's method has recently been given by Malgrange [3]. From that paper we have taken over the decisive use in a similar context of the parameter δ introduced below.

We shall study the solutions of a differential inequality,

$$|P(x, D)u| \leqq K \sum_{|\alpha| < m} |D_\alpha u| , \tag{1.1}$$

where m is the degree of the homogeneous operator P. Here u is a functions vanishing for

$$x^1 < x^{2^2} + \ldots + x^{\nu^2} , \tag{1.2}$$

when x is in a neighbourhood of the origin. When using the methods of [2] we shall first as weight function in the exponents choose

$$\varphi_\delta(x) = (x^1 - \delta)^2 + \delta(x^{2^2} + \ldots + x^{\nu^2}) , \tag{1.3}$$

where δ will be taken sufficiently small. Note that the surface $\varphi_\delta(x) = \varphi_\delta(0)$ for all δ has a contact of the second order at the origin with the paraboloid

$$x^1 = \tfrac{1}{2}(x^{2^2} + \ldots + x^{\nu^2}) ,$$

which apart from the origin lies in the set defined by (1.2). We shall also study more general elliptic operators by means of the weight function

$$\psi_\delta(x) = (x^1 - \delta)^2 + x^{2^2} + \ldots + x^{\nu^2} . \tag{1.4}$$

Since the radius δ of the sphere $\psi_\delta(x) = \psi_\delta(0)$ must be chosen small, we can then only obtain unique continuation across "sufficiently convex" surfaces.

The basic tool in [2] was an inequality of Trèves [5]. In the next section we shall examine what the arguments of Trèves yield in the case

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of variable coefficients. In the third section we shall then employ the arguments involving a partition of the unity used in [2], in order to prove the desired estimates for operators related to those studied by Calderón. In section 4 we study more general operators obtaining the weaker results referred to above. In section 5, finally, we prove the unique continuation of the solutions of (1.1) when $P(x, \xi)$ is the product of two factors satisfying the hypotheses of section 3 or 4. Such results have earlier been indicated by Mizohata [4]. Malgrange has (unpublished) given a simple and precise form to the arguments of [4], using the results of [3]. We here use the inequalities proved in sections 3 and 4 in a similar way.

2. The method of Trèves. We first recall Trèves' arguments for a differential operator with constant coefficients and shall afterwards discuss what it yields in the case of variable coefficients.

As usual we denote by D_j the differential operator $-i\partial/\partial x^j$ and by D_α, where $\alpha = (\alpha_1, \ldots, \alpha_j)$ is a multi-index with components varying from 1 to ν, we denote the product $D_{\alpha_1} \ldots D_{\alpha_j}$. (For a multiindex (α_1) of length 1 we only write α_1.) Similarly we define x^α.

With real numbers t_j we shall use the scalar product

$$(2.1)\qquad T(u, v) = \int u(x)\,\overline{v(x)}\,\exp(t_1^2 x^{1^2} + \ldots + t_\nu^2 x^{\nu^2})\,dx\,.$$

The formal adjoint of D_j with respect to this scalar product is the operator

$$(2.2)\qquad \delta_j = D_j - 2it_j^2 x^j\,.$$

For if $u, v \in C_0^\infty$, we have

$$T(u, D_j v) = \int u\,\overline{D_j v}\,\exp(t_1^2 x^{1^2} + \ldots + t_\nu^2 x^{\nu^2}) = T\big((D_j - 2it_j^2 x^j)u, v\big)\,.$$

The operators D_j and δ_k satisfy the following commutation relations

$$(2.3)\qquad D_j\delta_j - \delta_j D_j = -2t_j^2, \qquad D_j\delta_k - \delta_k D_j = 0, \quad j \neq k\,.$$

If $P(D)$ is a differential operator with constant coefficients, Trèves proved that the commutation relations imply the formula

$$(2.4)\quad T\big(P(D)u, P(D)u\big) = \Sigma \frac{2^{|\alpha|}}{|\alpha|!}\, t_\alpha^2\, T\big(\overline{P}^{(\alpha)}(\delta)u, \overline{P}^{(\alpha)}(\delta)u\big), \qquad u \in C_0^\infty\,.$$

Our purpose is to examine the character of the additional terms which enter in the case of variable coefficients. First we have to prove the uniqueness of the right hand side of (2.4).

LEMMA 1. *Suppose that $a_{\alpha\beta}$ are continuous functions in Ω and that*

$$\sum\sum T(a_{\alpha\beta}\delta_\alpha u, \overline{\delta_\beta u}) = 0, \qquad u \in C_0^\infty(\Omega)\,, \tag{2.5}$$

where the sum is finite and $a_{\alpha\beta}$ is symmetric for permutations within α or β. Then we have $a_{\alpha\beta}=0$ for all $x\in\Omega$, α and β, if $t_1 \ldots t_\nu \neq 0$.

