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SELF-ADJOINTNESS AND SPECTRA OF STURM-LIOUVILLE OPERATORS

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1. Introduction. Let q be a real locally integrable function, defined on the real axis R, and consider the linear differential operator

$$L = -d^2/dx^2 + q(x), \qquad -\infty < x < \infty .$$

The set of (all equivalence classes of) functions square integrable over R form a Hilbert space L^2 with the inner product

$$(u, v) = \int_{-\infty}^{\infty} u(x)\, \overline{v(x)}\, dx .$$

Let $\mathscr{D}$ be the set of all u in L^2 for which $u' = du/dx$ is absolutely continuous (in every closed interval) and $Lu = -u'' + qu$ is in L^2. It is a classical fact that L with domain $\mathscr{D}$ gives rise to a closed, densely defined operator A from L^2 to L^2 which is an extension of its adjoint A^*, which, in turn, is the closure of the restriction of A to the elements in $\mathscr{D}$ with compact supports. (See [1] and [12].)

There exist a number of criteria on q for A to be self-adjoint in which the negative part of q is subject to pointwise or integral restrictions. We shall in this paper establish the self-adjointness of A under conditions on q that allow large negative values of q to be "compensated" by large positive values in a neighbourhood. Our criteria will thus be applicable to functions with arbitrarily large integral mean values of the negative part. One of them is the following:

There exists a finite C so that for all intervals J of length $\leqq 1$ we have

$$\int_J q(x)\, dx \geqq -C . \tag{1.1}$$

The principal tool enabling us to treat these general functions q is a lemma by Ganelius [3], cited below.

Further, we shall show that if (1.1) holds, then u' is in L^2 for every $u \in \mathscr{D}$ and there exists a well-defined "potential energy"

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$$\lim_{M,N\to\infty} \int_{-M}^{N} q(x)\ |u(x)|^2\, dx\ ,$$

even though the function $q|u|^2$ may not be integrable.

Finally we shall prove that under the assumption (1.1) the operator A has discrete spectrum if and only if

$$\liminf_{|a|\to\infty} \int_{a}^{a+h} q(x)\, dx = +\infty$$

for all $h > 0$. — In the case when q is continuous and bounded from below, this criterion for a discrete spectrum has been found by Molčanov [7], and our result is thus an extension to general locally integrable functions satisfying (1.1).

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NOTATION. Throughout the paper the integrals for which no range of integration is stated are taken over the whole axis. Norm and inner product in L^2 will be denoted by $\|\ \|$ and $(,)$ respectively, and for any interval I we shall often write

$$\|u\|_I \quad \text{for} \quad \int_I |u(x)|^2\, dx\ .$$

$\mathscr{H}_1(I)$ will denote the class of absolutely continuous functions in $L^2(I)$ with first derivative in $L^2(I)$. In stead of $\mathscr{H}_1(R)$ we shall write $\mathscr{H}_1$.

2. Some preliminary results. In this section we shall study the real and locally integrable functions $q(x)$ which satisfy (1.1). If η_l denotes the characteristic function of the interval $|x| \leqq l$, the condition (1.1) is equivalent to the following: The convolutions

$$(\eta_l * q)(x) = \int \eta_l(x-t)\, q(t)\, dt$$

are bounded from below, uniformly for all x and $l \leqq \frac{1}{2}$. It is obvious that if x is restricted to any compact set these convolutions are always uniformly bounded for any locally integrable $q(x)$ and all $l \leqq \frac{1}{2}$. Thus the condition (1.1) is a global one only and involves no restriction on the local behaviour of q.

There is clearly no loss of generality in assuming that C in (1.1) is $\geqq 2$

and we shall always do so in the sequel. We should also remark that for a q satisfying (1.1) the relation $(\eta_l*(q+M))(x)\geqq -C+\varepsilon$ may not hold uniformly for small l and arbitrarily small positive ε, however large M is chosen, as is illustrated by $q(x)=x^n \sin x^{n+1}$.

Our starting point is the following lemma, due to Ganelius:

LEMMA 1. *Let $f\geqq 0$ and g be functions of bounded variation on the closed interval J. Then*

$$\int_J f\,dg \leqq \left[\inf_J f+\operatorname{var}_J f\right] \sup_{K\subset J} \int_K dg .$$

We shall also need the following simple

LEMMA 2. *Let J be a closed interval of length l. Then, for all $x\in J$ and any f in $\mathscr{H}_1(J)$ we have*

$$\tfrac12 l^{-1}\|f\|_J^2 - \tfrac12 l\|f'\|_J^2 \leqq |f(x)|^2 \leqq 2t^{-1}\|f\|_J^2 + t\|f'\|_J^2; \quad 0<t\leqq l, \tag{2.1}$$

and

$$\inf_J |f(x)|^2 \leqq l^{-1}\|f\|_J^2 . \tag{2.2}$$

PROOF. By applying Schwarz' inequality to the identity

$$f(x) = f(y) + \int_y^x f'(t)\,dt$$

we get

$$|f(x)|^2 \leqq 2|f(y)|^2 + 2|x-y|\int_y^x |f'(t)|^2\,dt .$$

Integrating in y over an interval containing x and of length $t\leqq l$ we get the upper bound in (2.1), and integrating in x over J we get the lower bound.—The estimate (2.2) is obvious.

LEMMA 3. *Let q be a real and locally integrable function satisfying* (1.1). *If I is a finite interval of length l and f any function in $\mathscr{H}_1(I)$*

$$\int_I q(x)\,|f(x)|^2\,dx \geqq -C\{2(hl/n)^{-1}\|f\|_I^2 + (hl/n)\|f'\|_I^2\} , \tag{2.3}$$

where n is the integer determined by $n-1<l\leqq n$ and h is any number in the interval $0<h\leqq 1$.