PROOF. Assuming that not all $a_{\alpha\beta}$ vanish at a point x in Ω, which may be assumed to be the origin, we denote by m the maximum of $|\alpha|+|\beta|$ when $a_{\alpha\beta}(0)\neq 0$. Let η be a fixed vector an replace u in (2.5) by the function

$$x \to u(x^1/\varepsilon - \eta^1/t_1^2\varepsilon^2, \ldots)\,.$$

After a change of variables and multiplication by a factor independent of x, we obtain in the limit when $\varepsilon \to 0$

$$\sum_{|\alpha|+|\beta|=m}\sum a_{\alpha\beta}(0)\int D_\alpha u\, \overline{D_\beta u}\, e^{2\langle x,\eta\rangle}\, dx = 0, \qquad u \in C_0^\infty\,.$$

If we here set $u=ve^{i\langle x, i\eta\rangle}$, we obtain

$$\sum_{|\alpha|+|\beta|=m}\sum a_{\alpha\beta}(0)\int (D+i\eta)_\alpha v\, \overline{(D+i\eta)_\beta v}\, dx = 0, \qquad v \in C_0^\infty\,.$$

Application of the Fourier transformation now gives immediately

$$\sum_{|\alpha|+|\beta|=m}\sum a_{\alpha\beta}(0)(\xi+i\eta)_\alpha(\xi-i\eta)_\beta = 0\,,$$

for real ξ and η. But then the last equation also holds for indeterminate ξ and η, and since $\xi+i\eta$ and $\xi-i\eta$ are then independent indeterminates, it follows that $a_{\alpha\beta}=0$ when $|\alpha|+|\beta|=m$. This is a contradiction and so the lemma is proved.

Now consider a homogeneous differential operator

$$P(x, D) = \sum_{|\alpha|=m} a_\alpha(x) D_\alpha$$

with variable coefficients which are bounded and Lipschitz continuous uniformly in a domain Ω. Write $\bar{P}(x, \xi)=\overline{P(x, \xi)}$ when $\xi\in R^\nu$. We shall prove the following modification of (2.5) for $u\in C_0^\infty(\Omega)$

$$T(P(x, D)u,\ P(x, D)u) = \sum \frac{2^{|\alpha|}}{|\alpha|!}\, t_\alpha^2\, T(\bar{P}^{(\alpha)}(x, \delta)u,\ \bar{P}^{(\alpha)}(x, \delta)u) + R\,. \tag{2.6}$$

Here R is a sum of terms of the form

$$t_\gamma\, T((\bar{a}_{\alpha'}a_{\alpha''})'(D, \delta)_{\beta'}u,\ (D, \delta)_{\beta''}u) \tag{2.7}$$

with $|\beta'| \leqq m$, $|\beta''| \leqq m$ and $|\beta'| + |\beta''| + |\gamma| = 2m-1$. The abbreviation f' means $D_j f$ for some j, and $(D, \delta)_\beta$ means an arbitrary product of the form $D_{\beta_1} D_{\beta_2} \delta_{\beta_3} D_{\beta_4} \cdots$.

To prove (2.6) we first note that

$$T(P(x, D)u,\ P(x, D)u) = \sum_{\alpha,\beta} T(a_\alpha D_\alpha u,\ a_\beta D_\beta u)\ . \tag{2.8}$$

We rewrite a typical term in the sum by integrating by parts

$$\begin{aligned} T(a_\alpha D_j D_{\alpha'} u,\ a_\beta D_k D_{\beta'} u) &= T(\delta_k(\bar{a}_\beta a_\alpha D_j D_{\alpha'} u),\ D_{\beta'} u) \\ &= T(\bar{a}_\beta a_\alpha \delta_k D_j D_{\alpha'} u,\ D_{\beta'} u) + T\big((D_k(\bar{a}_\beta a_\alpha)) D_j D_{\alpha'} u,\ D_{\beta'} u\big)\ . \end{aligned}$$

The last term we include in the sum R in the right hand side of (2.6). In the other term we use the commutation relations (2.3) in order to replace $\delta_k D_j D_{\alpha'}$ by $D_j D_{\alpha'} \delta_k$ + a combination of operators D_γ with $|\gamma| = m-1$ having coefficients which are multiplies of $t_k{}^2$. When this procedure has been applied to all terms in (2.8) we operate again on the terms which have not been included in the sum R, this time shifting a differential operator D_j from left to right. Continuing in this fashion by transporting components of D alternatively from right to left and left to right, we can finally write

$$T(P(x, D)u,\ P(x, D)u) = \sum_{\alpha,\beta} T(c_{\alpha\beta} \delta_\alpha u,\ \delta_\beta u) + R\ ,$$

where $c_{\alpha\beta}$, besides being polynomials in t_j, are quadratic expressions in the coefficients a_γ alone (*not* involving their derivatives). But since $R = 0$ if the coefficients are constant, the sum

$$\sum_{\alpha,\beta} T(c_{\alpha\beta} \delta_\alpha u,\ \delta_\beta u)$$

must then be identical to the right hand side of (2.4) in view of Lemma 1. This completes the proof of (2.6).