PROOF. There is no loss of generality in supposing that $I=(0, l)$. We first suppose $l=1$ and invoke lemma 1. Thus

$$-\int_I q\,|f|^2 \leqq \left[\inf_I |f|^2+\operatorname{var}_I |f|^2\right] \sup_{K\subset I} \int_K q\,.$$

By (1.1) the last factor is majorized by C, and from lemma 2 we get

$$\inf_I |f(x)|^2 \leqq \|f\|_I^2 \leqq h^{-1}\|f\|_I^2$$

for any h in the interval $0<h\leqq 1$. We now write $f(x)=f_1(x)+if_2(x)$, where f_1 and f_2 are real, and get

$$\operatorname{var}_I |f(x)|^2 = \int_I \left|\frac{d}{dx}|f(x)|^2\right| dx = \int_I |2f_1f_1'+2f_2f_2'| \leqq 2\|f\|_I\|f'\|_I\,,$$

by Cauchy's inequality, and hence

$$-\int_I q\,|f|^2 \leqq C\,h^{-1}[\|f\|_I^2+2h\|f\|_I\|f'\|_I] \leqq C\,h^{-1}[2\|f\|_I^2+h^2\|f'\|_I^2]\,,$$

which proves the lemma for $l=1$.

To prove the lemma for arbitrary l we put $Q(x)=ln^{-1}q(ln^{-1}x)$. Then

$$\int_0^l q(x)\,|f(x)|^2\,dx = \int_0^n Q(t)\,|f(ln^{-1}t)|^2\,dt = \sum_{k=1}^n \int_{k-1}^k Q(t)\,|f(ln^{-1}t)|^2\,dt\,.$$

We now observe that the function Q satisfies a condition of the type (1.1) with the same value of C for all intervals of length $\leqq n/l$ and hence for all intervals of length $\leqq 1$. The proof above for intervals of unit length therefore implies

$$\int_{k-1}^k Q(t)\,|f(ln^{-1}t)|^2\,dt \geqq -C\left\{2h^{-1}\int_{k-1}^k |f(ln^{-1}t)|^2\,dt + h\int_{k-1}^k \left|\frac{d}{dt}f(ln^{-1}t)\right|^2 dt\right\},$$

and hence, summing over k,

$$\int_0^n Q(t)\,|f(ln^{-1}t)|^2\,dt \geqq -C\left\{2h^{-1}\int_0^n |f(ln^{-1}t)|^2\,dt + h\int_0^n \left|\frac{d}{dt}f(ln^{-1}t)\right|^2 dt\right\}$$

$$= -C\left\{2h^{-1}l^{-1}n\int_0^l |f(x)|^2\,dx + hln^{-1}\int_0^l |f'(x)|^2\,dx\right\},$$

which proves the lemma.

COROLLARY. *If the length of the interval I is $\geqq 1$*

$$\int_I \{|f'(x)|^2+(2C^2+q)\,|f(x)|^2\}\,dx \geqq 0$$

for any f in $\mathscr{H}_1(I)$.

PROOF. It follows from the definition of n in lemma 3 that $n/lC < (l+1)/lC$. Since C is always assumed $\geqq 2$ this implies $n/lC < 1$ if $l \geqq 1$, and so we can put $h = n/lC$ in (2.3), and the corollary follows.

THEOREM 1. *If q is a real and locally integrable function satisfying* (1.1) *and f any function in $\mathscr{H}_1$ with compact support,*

$$\int q(x)\,|f(x)|^2\,dx \geqq -C\{2h^{-1}\|f\|^2 + h\|f'\|^2\} \tag{2.4}$$

for all h in the interval $0 < h \leqq 1$.

PROOF. We divide the real axis into a sum of disjoint intervals of unit length. Then (2.3) holds in each of these intervals and a summation gives (2.4)

REMARK. If the support of f is not compact, theorem 1 still obviously holds if $q|f|^2$—possibly without being integrable—is such that

$$\lim_{M, N \to \infty} \int_{-M}^{N} q(x)\,|f(x)|^2\,dx$$

exists. The integral in (2.4) must then of course be interpreted accordingly.

If the real locally integrable function q satisfies an upper estimate of a type corresponding to (1.1), that is

$$\int_J q(x)\,dx \leqq C_1 \tag{2.5}$$

for all intervals J of length $\leqq 1$, the function $-q$ satisfies (1.1) with C replaced by C_1 and hence lemma 3 and theorem 1 give upper bounds for $\int q|f|^2$. For convenience we state these in a separate theorem.

THEOREM 2. *Let q be a real locally integrable function satisfying* (2.5). *If I is any finite interval of length l and f a function in $\mathscr{H}_1(I)$*

$$\int_I q(x)\,|f(x)|^2\,dx \leqq C_1\{2(hl/n)^{-1}\|f\|_I^2 + (hl/n)\|f'\|_I^2\},$$

where n is determined by $n-1 < l \leqq n$, for any h in the interval $0 < h \leqq 1$.
If f is in $\mathscr{H}_1$ and has compact support, then also

$$\int q(x)\,|f(x)|^2\,dx \leqq C_1\{2h^{-1}\|f\|^2 + h\|f'\|^2\}$$

for any positive $h \leqq 1$.

The following lemma will be used in section 5.

LEMMA 4. *Assume that I is an interval of length $\leqq 1$, that q satisfies* (1.1) *and that*

$$\int_I q(x)|h(x)|^2\,dx \leqq C_1$$

for some h in $\mathscr{H}_1(I)$, such that

$$0 < m \leqq |h(x)| \leqq M$$

for all x in I. Then

$$\int_I q(x)\,dx \leqq C_0\,,$$

where C_0 depends only on C, C_1, m, M and $\|h'\|_I^2$.

PROOF. We apply lemma 1 with $f=|h|^{-2}$ and $dg=q|h|^2dx$ to obtain

$$(2.6)\qquad \int_I q(x)\,dx = \int_I q|h|^2|h|^{-2} \leqq \left[\inf_I |h|^{-2} + \operatorname*{var}_I |h|^{-2}\right] \sup_{J\subset I}\int_J q|h|^2\,,$$

and we shall exhibit a bound for each of the factors on the right.

For any $J\subset I$ the set $I-J$ consists of at most two intervals K and L, of length k and l respectively. From lemma 3 with $h=1$ we find

$$\int_K q|h|^2 \geqq -C[2k^{-1}\|h\|_K^2 + k\|h'\|_K^2]\,.$$

Since $\|h\|_K^2\leqq kM^2$ and $k\leqq 1$ this yields

$$\int_K q|h|^2 \geqq -C[2M^2+\|h'\|_K^2]\,.$$

Because a similar estimate holds for the interval L it follows that

$$\int_{I-J} q|h|^2 \geqq -C[4M^2+\|h'\|_I^2]$$

and hence

$$\int_J q|h|^2 = \int_I q|h|^2 - \int_{I-J} q|h|^2 \leqq C_1 + C[4M^2+\|h'\|_I^2]\,.$$

Thus there exists a bound of the required type for the second factor in (2.6).

On the other hand, $\inf_I |h|^2\leqq m^{-2}$ and

$$\operatorname*{var}_I |h|^{-2} = \int_I \left|\frac{d}{dx}|h(x)|^{-2}\right| dx = \int_I 2|h|^{-4}|\operatorname{Re}(h\,\overline{h'})|$$
$$\leqq 2m^{-4}\|h\|_I\|h'\|_I \leqq 2m^{-4}M\|h'\|_I\,,$$

and so we have a bound of the desired type for $\int_I q(x)dx$.

3. A criterion for self-adjointness. The operator A defined in section 1 has the property $A^* \subset A$, and the self-adjointness of A is therefore established if we prove that A is symmetric, i.e. that

$$A \subset A^* . \tag{3.1}$$

Various criteria for (3.1) are already known. Usually they are expressed in the terms of "limit point" and "limit circle", introduced by H. Weyl in his fundamental paper [14], and then the property (3.1) is equivalent to L being of the limit point type at $-\infty$ and $+\infty$. Weyl proves that (3.1) holds if q is bounded from below. The first improvement on Weyl's condition was given by Hartman and Wintner [5] who found that L is of the limit point type at $+\infty$ if

$$\int_0^x q^-(t)\,dt = O(x^3), \qquad x \to \infty , \tag{3.2}$$

where $q^-(t) = \max\big(-q(t), 0\big)$. Since then, further improvements have been published, one being the following by Hartman [4]:

L is of the limit point type at $+\infty$ if

$$\int_0^x q^-(t)\,dt \leqq x^{-1}\left(\int_0^x M(t)\,dt\right)^2 , \tag{3.3}$$

where $M(t)$ is monotonic for large t and such that $\int^\infty M(t)\,dt = \infty$. See also [2], [9], [11], and [13].

In the sequel we shall find another criterion for $A = A^*$, which will be an immediate consequence of two lemmas.

Lemma 5 (*cf.* [6]). *Let* $\omega \geqq 0$ *be a locally square integrable function, such that*

$$\int_0^\infty \omega = \int_{-\infty}^0 \omega = +\infty . \tag{3.4}$$

Then A is symmetric if

$$\int \omega^2 |u'|^2 < \infty \tag{3.5}$$

for all u in the domain $\mathscr{D}$ of A.

Proof. Given any number N we can, according to (3.4), find a number $N^* > N$ such that

$$\alpha_N = \int_N^{N^*} \omega(x)\,dx \geqq 1 \qquad \text{and} \qquad \beta_N = \int_{-N^*}^{-N} \omega(x)\,dx \geqq 1 .$$

We define an absolutely continuous function μ_N by

$$\mu_N(x) = \begin{cases} 1 & , \quad |x| \leqq N , \\ \alpha_N^{-1} \int_x^{N^*} \omega(t)\, dt, & N \leqq x \leqq N^* , \\ \beta_N^{-1} \int_{-N^*}^{x} \omega(t)\, dt, & -N^* \leqq x \leqq -N , \\ 0 & , \quad |x| \geqq N^* . \end{cases}$$

It follows that $\mu_N(x) \to 1$ as $N \to \infty$ and that the derivative $\mu_N'(x)$ exists almost everywhere and satisfies

$$|\mu_N'(x)| \leqq \omega(x) .$$

Now, let u and v be any two elements in $\mathscr{D}$. Then $|u\,\overline{Av} - Au\,\overline{v}|$ is integrable, and as μ_N tends boundedly to 1

$$(u,Av) - (Au,v) = \int (u\,\overline{Av} - Au\,\overline{v}) = \lim_{N\to\infty} \int \mu_N(u\,\overline{Av} - Au\,\overline{v}) .$$

Integrating by parts we obtain

$$\left| \int \mu_N(u\,\overline{Av} - Au\,\overline{v}) \right| = \left| \int \mu_N'(u\,\overline{v}' - u'\,\overline{v}) \right| \leqq \int_{I+J} \omega |u\,\overline{v}' - u'\,\overline{v}| ,$$

where $I = (N, N^*)$ and $J = (-N^*, -N)$. But u and v are both in L^2 and so are $\omega v'$ and $\omega u'$ according to (3.5), and hence $|\omega u\,\overline{v}'|$ and $|\omega u'\,\overline{v}|$ are both integrable. Therefore

$$\int_I \omega\, |u\, v' - u'\, v| \leqq \int_N^{N^*} (|\omega u\,\overline{v}'| + |\omega u'\,\overline{v}|) \to 0$$

as $N \to \infty$, and similarly for the integral over J. Consequently

$$(u\,\overline{Av} - Au\,\overline{v}) = \lim_{N\to\infty} \int \mu_N(u\,\overline{Av} - Au\,\overline{v}) = 0 ,$$

which proves the lemma.

In the sequel we shall make use of sets of functions $\varphi(x)$ with compact supports and uniformly bounded derivatives. To fix the ideas we define φ as follows:

(3.6)

i. $\varphi(x) = \varphi(x,r,R) = \begin{cases} 1 & \text{for} \quad -r \leqq x \leqq R , \\ 0 & \text{for} \quad x < -r-1 \text{ and } x > R+1 . \end{cases}$

ii. For every x, $\varphi(x)$ is increasing in r and R.

iii. The derivatives $\varphi'(x)$ and $\varphi''(x)$ are continuous and uniformly bounded in x, r and R.

From the definition follows that $0 \leqq \varphi(x) \leqq 1$ and that $\varphi \to 1$ as $\min(r,R) \to \infty$.

LEMMA 6. *Let ω be a bounded, twice continuously differentiable function with bounded first and second derivatives. Then*

$$\int_J \omega(x)\, q(x)\, dx \geqq -C \tag{3.7}$$

for all intervals J of length $\leqq 1$ implies

$$\int \omega^2\, |u'|^2 < \infty$$

for all u in $\mathscr{D}$.