Next we shall estimate R. Writing

$$T_k(u, u) = \sum_{|\alpha|=k} T(D_\alpha u,\ D_\alpha u)$$

we shall prove that

$$|R|^2 \leqq C T_m(u, u)\, T_{m-1}(u, u)\ . \tag{2.9}$$

To do so we first note that it follows from Theorem 4.4 in Trèves [5] that

$$T_j(u, u) \leqq |t|^{2(j-k)}\, T_k(u, u), \qquad j \leqq k\ . \tag{2.10}$$

Using this inequality we shall prove

$$|t|^{2(k-|\alpha|)}\, T\big((D, \delta)_\alpha u,\ (D, \delta)_\alpha u\big) \leqq C T_k(u, u), \qquad |\alpha| \leqq k\ . \tag{2.11}$$

Once this inequality is proved, the estimate (2.9) follows immediately from the fact that the terms of R are of the form (2.7) by using Cauchy–Schwarz' inequality and (2.10).

The left side of (2.11) can be rewritten by shifting components of δ to the other side, using the commutation relations. Noting that when the commutation relations give a factor t_j^2, we loose at the same time a factor D_j and a factor δ_j, we find that to prove (2.11) it is enough to prove that

$$(2.12)\quad |t|^{2k-|\alpha'|-|\alpha''|}\,|T(D_{\alpha'}u, D_{\alpha''}u)| \leqq CT_k(u, u), \qquad |\alpha'| \leqq k,\ |\alpha''| \leqq k\,.$$

But this follows immediately from Cauchy–Schwarz' inequality and (2.10).

We shall now besides (2.6) and (2.9) use the corresponding identity and inequality obtained by replacing P by $P^{(\beta)}$. With a trivial comparison between the first sum occurring in the right hand side of (2.6) in the two cases, we obtain

$$(2.13)\quad t_\beta^2 T\big(P^{(\beta)}(x, D)u,\ P^{(\beta)}(x, D)u\big) \leqq C\{T\big(P(x, D)u,\ P(x, D)u\big) + \big(T_m(u, u)\,T_{m-1}(u, u)\big)^{\frac12} + t_\beta^2\big(T_{m-|\beta|}(u, u)\,T_{m-|\beta|-1}(u, u)\big)^{\frac12}\}\,.$$

Using (2.10) again we get

THEOREM 1. *Let $P(x, D)$ be a homogeneous differential operator of order m with uniformly bounded and Lipschitz continuous coefficients in Ω. Then we have with a constant C depending only on m, ν and the bounds for the coefficients a_α and their first derivatives*

$$(2.14)\quad t_\beta^2\, T\big(P^{(\beta)}(x, D)u,\ P^{(\beta)}(x, D)u\big) \leqq C\{T\big(P(x, D)u,\ P(x, D)u\big) + \big(T_m(u, u)T_{m-1}(u, u)\big)^{\frac12}\}, \qquad u \in C_0^\infty(\Omega)\,.$$

REMARK. We obtain the same result with the same constant C if we modify the definition (2.1) of $T(u, v)$ by a factor $e^{2\langle x, \eta\rangle}$ with fixed η under the integral sign. This follows immediately from the fact that C does not depend on the choice of the origin.

3. Elliptic operators with "simple characteristics". In this section we use our methods to prove an extension of the results of Calderón concerning elliptic operators. Thus we assume in the whole section, besides the regularity assumption in Theorem 1, that P is elliptic at the origin,

$$(3.1)\qquad P(0, \xi) \neq 0, \qquad 0 \neq \xi \in R^n\,,$$

and that the equation

$$P(0, \zeta_1, \xi_2, \ldots, \xi_\nu) = 0 \tag{3.2}$$

has distinct zeros for real $(\xi_2, \ldots, \xi_\nu) \neq 0$. We then have with $N_0 = (-1, 0, \ldots, 0)$

$$|\xi + i\tau N_0|^{2m} \leq C\big(|P(0, \xi + i\tau N_0)|^2 + \tau^2 |P^{(1)}(0, \xi + i\tau N_0)|^2\big) .$$

In fact, the polynomial in ξ and τ on the right is homogeneous of degree m and is $\neq 0$ for real $(\xi, \tau) \neq (0, 0)$. It follows by continuity and homogeneity arguments that there is a neighbourhood U of 0 and an open cone $V \ni N_0$ such that for real (ξ, τ)

$$|\xi + i\tau N|^{2m} \leq C_1\big(|P(x, \xi + i\tau N)|^2 + \tau^2 |N|^2 |P^{(1)}(x, \xi + i\tau N)|^2\big), \quad x \in U,\ N \in V . \tag{3.3}$$

Similarly we get for real ξ and τ

$$|\xi + i\tau N|^{2(m-1)} \leq C_2 \sum_1^\nu |P^{(j)}(x, \xi + i\tau N)|^2, \quad x \in U,\ N \in V . \tag{3.4}$$

For if the right hand side should vanish for some real $(\xi, \tau) \neq (0, 0)$ when $x = 0$ and $N = N_0$, we have also $P = 0$ and hence the equation (3.2) would have a multiple zero.