PROOF. Let φ be one of the functions introduced by (3.6) and put $\psi = \varphi^2\omega^2$. If u is any function in $\mathscr{D}$ we get, by partial integration,

$$\int \psi\, Au\, \bar{u} = \int \psi'\, u'\, \bar{u} + \int \psi\, |u'|^2 + \int \psi q\, |u|^2. \tag{3.8}$$

Now, let u be a *real* function in $\mathscr{D}$. Then the first integral on the right can be integrated by parts, yielding

$$\int \psi\, Au\, \bar{u} = -\tfrac{1}{2} \int \psi''\, |u|^2 + \int \psi\, |u'|^2 + \int \psi q\, |u|^2 . \tag{3.9}$$

The functions ψ and ψ'' tend boundedly to ω^2 and $(\omega^2)''$ respectively as $\varphi \to 1$, that is as $\min(r,R) \to \infty$, and as $|\omega^2\, Au\, \bar{u}|$ and $|(\omega^2)''|\, |u|^2$ are both integrable, the first two integrals in (3.9) tend to the finite limits $\int \omega^2 Au\, \bar{u}$ and $\int (\omega^2)''\, |u|^2$ respectively, as $\varphi \to 1$. Since the convergence of ψ is also monotone, we conclude that $\int \psi |u'|^2$ must tend to $\int \omega^2 |u'|^2$, although this limit may not be finite, and therefore $\int \psi q |u|^2$ must also have a limit (possibly $-\infty$).

We put $Q(x) = \omega(x) q(x)$. From (3.7) it then follows that Q satisfies a condition of the type (1.1) and we apply lemma 1 as in the proof of lemma 3 and theorem 1 to obtain

$$\begin{aligned} -\int \psi q |u|^2 &= -\int Q(x)\, [\omega(x)\, \varphi^2(x)\, |u(x)|^2\, dx] \\ &\leqq C \left[2 \int \varphi^2 \omega\, |u|^2 + \operatorname{var} \omega\, \varphi^2\, |u|^2 \right] . \end{aligned}$$

But $\operatorname{var} \omega\varphi^2 |u|^2$ is bounded by

$$\int \varphi^2\, |\omega'|\, |u|^2 + 2 \int \varphi\, \omega\, |\varphi'|\, |u|^2 + 2 \int \omega\, \varphi^2\, |u\, u'| ,$$

which in turn is majorized by

$$M\|u\|^2 + 2\|u\|\,\|\varphi\omega u'\|\,,$$

where the coefficient M depends only on the bounds for ω, ω', and φ'. Hence from (3.9)

$$\|\varphi\omega u'\|^2 \leqq O(1) + 2\|u\|\,\|\varphi\omega u'\|\,,$$

and thus $\|\varphi\omega u'\|^2 = \int\varphi^2\omega^2|u'|^2 = \int\psi|u'|^2$ must be bounded. Therefore

$$\int\omega^2|u'|^2 < \infty\,, \tag{3.10}$$

and the lemma is proved for real u in $\mathscr{D}$.

Since every u in $\mathscr{D}$ may be written $u_1 + iu_2$, where u_1 and u_2 are real and in $\mathscr{D}$, the proof for real u shows that $\int\omega^2|u_1'|^2 < \infty$ and $\int\omega^2|u_2'|^2 < \infty$. Hence $\int\omega^2|u'|^2 < \infty$ for all u in $\mathscr{D}$, and the proof of the lemma is complete.

We observe that $\int\psi q|u|^2$ has a finite limit for all u in $\mathscr{D}$, and further that $|u'\,\bar{u}|$ is integrable and hence that $\int\psi' u'\bar{u}$ in (3.8) tends to $\int(\omega^2)'\,u'\,\bar{u}$ for all u in $\mathscr{D}$.

As was stated above the operator A must be self-adjoint if it is symmetric, and so the two last lemmas yield

THEOREM 3. *The operator A is self-adjoint if there exists a positive, twice continuously differentiable function $\omega(x)$ satisfying*

$$\omega(x),\ \omega'(x)\ \textit{and}\ \omega''(x)\ \textit{bounded}$$

and

$$\int_0^\infty \omega(x)\,dx = \int_{-\infty}^0 \omega(x)\,dx = +\infty\,,$$

such that

$$\int_J \omega(x)\,q(x)\,dx \geqq -C$$

for all intervals J of length $\leqq 1$.

REMARK. If ω satisfies the conditions of the last theorem and if there s given a pointwise estimate

$$\omega^2(x)q(x) \geqq -C\,, \tag{3.11}$$

the last integral in (3.8) is immediately seen to be bounded from below, which implies that $\int\omega^2|u'|^2$ is finite and hence that A is self-adjoint according to lemma 5.

To conclude this section we shall compare our criterion (3.7) with the previously known criteria (3.2) and (3.3). Obviously $q(x) \geqq -Cx^2$ is a special case of (3.2)—and also of (3.11) above—and by putting $q(x) = x^2$ we see that (3.2) can hold without (3.7) being satisfied, so for some functions q our new criterion is less general than those previously known. On the other hand, the criterion (3.7) is for some oscillating q considerably more general than (3.2) and (3.3). To illustrate this, put $q(x) = x^n \sin x^{n+1}$ for some integer n. Then

$$\int_J q(x)\,dx \geqq -2(n+1)^{-1} \geqq -1$$

for all intervals J, and so (3.7) and even (1.1) is satisfied. But for large x

$$\int_0^x q^-(t)\,dt \geqq C x^n$$

for some positive constant C.—In general, it is easily seen that for functions q satisfying (3.7), the integral

$$\int_0^x q^-(t)\,dt$$

can tend to infinity (as $x \to \infty$) with arbitrary rapidity.