Let $U_\delta \subset U$ be a neighbourhood of 0 such that $\operatorname{grad} \varphi_\delta(x) \in V$ and $|\operatorname{grad} \varphi_\delta(x) - \operatorname{grad} \varphi_\delta(0)| < \delta$ when $x \in U_\delta$. (The function φ_δ is defined by (1.3).) We shall prove

THEOREM 2. *Suppose that the coefficients of $P(x, D)$ are Lipschitz continuous, that* (3.1) *is valid and that the equation* (3.2) *has simple zeros. Then we have when $u \in C_0^\infty(U_\delta)$ and $|\alpha| \leq m$*

$$(1 + \delta^2 \tau)^{m-|\alpha|-1} \tau^{m-|\alpha|} \int |D_\alpha u|^2 e^{2\tau\varphi_\delta} \, dx \leq C \int |P(x, D)u|^2 e^{2\tau\varphi_\delta} \, dx , \tag{3.5}$$

provided that $\tau\delta > M$ and $\delta < \delta_0$, where M and δ_0 are constants.

PROOF. We shall use essentially the same partition of the unity as in [2], section 4. Thus take a function $\Theta \in C_0^\infty$ such that

$$\sum_g \Theta(x - g) = 1 \tag{3.6}$$

where g runs through all lattice points, and the support of Θ is contained in the cube $\max |x^j| < 1$. We then have $u = \Sigma u_g$ if we set

$$u_g(x) = \Theta\big(x^1 \tau^{\frac{1}{2}} - g^1,\ x^2 (\tau\delta)^{\frac{1}{2}} - g^2, \ldots, x^\nu (\tau\delta)^{\frac{1}{2}} - g^\nu\big)\, u(x) . \tag{3.7}$$

(The definition is chosen so that $\tau\varphi_\delta$ is nearly linear in the support of u_g.) By x_g we denote the point

$$x_g = \left(g^1/\tau^{\frac{1}{2}},\ g^2/(\tau\delta)^{\frac{1}{2}},\ \ldots,\ g^\nu/(\tau\delta)^{\frac{1}{2}}\right),$$

and we write $N_g = \operatorname{grad}\varphi_\delta(x_g)$.

Now take $x = x_g$, $N = N_g$ in (3.3), multiply by $|\hat{u}_g(\xi + i\tau N_g)|^2$ and integrate. Since $\hat{v}(\xi + i\eta)$ is the Fourier transform of $v(x)e^{\langle x,\eta\rangle}$ if $v \in C_0^\infty$, Parseval's formula gives

$$(3.8)\quad \int |D^m u_g|^2\, e^{2\tau\langle x, N_g\rangle}\, dx \leqq C_1 \int \{|P(x_g, D)u_g|^2 + \tau^2 |N_g|^2 |P^{(1)}(x_g, D)u_g|^2\}\, e^{2\tau\langle x, N_g\rangle}\, dx,$$

where we have used the abbreviation

$$|D^m u|^2 = \sum_{|\alpha|=m} |D_\alpha u|^2 .$$

Similarly, (3.4) gives

$$(3.9)\quad \int |D^{m-1}u_g|^2\, e^{2\tau\langle x, N_g\rangle}\, dx \leqq C_2 \int \sum_1^\nu |P^{(j)}(x_g, D)u_g|^2\, e^{2\tau\langle x, N_g\rangle}\, dx .$$

We multiply these two inequalities by $e^{2\tau(\varphi(x_g) - \langle x_g, N_g\rangle)}$. From Taylor's formula we get

$$\varphi(x_g) + \langle x - x_g,\ \operatorname{grad}\varphi(x_g)\rangle = \varphi(x) - (x^1 - x_g^1)^2 - \delta \sum_2^\nu (x^j - x_g^j)^2 .$$

In the support of u_g we have $|x^1 - x_g^1| < 1/\tau^{\frac{1}{2}}$ and $|x^j - x_g^j| < 1/(\tau\delta)^{\frac{1}{2}}$, $j > 1$. Hence

$$0 \leqq (x^1 - x_g^1)^2 + \delta \sum_2^\nu (x^j - x_g^j)^2 < \nu/\tau .$$

(3.8) and (3.9) may thus be rewritten in the form

$$(3.8')\quad \int |D^m u_g|^2\, e^{2\tau\varphi_\delta}\, dx \leqq C_1 e^{2\nu} \int \{|P(x_g, D)u_g|^2 + \tau^2 |N_g|^2 |P^{(1)}(x_g, D)u_g|^2\}\, e^{2\tau\varphi_\delta}\, dx,$$

$$(3.9')\quad \int |D^{m-1}u_g|^2\, e^{2\tau\varphi_\delta}\, dx \leqq C_2 e^{2\nu} \int \sum_1^\nu |P^{(j)}(x_g, D)u_g|^2\, e^{2\tau\varphi_\delta}\, dx .$$

Next note that

$$P^{(\alpha)}(x_g, D)u_g = P^{(\alpha)}(x, D)u_g + \left(P^{(\alpha)}(x_g, D) - P^{(\alpha)}(x, D)\right)u_g$$

where the coefficients of $P^{(\alpha)}(x_g, D) - P^{(\alpha)}(x, D)$ are $O\left(1/(\tau\delta)^{\frac{1}{2}}\right)$ when x is in the support of u_g. (We assume for example $\delta < 1$ so that $\tau\delta < \tau$.) Thus (3.8) gives if we also note that $|N_g| < 3\delta$ when $u_g \not\equiv 0$, in view of the definition of U_δ,

$$\int |D^m u_g|^2 e^{2\tau\varphi\delta}\, dx$$

$$\leqq C_1' \int \{|P(x,D)u_g|^2 + (\delta\tau)^{-1}|D^m u_g|^2 + \delta^2\tau^2 |P^{(1)}(x,D)u_g|^2 + \delta\tau |D^{m-1}u_g|^2\} e^{2\tau\varphi\delta}\, dx .$$