4. A criterion for finite kinetic and potential energy. If the function q is bounded from below it follows, as Wintner [15] has shown, that u' is in L^2 for every u in $\mathscr{D}$. A weaker condition, (1.1) permits the same conclusion in view of lemma 6, for with $\omega(x) \equiv 1$ we see from (3.8), (3.9) and (3.10) that $\|u'\|^2$ is finite, that

$$\lim_{\varphi \to 1} \int \varphi^2 q\,|u|^2 \quad \text{exists}$$

and that

$$(Au,u) = \int Au\,\bar{u} = \int |u'|^2 + \lim_{\varphi \to 1} \int \varphi^2 q\,|u|^2 .$$

This enables us to prove that the "potential energy"

$$(4.1) \qquad Q(u) = \lim_{M,N \to \infty} \int_{-M}^{N} q(x)\,|u(x)|^2\,dx$$

exists and is finite for every u in $\mathscr{D}$.

Let $\varphi_1 = \varphi^2(x,r,R)$ and $\varphi_2 = \varphi^2(x,r-1,R-1)$ as defined by (3.6). Then obviously

$$\int_{-r}^{R} q|u|^2 = \int \varphi_1 q\,|u|^2 - \int_{-r-1}^{-r} \varphi_1 q\,|u|^2 - \int_{R}^{R+1} \varphi_1 q\,|u|^2$$

and

$$\int_{-r}^{R} q\,|u|^2 = \int \varphi_2 q|u|^2 + \int_{-r}^{-r+1} (1-\varphi_2)\, q|u|^2 + \int_{R-1}^{R} (1-\varphi_2) q\,|u|^2 .$$

In these two identities the four integrals over intervals of unit length can each be one-sidedly estimated by the norms of u and u' over the interval, by lemma 3. Since u and u' are both in L^2, those norms tend to 0 with increasing r and R. Thus

$$\int \varphi_2 q\,|u|^2 - o(1) \leqq \int_{-r}^{R} q\,|u|^2 \leqq \int \varphi_1 q\,|u|^2 + o(1)\,,$$

and hence

$$\int_{-r}^{R} q\,|u|^2 \to \lim_{\varphi\to 1} \int \varphi^2 q\,|u|^2$$

as $\min(r,R) \to \infty$, and so the limit in (4.1) exists. It also follows that

$$Q(u) = \int q\,|u|^2 = (Au,u) - (u'.u') \tag{4.2}$$

for all u in $\mathscr{D}$, which is equivalent to

$$(Au,u) = \int \{|u'|^2 + q|u|^2\}\,.$$

We have thus proved the first half of the following

THEOREM 4. *If q satisfies* (1.1) *the potential energy $Q(u)$ defined by* (4.1) *exists and is finite for any u in $\mathscr{D}$. Moreover, for any h in the interval $0 < h \leqq C^{-1}$ and every u in $\mathscr{D}$*

$$(1-Ch)(u',u') \leqq 2Ch^{-1}(u,u) + (Au,u) \tag{4.3}$$

and

$$(1-Ch).Q(u) \geqq -2Ch^{-1}(u,u) - Ch(Au,u)\,. \tag{4.4}$$

PROOF. For all $h \leqq C^{-1}$ (<1) and every u in $\mathscr{D}$

$$Q(u) = \int q\,|u|^2 \geqq -2Ch^{-1}\|u\|^2 - Ch\|u'\|^2\,,$$

as is seen from theorem 1 and the remark following it. Then (4.3) and (4.4) follow from (4.2).

We remark that although $Q(u)$ is well defined for all u in $\mathscr{D}$, the function $q|u|^2$ need not be integrable. A counterexample can be constructed in the following way. Put

$$q(x) = -2\varepsilon x \cos(x^2)\, \mathrm{tg}\,\theta(x)$$

with

$$\theta(x) = \pi/4 + \varepsilon \int_0^x \sin(t^2)dt$$

and ε sufficiently small. Put $v(x) = x^{-1} \cos \theta(x)$ and let φ be one of the functions introduced by (3.6), so that $1-\varphi$ vanishes in some neighbourhood of the origin. Then the function $u(x) = (1-\varphi)v(x)$ is in $\mathcal{D}$, but $q|u|^2$ is not integrable.

If we put $h = C^{-1}$ in (4.3) we get the following

COROLLARY. *The operator A is bounded from below; more precisely*

$$(4.5) \qquad (Au,u) \geqq -2C^2(u,u) .$$

5. A necessary and sufficient condition for discrete spectrum. Under the assumption (1.1) the operator A is self-adjoint, as has been shown in the preceding sections, and we shall now investigate the spectrum of A.

H. Weyl [14] proved that the spectrum of A is discrete if

$$\liminf_{|x|\to\infty} q(x) = +\infty .$$

Later, Molčanov has studied potentials q that are continuous and bounded from below and proved that for such potentials

$$(5.1) \qquad \liminf_{|a|\to\infty} \int_a^{a+h} q(x)\,dx = +\infty, \qquad \text{all } h > 0 ,$$

is a necessary and sufficient condition for discrete spectrum. The results of the preceding sections will enable us to follow Molčanov's approach for general locally integrable q satisfying (1.1) and prove that the spectrum is discrete if and only if (5.1) holds.

THEOREM 5. *If q satisfies* (1.1) *the spectrum of A is discrete if and only if*

$$(5.2) \qquad \liminf_{|a|\to\infty} \int_a^{a+h} q(x)\,dx = +\infty$$

for all $h > 0$.

PROOF. Consider the operator

$$B = A + (2C^2+1)I ,$$

I = identity operator, with domain $\mathcal{D}$. According to (4.5) B is $\geqq I$ and therefore $B^{\frac{1}{2}}$ is a self-adjoint operator $\geqq I$. Further, $B^{\frac{1}{2}}$ has discrete spectrum if and only if B and hence also A has discrete spectrum. Now, Rellich [8] has proved that the spectrum of $B^{\frac{1}{2}}$ is discrete if and only if the set of all u in the domain of $B^{\frac{1}{2}}$ satisfying

$$\|B^{\frac{1}{2}}u\|^2 + \|u\|^2 \leqq 1$$

is precompact (i.e. every infinite sequence contains a Cauchy-sequence). But

$$\|B^{\frac{1}{2}}u\|^2 = (Bu,u) \geqq (u,u) \tag{5.3}$$

for all u in $\mathscr{D}$, and it is well known that $\mathscr{D}$ is a dense subset of the domain of $B^{\frac{1}{2}}$ in the topology of the graph. Therefore A has discrete spectrum if and only if the set

$$\mathscr{M} = \{u \in \mathscr{D};\ (Bu,u) \leqq 1\}$$

is precompact.

The elements of $\mathscr{M}$ have uniformly bounded norms, according to (5.3), and with an appropriate choice of h in (4.3) we see that $\|u'\|^2$ is also uniformly bounded in $\mathscr{M}$. Thus $\mathscr{M}$ is an equicontinuous family of functions in L^2 in the sense that

$$\|u(x+h) - u(x)\|^2$$

tends uniformly to 0 as h tends to 0. A compactness theorem of M. Riesz [10] can now be applied: The set $\mathscr{M}$ is precompact if and only if

$$\lim_{n\to\infty}\left(\sup_{u\in\mathscr{M}} \int_{x>n} |u|^2\right) = 0\,. \tag{5.4}$$

We shall now first prove that (5.2) is a sufficient condition for discrete spectrum, and to do this we suppose that (5.4) is not fulfilled. This means that we assume the existence of a sequence of functions u_n in $\mathscr{M}$ for which

$$\int_{|x|>n} |u_n(x)|^2\,dx \geqq \eta^{-1} > 0 \tag{5.5}$$

for some η independent of n. Now

$$(Bu_n,u_n) = \int \{|u_n'|^2 + (q+2C^2+1)|u_n|^2\} \leqq 1\,,$$

according to (4.2), and if $n \geqq 1$

$$\int_{-n}^{n} \{|u_n'|^2 + (q+2C^2+1)|u_n|^2\} \geqq 0\,,$$

as is seen from the corollary to lemma 3, and therefore, in view of (5.5),

$$\int \{|u_n'|^2 + (q+2C^2+1)|u_n|^2\} \leqq 1 \leqq \eta \int_{x>n} |u_n|^2 .$$

We break up the intervals $|x|>n$ into a sum of disjoint intervals J_k of equal length $l \leqq 1$. (This number l shall be the same for all n. It will be clear below how l is most suitably chosen, depending on the numbers C and η only.) Then

$$\sum_k \int_{J_k} \{|u_n'|^2 + (q+2C^2+1)|u_n|^2\} \leqq \eta \cdot \sum_k \int_{J_k} |u_n|^2 . \tag{5.6}$$

It follows that there exists at least one interval $I_n = I$ among the J_k for which

$$\|u_n'\|_I^2 + (2C^2+1)\|u_n\|_I^2 + \int_I q|u_n|^2 \leqq \eta \|u_n\|_I^2 . \tag{5.7}$$

Lemma 3 and (5.7) yield

$$(1-Cl)\|u_n'\|_I^2 + (2C^2+1-2Cl^{-1})\|u_n\|_I^2 \leqq \eta \|u_n\|_I^2 .$$

Let v_n be a multiple of u_n so determined that $\|v_n\|_I^2 = l$ and let $l < 1/C$; then

$$\|v_n'\|_I^2 \leqq (1-Cl)^{-1}(\eta + 2Cl^{-1} - 2C^2 - 1).l ,$$

and

$$l\|v_n'\|_I^2 \leqq l(1-Cl)^{-1}\big(\eta l + 2C - l(2C^2+1)\big) .$$

Here the expression on the right tends to 0 as l tends to 0, and thus there exists a number $l_0(\eta, C)$, depending only on η and C, such that $l \leqq l_0$ implies $l\|v_n'\|_I^2 \leqq \frac{1}{2}$. Letting the intervals in (5.6) have precisely the length l_0 we conclude from lemma 2 that

$$1/4 \leqq |v_n(x)|^2 \leqq 9/4 \tag{5.8}$$

for all x in I. Finally we obtain from (5.7), which also holds for v_n by homogenity,

$$\begin{aligned} \int_I q(x)\,|v_n(x)|^2\,dx &\leqq \|v_n\|_I^2(\eta - 2C^2 - 1) - \|v_n'\|_I^2 \\ &\leqq l_0(\eta - 2C^2 - 1) = K . \end{aligned} \tag{5.9}$$

In view of (5.8) and (5.9), the hypotheses of lemma 4 are satisfied and hence

$$\int_I q(x)\,dx \leqq C_0 ,$$

where C_0 depends only on $\|v_n'\|_I^2$, C and K; that is on η and C only.

Accordingly, if $\mathscr{M}$ is not precompact we can find a sequence of intervals I_n of equal length l_0 and with I_n outside the interval $|x| \leqq n$, such that $\int_{I_n} q(x)dx$ is uniformly bounded. Then (5.2) cannot be true, and hence $\mathscr{M}$ must be precompact if (5.2) holds. This proves the sufficiency half of theorem 5.

It remains to prove that our condition for discrete spectrum is necessary, and to do this we suppose that the condition is not satisfied. This is equivalent to assuming the existence of an infinite sequence $\{\Delta_\nu\}_1^\infty$ of disjoint intervals, all of equal length $\varkappa > 0$, in which

$$(5.10) \qquad \int_{\Delta_\nu} q(x)\,dx \leqq C_1$$

for all ν. Obviously there is no loss of generality in supposing $\varkappa \leqq 1$, for if $\varkappa > 1$ we can find a sequence of intervals contained in the Δ_ν of length $\leqq 1$ in which (5.10) holds.

We first observe that (5.10) implies the existance of an upper bound for the corresponding integral over any subinterval J contained in Δ_ν, for in view of (1.1)

$$(5.11) \qquad \int_J q(x)\,dx = \int_{\Delta_\nu} q(x)\,dx - \int_{\Delta_\nu - J} q(x)\,dx \leqq C_1 + 2C = K\,.$$

Let φ_1 ($\neq 0$) be a twice continuously differentiable function with support contained in Δ_1 and let φ_ν be the translate of φ_1 to the interval Δ_ν. Applying theorem 2 we then get

$$(5.12) \int \left(|\varphi_\nu'|^2 + (q + 2C^2 + 1)|\varphi_\nu|^2\right) \leqq (1 + K\varkappa)\,\|\varphi_\nu'\|^2 + (2C^2 + 1 + 2K\varkappa^{-1})\,\|\varphi_\nu\|^2$$
$$= (1 + K\varkappa)\,\|\varphi_1'\|^2 + (2C^2 + 1 + 2K\varkappa^{-1})\,\|\varphi_1\|^2$$

for all ν. Since the functions φ_ν have disjoint supports, it follows that

$$\|\varphi_j - \varphi_k\|^2 = 2\|\varphi_1\|^2 > 0 \qquad \text{when} \quad j \neq k\,,$$

and hence a set containing the functions φ_ν cannot be precompact.