In the same way we get from (3.9)′ that

$$\int |D^{m-1}u_g|^2 e^{2\tau\varphi\delta}\, dx \leqq C_2' \int \left\{ \sum_1^{\nu} |P^{(j)}(x,D)u_g|^2 + (\delta\tau)^{-1}|D^{m-1}u_g|^2 \right\} e^{2\tau\varphi\delta}\, dx .$$

When $2C_2' < \delta\tau$ we get, writing $C_2'' = 2C_2'$

$$\int |D^{m-1}u_g|^2 e^{2\tau\varphi\delta}\, dx \leqq C_2'' \int \sum_1^{\nu} |P^{(j)}u_g|^2 e^{2\tau\varphi\delta}\, dx , \tag{3.10}$$

and using this result we obtain when $2C_1' < \delta\tau$ also

$$\int |D^m u_g|^2 e^{2\tau\varphi\delta}\, dx \leqq C_1'' \int \left\{ |Pu_g|^2 + \delta^2\tau^2 |P^{(1)}u_g|^2 + \delta\tau \sum_2^{\nu} |P^{(j)}u_g|^2 \right\} e^{2\tau\varphi\delta}\, dx . \tag{3.11}$$

Here we have started to write $P^{(\alpha)}$ instead of $P^{(\alpha)}(x, D)$ which should not cause any ambiguities since $P(x, D)$ is the only differential operator studied from now on.

In view of Cauchy's inequality we have

$$|D^k u|^2 \leqq 2^{\nu} \sum_g |D^k u_g|^2 \tag{3.12}$$

since $D_\alpha u = \sum D_\alpha u_g$ and at most 2^ν of the supports of the functions u_g can meet at any point. Further, if we write β^* for the set of indices > 1 in β and put $\Theta^{(\beta)} = D^\beta \Theta / |\beta|!$ we get

$$P^{(\alpha)}u_g = \sum_\beta P^{(\alpha+\beta)}u\, \tau^{|\beta|/2}\, \delta^{|\beta^*|/2}\, \Theta^{(\beta)} .$$

Denoting by C an upper bound for $2^\nu \sum |\Theta^{(\beta)}|^2$, the summation being performed for $|\beta| \leqq m$, we obtain

$$\tau^{|\alpha|}\, \delta^{|\alpha^*|} \sum_g |P^{(\alpha)}u_g|^2 \leqq C \sum_\beta |P^{(\alpha+\beta)}u|^2\, \tau^{|\alpha|+|\beta|}\, \delta^{|\alpha^*|+|\beta^*|} .$$

Integrating and using Theorem 1 with $t_1^2 = 2\tau$, $t_2^2 = \ldots = t_\nu^2 = 2\tau\delta$, we get

$$\tau^{|\alpha|}\, \delta^{|\alpha^*|} \sum_g \int |P^{(\alpha)}u_g|^2 e^{2\tau\varphi\delta}\, dx \leqq C'(B^2 + A_m A_{m-1}) \tag{3.13}$$

where we have used the notation

$$A_k = \left(\int |D^k u|^2 e^{2\tau\varphi\delta}\, dx \right)^{\frac{1}{2}} \tag{3.14}$$

and

$$B = \left(\int |Pu|^2 e^{2\tau\varphi\delta} dx \right)^{\frac{1}{2}} . \tag{3.15}$$

Adding the inequalities (3.10) or (3.11), using (3.12) and (3.13), we now get

$$A_{m-1}{}^2 \leqq C_3(\tau\delta)^{-1} (B^2 + A_m A_{m-1}) \tag{3.16}$$

$$A_m{}^2 \leqq C_4(1+\delta^2\tau) (B^2 + A_m A_{m-1}) . \tag{3.17}$$

It now only remains to use the inequality between geometric and arithmetic means twice. First, (3.17) gives

$$A_m{}^2 \leqq C_4(1+\delta^2\tau)B^2 + A_m{}^2/2 + C_4{}^2(1+\delta^2\tau)^2 A_{m-1}{}^2/2$$

or

$$A_m{}^2 \leqq 2C_4(1+\delta^2\tau)B^2 + C_4{}^2(1+\delta^2\tau)^2 A_{m-1}{}^2 . \tag{3.18}$$

Similarly (3.16) gives

$$A_{m-1}{}^2 \leqq C_3(\tau\delta)^{-1}\left(B^2 + A_m{}^2/2(1+\delta^2\tau) + (1+\delta^2\tau)A_{m-1}{}^2/2\right)$$

or combined with (3.18)

$$A_{m-1}{}^2\left(1 - 2^{-1}C_3(1+C_4{}^2)(1/\tau\delta+\delta)\right) \leqq C_3(1+C_4)(\tau\delta)^{-1}B^2 . \tag{3.19}$$

Now choose δ_0 and M so that

$$C_3(1+C_4{}^2)(M^{-1}+\delta_0) < 1$$

and so that all previous requirements on δ and τ are met when $\delta < \delta_0$ and $\tau\delta > M$. Then we get for such δ and τ

$$A_{m-1}{}^2 \leqq C_5 B^2/\tau\delta . \tag{3.20}$$

Using this estimate in (3.18) we obtain

$$A_m{}^2 \leqq C_6(1+\delta^2\tau)B^2 . \tag{3.21}$$

Now we have

$$\tau(1+\delta^2\tau)\int |v|^2 e^{2\tau\varphi\delta} dx \leqq \int |D_1 v|^2 e^{2\tau\varphi\delta} dx, \qquad v \in C_0^\infty(U_\delta) . \tag{3.22}$$

In fact, if we write $ve^{\tau\varphi\delta} = w$, this is equivalent to

$$\tau(1+\delta^2\tau) \int |w|^2 dx \leqq \int |\partial w/\partial x^1 - 2\tau(x^1-\delta)w|^2 dx .$$

Partial integration gives

$$\int |\partial w/\partial x^1 - 2\tau(x^1-\delta)w|^2 dx$$

$$= \int |\partial w/\partial x^1|^2 dx + 4\tau^2 \int (x^1-\delta)^2 |w|^2 dx + 2\tau \int |w|^2 dx \geqq (\delta^2\tau^2 + 2\tau) \int |w|^2 dx$$

since $|2(x^1-\delta)|>\delta$ in U_δ. From (3.22) we get if $|\alpha|<m$

$$(\tau+\delta^2\tau^2)^{m-|\alpha|}\int |D_\alpha u|^2\, e^{2\tau\varphi_\delta}\, dx \leqq A_m ,$$

which combined with (3.21) completes the proof of Theorem 2.

From Theorem 2 one gets a result on unique continuation by standard arguments. For the sake of convenience we only consider rather smooth functions.

THEOREM 3. *Let $P(x, D)$ satisfy the hypotheses of Theorem 2 and let $u\in C^m$ be a solution of the differential inequality* (1.1) *which vanishes in the intersection of a neighbourhood of* 0 *and the set* (1.2). *Then u vanishes in a full neighbourhood of* 0.

PROOF. Choose δ so small that (3.5) holds and U_δ belongs to the neighbourhood given in the theorem. Take a function $\chi\in C_0^\infty(U_\delta)$ so that $\chi=1$ in a neighbourhood U' of 0 and set $v=\chi u$. Then we have in U'

$$|Pv| = |Pu| \leqq K \sum_{|\alpha|<m} |D_\alpha u| = K \sum_{|\alpha|<m} |D_\alpha v| .$$

Since $x^1 \geqq x^{2^2}+\ldots+x^{\nu^2}$ in the support of v, we have there

$$\varphi_\delta(x) \leqq x^{1^2}-\delta x^1+\delta^2 ,$$

and since $x^1<\delta$ in U_δ we have $\varphi_\delta(x)<\varphi_\delta(0)$ in the support of v except when $x=0$. If $x\in \complement U'$ and is in the support of Pv we thus have

$$\varphi_\delta(x) \leqq \varphi_\delta(0)-\varkappa ,$$

for some $\varkappa>0$. Now we apply (3.5) to v; that this inequality is valid for all function in C^m follows immediately by approximation. This gives

$$\sum_{|\alpha|<m}\int |D_\alpha v|^2\, e^{2\tau\varphi_\delta}\, dx \leqq C\tau^{-1}\int |Pv|^2\, e^{2\tau\varphi_\delta}\, dx$$
$$\leqq C'\tau^{-1}\sum_{|\alpha|<m}\int_{U'} |D_\alpha v|^2\, e^{2\tau\varphi_\delta}\, dx + C\tau^{-1}\int_{\complement U'} |Pv|^2\, e^{2\tau\varphi_\delta}\, dx ,$$

or restricting the integration in the left hand side to U'

$$(1-C'/\tau)\sum_{|\alpha|<m}\int_{U'} |D_\alpha v|^2\, e^{2\tau\varphi_\delta}\, dx \leqq C\tau^{-1}\int_{\complement U'} |Pv|^2\, e^{2\tau\varphi_\delta}\, dx .$$

If $U''\subset U'$ is a neighbourhood of 0 where $\varphi_\delta(x)>\varphi_\delta(0)-\varkappa/2$ we get when $\tau>C'$

$$(1-C'/\tau)\int_{U''} |v|^2\, dx \leqq C\tau^{-1}e^{-\varkappa\tau}\int_{\complement U'} |Pv|^2\, dx$$

and when $\tau\to\infty$ we get $v=0$ in U''. This proves the theorem.

The hypotheses in this theorem are weaker than those made by Calderón [1] for elliptic operators. In fact, he requires that $P(x, \xi)$ is real and that a_α have Hölder continuous derivatives. Furthermore he has to exclude the case $\nu = 3$.