Now, the functions φ_ν need not be in $\mathscr{D}$, for q is only assumed locally integrable and therefore $-\varphi_\nu'' + q\varphi_\nu$ is not necessarily in L^2. We shall presently prove, however, that they are in the domain of $B^{\frac{1}{2}}$ and that $(B^{\frac{1}{2}}\varphi_\nu, B^{\frac{1}{2}}\varphi_\nu)$ is given by the integral on the left in (5.12). Supposing φ_1 so normed that the right hand side in (5.12) is $\leqq \frac{1}{2}$, say, and using the fact that $B^{\frac{1}{2}} \geqq I$, we then conclude that the set

$$\mathscr{M}' = \{u \in \mathscr{D}(B^{\frac{1}{2}});\ \|B^{\frac{1}{2}}u\|^2 + \|u\|^2 \leqq 1\}$$

contains the sequence $\{\varphi_\nu\}_1^\infty$. Therefore $\mathscr{M}'$ is not precompact, and so the spectrum of A cannot be discrete. Thus the assumption (5.10) must be false if A has a discrete spectrum and consequently (5.2) is a necessary condition. The proof of theorem 5 is thereby complete, although we have yet to show that smooth functions with compact supports are in the domain of the operator $B^{\frac{1}{2}}$.

6. The domain of the operator $B^{\frac{1}{2}}$. We recall that the operator B was defined as $A+(2C^2+1)I$, where I is the identity operator, and that under the condition (1.1) it is a self-adjoint operator $\geqq I$. In this section we shall give a characterisation of the elements in the domain of the operator $B^{\frac{1}{2}}$. This domain will be denoted by $\mathscr{D}(B^{\frac{1}{2}})$.

Define $\mathscr{R}_0$ as the set of all absolutely continuous functions with first derivatives in L^2 and with compact supports, and define for arbitrary f and g in $\mathscr{R}_0$

$$\langle f, g\rangle = \int (f'\bar{g}' + p f \bar{g}) , \tag{6.1}$$

where $p(x)=q(x)+2C^2+1$. Then, in view of theorem 1,

$$\langle f,f\rangle = \int \left(|f'|^2 + p|f|^2\right) \geqq (1-Ch)\|f'\|^2 + (2C^2+1-2Ch^{-1})\|f\|^2 , \tag{6.2}$$

for all positive $h \leqq 1$, and therefore with proper choice of h

$$\langle f,f\rangle \geqq C_1(\|f'\|^2+\|f\|^2) \tag{6.3}$$

for some positive constant C_1. Closing $\mathscr{R}_0$ in the norm (6.2) we get a Hilbert space $\mathscr{R}$.

Lemma 7. *Every element in the Hilbert space $\mathscr{R}$ is an absolutely continuous function in L^2 with first derivative in L^2. For any h in $\mathscr{R}_0$ and an arbitrary f in $\mathscr{R}$ the inner product in $\mathscr{R}$ is given by*

$$\langle f,h\rangle = \int (f'\bar{h}' + p f \bar{h}) . \tag{6.4}$$

Proof: The first assertion of the lemma follows immediately from (6.3). To prove the second one, let f be defined by a sequence $\{f_\nu\}_1^\infty$ of elements in $\mathscr{R}_0$. Then

$$\langle f_\nu,h\rangle = \int (f_\nu'\bar{h}' + p f_\nu \bar{h}) \tag{6.5}$$

by definition. But f_ν' and f_ν converge in L^2 to f' and f, and hence f_ν tends uniformly to f on the support of h, and so the integral in (6.5) tends to the integral in (6.4), which proves the lemma.

Lemma 8. *The set of all elements in $\mathscr{D}$ with compact supports is ($\mathscr{R}$-)dense in $\mathscr{R}$.*

Proof. Denote the set of all functions in $\mathscr{D}$ with compact supports with $\mathscr{D}_0$, and suppose that $\langle f, u\rangle = 0$ for all u in $\mathscr{D}_0$ and some f in $\mathscr{R}$. Integrating by parts we then obtain, according to lemma 7,

$$0 = \langle f,u\rangle = \int (f'u' + pfu) = \int (-fu'' + pfu) = (f,Bu)\,.$$

But $B(\mathscr{D}_0)$ is dense in L^2, hence $f=0$ and the lemma follows.

THEOREM 6. *The domain of the operator $B^{\frac{1}{2}}$ coincides with $\mathscr{R}$.*

PROOF. We first note that $\mathscr{D}_0$ is dense in $\mathscr{D}$ in the topology of the graph, for as we remarked in section 1, A is the closure of its restriction to $\mathscr{D}_0$. Using well-known functional calculus for operators we then conclude that $\mathscr{D}_0$ is also dense in the domain of $B^{\frac{1}{2}}$ in the corresponding graph-topology. Since

$$(B^{\frac{1}{2}}u, B^{\frac{1}{2}}u) = (Bu,u) = \langle u,u\rangle$$

for all u in $\mathscr{D}_0$ it follows that the domain of $B^{\frac{1}{2}}$ is obtained by closing $\mathscr{D}_0$ in the norm of $\mathscr{R}$, and thus

$$\mathscr{D}(B^{\frac{1}{2}}) \subset \mathscr{R}\,.$$

But lemma 8 shows that $\mathscr{D}(B^{\frac{1}{2}})$ cannot be a proper subset of $\mathscr{R}$, for then some f in $\mathscr{R}$ would be orthogonal ($\mathscr{R}$) to all h in $\mathscr{D}(B^{\frac{1}{2}})$ and hence a fortiori to all u in $\mathscr{D}_0$, which is possible only for $f=0$. Thus

$$\mathscr{D}(B^{\frac{1}{2}}) = \mathscr{R}\,,$$

and the theorem is proved.

It is clear that lemma 7 and theorem 6 contain the left-out parts of the proof of theorem 5 in the preceding section.—We should also remark that we have not given any explicit form for the inner product $\langle f,g\rangle$ between two arbitrary elements in $\mathscr{R}$, and we have needed no such knowledge in the proofs.

It may be of interest to note, however, that an integral expression corresponding to (6.1) does in fact give the inner product $\langle f,g\rangle$ for arbitrary f and g in $\mathscr{R}$. This has been shown to the author by professor William F. Donoghue, who communicated the following proof.

LEMMA 9. *The inner product in $\mathscr{R}$ is given by*

$$\langle f,g\rangle = \lim_{M,N\to\infty} \int_{-M}^{N} (f'\bar{g}' + pf\bar{g})\,. \tag{6.6}$$

PROOF. It is sufficient to prove that for all f in $\mathscr{R}$

$$\langle f,f\rangle = \lim_{M,N\to\infty} \int_{-M}^{N} (|f'|^2 + p|f|^2)\,, \tag{6.7}$$

for then

$$4\langle f,g\rangle = \langle f+g,f+g\rangle-\langle f-g,f-g\rangle+i\langle f+ig,f+ig\rangle-i\langle f-ig,f-ig\rangle$$
$$= \lim_{M,N\to\infty} 4\int_{-M}^{N}(f'\bar{g}'+p\,f\,\bar{g})\,.$$

We define

$$\langle f,f\rangle_n = \int_{n-1}^{n}(|f'|^2+p|f|^2)$$

for any f in $\mathscr{R}$ and infer from the corollary to lemma 3 that the number $\langle f,f\rangle_n$ is non-negative for all n. We proceed to prove that the series

$$P(f) = \sum_{n=-\infty}^{\infty}\langle f,f\rangle_n \tag{6.8}$$

always converges to the sum $\langle f,f\rangle$.

For any h in $\mathscr{R}_0$ the series in (6.8) is finite and $P(h)=\langle h,h\rangle$. Now, let f be an arbitrary element in $\mathscr{R}$, defined by a Cauchy-sequence $\{f_\nu\}_1^\infty$ of elements in $\mathscr{R}_0$. Then, as we have seen, f_ν' converges in L^2 to f' and f_ν converges uniformly to f on compacts. Thus the individual terms $\langle f_\nu,f_\nu\rangle_n$ converge to $\langle f,f\rangle_n$ for every n. But $\langle f_\nu,f_\nu\rangle$ converges to $\langle f,f\rangle$, and hence Fatou's lemma shows that

$$P(f) = \sum_{-\infty}^{\infty}\langle f,f\rangle_n = \sum_{-\infty}^{\infty}\lim_{\nu\to\infty}\langle f_\nu,f_\nu\rangle_n \leqq \lim_{\nu\to\infty}\sum_{-\infty}^{\infty}\langle f_\nu,f_\nu\rangle_n = \lim_{\nu\to\infty}P(f_\nu)$$
$$= \lim_{\nu\to\infty}\langle f_\nu,f_\nu\rangle = \langle f,f\rangle\,.$$

Thus the series $P(f)$ certainly converges, for each term is non-negative, and $P(f)\leqq\langle f,f\rangle$.

To obtain the opposite inequality, we define $\langle f,h\rangle_n$ for f in $\mathscr{R}$ and h in $\mathscr{R}_0$ by

$$\langle f,h\rangle_n = \int_{n-1}^{n}(f'\,\bar{h}'+p\,f\,\bar{h})\,.$$

Lemma 7 shows that

$$\langle f,h\rangle = \sum_{-\infty}^{\infty}\langle f,h\rangle_n\,,$$

the series in fact being finite, and since $\langle f,f\rangle_n$ is positive definite we get by Schwarz' inequality

$$|\langle f,h\rangle_n|^2 \leqq \langle f,f\rangle_n\,\langle h,h\rangle_n\,,$$

and hence

$$|\langle f,h\rangle|^2 = \left|\sum_{-\infty}^{\infty}\langle f,h\rangle_n\right|^2 \leqq \sum_{-\infty}^{\infty}\langle f,f\rangle_n\sum_{-\infty}^{\infty}\langle h,h\rangle_n = P(f)\,\langle h,h\rangle\,.$$

As $\mathscr{R}_0$ is dense in $\mathscr{R}$ this proves that $P(f) \geqq \langle f,f \rangle$ and therefore, in view of the inequality obtained above, $P(f) = \langle f,f \rangle$.

We have thus proved that the integral in (6.7) converges to $\langle f,f \rangle$ when M and N tend to infinity through integral values. But f and f' are both in L^2, and for arbitrary M and N we can therefore apply lemma 3 as in the proof of theorem 4 to obtain

$$\int_{-[M]-1}^{[N]+1} (|f'|^2 + p|f|^2) + o(1) \geqq \int_{-M}^{N} (|f'|^2 + p|f|^2)$$
$$\geqq \int_{-[M]}^{[N]} (|f'|^2 + p|f|^2) - o(1) \, ,$$

$[N]$ denoting the greatest integer $\leqq N$. But we have just proved that the expressions on the left and on the right both tend to $\langle f,f \rangle$, and hence the lemma is proved.

In the following concluding theorem we shall leave out the details of the proof.

THEOREM 7. *The domain of the operator $B^{\frac{1}{2}}$ consists of precisely those f in L^2 which are absolutely continuous, for which f' is in L^2 and for which the potential energy $Q(u)$ in* (4.1) *is well-defined and finite.*

PROOF. We have just shown that the limit in (6.7) exists and is finite for all f in $\mathscr{R}$. Since f and f' are in L^2 it follows that the potential energy exists and is finite.

Conversely, if f is a function satisfying the conditions of the theorem, the formula

$$F(g) = \lim_{M,N \to \infty} \int_{-M}^{N} (g' \bar{f}' + p\, g\, \bar{f})$$

is seen to define a continuous functional on $\mathscr{R}$. This functional is realised by some element h in $\mathscr{R}$ and it is not difficult to prove that the function $f-h$ must then be an L^2-solution to the equation $Bu = 0$. Since B is positive this implies $f = h$, hence f is in R and the theorem is proved.

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