4. Elliptic operators with non singular characteristics. The assumption in section 3 that the equation (3.2) has simple zeros shall now be replaced by the weaker hypothesis

$$(4.1) \qquad \sum_1^\nu |P^{(j)}(0, \xi + i\tau N_0)|^2 \neq 0, \qquad \text{for real} \quad (\xi, \tau) \neq (0, 0) .$$

This is enough to prove (3.4), but (3.3) will be modified so that in the right hand side we have

$$\sum_1^\nu |P^{(j)}(x, \xi + i\tau N)|^2 \qquad \text{instead of} \qquad |P^{(1)}(x, \xi + i\tau N)|^2$$

only. We introduce U_δ as in section 3, now with φ_δ replaced by ψ_δ, and shall prove

THEOREM 4. *Suppose that the coefficients of $P(x, D)$ are Lipschitz continuous, and that (3.1), (4.1) are valid. Then we have when $u \in C_0^\infty(U_\delta)$ and $|\alpha| \leqq m$*

$$(4.2) \quad (1 + \delta^2 \tau)^{m - |\alpha| - 1} \, \tau^{m - |\alpha|} \int |D_\alpha u|^2 e^{2\tau\psi_\delta} \, dx \leqq C \int |P(x, D)u|^2 e^{2\tau\psi_\delta} \, dx ,$$

provided that $\tau\delta > M$ and $\delta < \delta_0$, where M and δ_0 are constants.

PROOF. Very few changes are required in the proof of Theorem 2, so we shall content ourselves with indicating them briefly. They mostly depend on the fact that in ψ_δ the coefficient of x^{j^2} is 1 for all j, so we shall deal with all the variables now as we dealt with x^1 in the proof of Theorem 2. Thus (3.7) is replaced by

$$u_g(x) = \Theta(x^1 \tau^{\frac{1}{2}} - g^1, \; x^2 \tau^{\frac{1}{2}} - g^2, \; \ldots) \, u(x)$$

and x_g is modified similarly. With φ_δ replaced by ψ_δ, (3.9) and (3.9)′ will hold without any other change, while in (3.8) and (3.8)′ we must write $\Sigma |P^{(j)} u_g|^2$ instead of $|P^{(1)} u_g|^2$. Similarly, (3.10) is not changed whereas in (3.11) the factor $\delta\tau$ of the sum should be replaced by $\delta^2 \tau^2$. We also get (3.13) if we now interpret $|\alpha^*|$ as 0 for all α. Altogether we get (3.17) unchanged and (3.16) with $(\tau\delta)^{-1}$ replaced by τ^{-1}. Since $\delta < 1$ this implies the old inequality (3.16) and hence (3.21). The proof is then completed as before.

An exact repetition of the proof of Theorem 3 now gives

THEOREM 5. *Let $P(x, D)$ satisfy the hypotheses of Theorem 4 and let $u \in C^m$ be a solution of the differential inequality* (1.1) *which vanishes outside the intersection of a neighbourhood of* 0 *and the sphere $\psi_\delta(x) \leqq \delta^2$ for some $\delta < \delta_0$, where δ_0 is the constant in Theorem* 4. *Then u vanishes in a full neighbourhood of* 0.

5. Some elliptic operators with double characteristics. We shall now consider operators of the form

$$P(x, \xi) = P_1(x, \xi)\, P_2(x, \xi)\,, \tag{5.1}$$

where P_j is of order m_j and satisfies the assumptions in Theorem 2. We define U_δ as the intersection of the U_δ associated with P_j, $j = 1, 2$, in Theorem 2.

THEOREM 6. *Suppose that the coefficients of P_1 are Lipschitz continuous, that those of P_2 have Lipschitz continuous derivatives of order $m_1 - 1$ and that P_1 and P_2 satisfy the assumptions of Theorem* 2. *Then we have, if $u \in C_0^\infty(U_\delta)$ and $|\alpha| \leqq m = m_1 + m_2$, provided that $\delta < \delta_0$ and $\tau\delta > M$*

$$(1+\delta^2\tau)^{m-|\alpha|-2}\, \tau^{m-|\alpha|} \int |D_\alpha u|^2 e^{2\tau\varphi_\delta}\, dx \leqq C \int |P(x, D)u|^2 e^{2\tau\varphi_\delta}\, dx\,. \tag{5.2}$$

PROOF. In view of Theorem 2 we have

$$\sum_{|\alpha'| \leqq m_1} (1+\delta^2\tau)^{m_1-|\alpha'|-1}\, \tau^{m_1-|\alpha'|} \int |D_{\alpha'} P_2(x, D)u|^2 e^{2\tau\varphi_\delta}\, dx$$
$$\leqq C \int |Q(x, D)u|^2 e^{2\tau\varphi_\delta}\, dx\,,$$

where $Q(x, D) = P_1(x, D)\, P_2(x, D)$. Now $D_{\alpha'} P_2(x, D)u = P_2(x, D) D_{\alpha'} u$ + a linear combination with bounded coefficients of $D_\beta u$ with

$$|\beta| \leqq |\alpha'| + m_2 - 1\,.$$

Since

$$m_1 - |\alpha'| \leqq m_1 + m_2 - |\beta| - 1 = m - |\beta| - 1\,,$$

we get with another constant C

$$\sum_{|\alpha'| \leqq m_1} (1+\delta^2\tau)^{m_1-|\alpha'|-1}\, \tau^{m_1-|\alpha'|} \int |P_2(x, D) D_{\alpha'} u|^2 e^{2\tau\varphi_\delta}\, dx$$
$$\leqq C \left\{ \int |Q(x, D)u|^2 e^{2\tau\varphi_\delta}\, dx + \sum_{|\beta| \leqq m-1} (1+\delta^2\tau)^{m-|\beta|-2}\, \tau^{m-|\beta|-1} \int |D_\beta u|^2 e^{2\tau\varphi_\delta}\, dx \right\}.$$

Using Theorem 2 again in the left hand side and moving terms from right to left, we get with still another constant C

$$\sum_{|\alpha|\leqq m} (1+\delta^2\tau)^{m-|\alpha|-2}\,\tau^{m-|\alpha|}(1-C/\tau)\int |D_\alpha u|^2 e^{2\tau\varphi_\delta}\,dx \leqq C\int |Q(x,D)u|^2 e^{2\tau\varphi_\delta}\,dx .$$

When $\tau > 2C$ we get (5.2) though with $P(x, D)$ replaced by $Q(x, D)$. These two operators differ by terms of order $< m$. Thus we get

$$\sum_{|\alpha|\leqq m} (1+\delta^2\tau)^{m-|\alpha|-2}\,\tau^{m-|\alpha|}\int |D_\alpha u|^2 e^{2\tau\varphi_\delta}\,dx$$
$$\leqq C\left\{\int |P(x,D)u|^2 e^{2\tau\varphi_\delta}\,dx + \sum_{|\alpha|<m} |D_\alpha u|^2 e^{2\tau\varphi_\delta}\,dx\right\}.$$

The coefficient in the left hand side in front of the term involving $D_\alpha u$ tends to 0 when $\tau \to \infty$ if $|\alpha| < m-1$, and if $|\alpha| = m-1$ it tends to $1/\delta^2$. When δ is so small that $\delta^{-2} > 2C$ we can thus move terms from the right hand side to the left hand side, obtaining for sufficiently large τ the inequality (5.2). The proof is complete.

The fact just noticed that the coefficients in the left hand side of (5.2) can be made arbitrarily large, when $|\alpha| < m$, by choosing δ and τ^{-1} sufficiently small, makes it possible to repeat the proof of Theorem 4 again to prove a theorem on unique continuation.

THEOREM 7. *Let $P(x, D)$ satisfy the assumptions of Theorem* 6. *Then the same conclusion as in Theorem* 3 *holds.*

REMARK. The regularity assumptions on the coefficients in P_2 may be weakened. This may be done by considering more carefully what Trèves' method gives when P is of the form (5.1) and then repeating the arguments of the proof of Theorem 2. Alternatively, we may use what we have proved for operators P_j with coefficients which are linear functions of x. Making a new suitable partition of the unity, replacing P_j by an operator with linear coefficients when we study a component in the partition of u, we get a proof of Theorem 6 and thus of Theorem 7 when the coefficients are only C^1. However, we do not carry out the proof here.

If we start from Theorem 4 instead of Theorem 2 and repeat the proof of Theorem 6 we evidently obtain the following result.

THEOREM 8. *Suppose that the coefficients of P_1 are Lipschitz continuous. that those of P_2 have Lipschitz continuous derivatives of order m_1-1 and that P_1 and P_2 satisfy the assumptions of Theorem* 4. *Then we have, if $u \in C_0^\infty(U_\delta)$ and $|\alpha| \leqq m$, provided that $\delta < \delta_0$ and $\tau\delta > M$,*

$$(5.3)\quad (1+\delta^2\tau)^{m-|\alpha|-2}\,\tau^{m-|\alpha|}\int |D_\alpha u|^2 e^{2\tau\varphi_\delta}\,dx \leqq C\int |P(x,D)u|^2 e^{2\tau\varphi_\delta}\,dx .$$

The corresponding result on unique continuation is the following.

THEOREM 9. *Let $P(x, D)$ satisfy the hypotheses of Theorem* 8 *and let $u \in C^m$ be a solution of the differential inequality* (1.1), *which vanishes outside the intersection of a neighbourhood of* 0 *and the sphere $\psi_\delta(x) \leqq \delta^2$ for some $\delta < \delta_0$, where δ_0 is the constant in Theorem* 8. *Then, if in addition $K\delta$, where K is the constant in* (1.1), *is sufficiently small, u vanishes in a full neighbourhood of* 0.

Note that whereas in Theorem 5 the choice of δ was only influenced by the coefficients in $P(x, D)$, it has been necessary here to take into account also the size of the constant K. This is due to the fact that in order to get a coefficient for the derivatives of order $m-1$ in (5.3) which is sufficiently large to take care of the constant K, it is necessary to choose δ small. Except for this point, however, the proof of Theorem 9 is again identical to the proof of Theorem 3, so we omit the details.

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