MATH. SCAND. 7 (1959), 305–320

PARTIALLY ORDERED RECURSIVE ARITHMETICS

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1. One of the chief assumptions of recursive number theory is that every number has exactly one successor. Although very natural, this assumption is not the only possible one. We can ask if it is also possible to build up recursive arithmetics in which every number has more than one successor.

In this article we shall develop such arithmetics as formal equation calculi, in which the only axioms are recursive function definitions and the only rules of proof are substitution rules and the rule about the uniqueness of a function defined by recursion. In general, such arithmetics are not commutative. To make them obey the law of commutativity some new axioms are necessary. They bear on the successor functions or on the initial equations in recursive definitions.

We hope that these new arithmetics will be useful as models in questions of foundations. They may be interesting also in themselves, as illustrations of the possibilities of recursive definitions.

2. We assume that the reader is familiar with the elements of the recursive number theory.

In the arithmetic which we shall develop, every number x will have n successors $S_0x, S_1x, S_2x, \ldots, S_{n-1}x$. As it makes no serious difference to our formalism we shall at the same time allow the possibility for n to be countably infinite. Then every number x will have ω successors S_0x, $S_1x, \ldots$.

We denote numeral variables by $x, y, z, u, \ldots$, and definite numerals by $a, b, c, d, \ldots$. The numerals are $0, S_0 0, S_1 0, S_2 0, \ldots, S_0 S_0 0, S_1 S_0 0$, $S_2 S_0 0, \ldots, S_0 S_1 0, \ldots$, and so on. In writing numerals other than zero we shall omit zero. So, for instance, we write $S_1 S_0 S_2$ for $S_1 S_0 S_2 0$.

For the construction of our arithmetic we assume we have two explicit functions: the zero-function $Z(x) = 0$ and the identity-function $Y(x) = x$, and also n (or ω) successor functions $S_\nu x$, $\nu = 0, 1, \ldots, n-1$ (or $0, 1, 2, \ldots$) which replace the numeral x by the numeral $S_\nu x$.

Received September 25, 1959.

From these functions we shall construct new functions by recursive definitions:

DEFINITION 2.1. *A function $F(x, y)$ is defined by simple recursion if there are given $n+1$ (or ω) equations*

$$(2.1) \qquad \begin{aligned} F(x, 0) &= a(x) \\ F(x, S_\nu y) &= b_\nu\big(x, y, F(x, y)\big), \ \nu = 0, 1, \ldots, \end{aligned}$$

in which $a(x)$ and $b_\nu(x, y, z)$ are functions previously defined.

To give an example let us define n (or ω) linear operations σ_ν, $\nu = 0, 1, \ldots, n-1$ (or $0, 1, \ldots$).

$$(2.2) \qquad \begin{aligned} x\sigma_\nu 0 &= x \\ x\sigma_\nu S_\mu y &= S_{\mu+\nu}(x\sigma_\nu y) , \end{aligned}$$

$\nu, \mu = 0, 1, 2, \ldots, (n-1)$ (or $0, 1, \ldots$). *The addition of indices ν and μ is to be taken modulo n for finite n.*

The most significant of these operations is σ_0 which we call addition and denote by $+$:

$$(2.2') \qquad \begin{aligned} x+0 &= x \\ x+S_\mu y &= S_\mu(x+y) . \end{aligned}$$

As the simplest example $S_0+S_1 = S_1S_0$, $S_1+S_0 = S_0S_1$ shows, addition is not commutative. To make it commutative we introduce new *axioms*:

$$(2.3) \qquad S_\nu S_\mu x = S_\mu S_\nu x, \qquad \nu, \mu = 0, 1, \ldots .$$

These axioms are equivalent to

$$(2.3') \qquad F(x, S_\nu S_\mu y) = F(x, S_\mu S_\nu y)$$

in the definition (2.1), or to

$$(2.3'') \qquad b_\nu\big(x, S_\mu y, b_\mu(x, y, F(x, y))\big) = b_\mu\big(x, S_\nu y, b_\nu(x, y, F(x, y))\big) .$$

For every new function, defined by recursion, we must verify that (2.3′) or (2.3″) are fulfilled.

To make clear what the axioms (2.3) mean for our arithmetic we take the special case $n=2$. A model for partially ordered recursive arithmetic with two successors without axioms (2.3) is given by the tree (fig. 1) .

After the introduction of the axiom

$$(2.3''') \qquad S_0S_1x = S_1S_0x$$

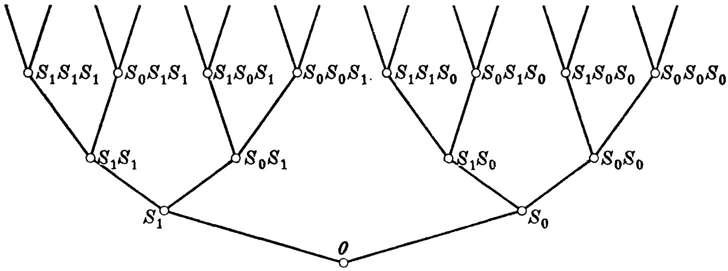

Fig. 1

to which axioms (2.3) reduce in this case, this tree reduces to a two-dimensional lattice (fig. 2).

Similarly, for $n = 3$ a tree reduces to the lattice of lattice-points of the first octant and similarly in others cases. Generally, axioms (2.3) make the order of successors S_ν irrelevant. We shall therefore write the numerals beginning with the successor of lowest index and then with higher ones, in lexicographical order.

To avoid "pathological" models we introduce as an *axiom* that

$$(2.3'''') \qquad S_a S_b S_c \ldots S_q = S_{a'} S_{b'} S_{c'} \ldots S_{q'}$$

with $a \leqq b \leqq c \leqq \ldots \leqq q$ *and* $a' \leqq b' \leqq c' \leqq \ldots \leqq q'$ *if and only if* $a = a'$, $b = b'$, $c = c'$, $\ldots$, $q = q'$.

The necessity for this axiom was pointed out to the author by R. L. Goodstein.

Definition (2.1) by single recursion is not the only one we shall assume. Later we shall introduce also definition by double recursion.

We shall allow the construction of new functions by substitution, in the known manner. Also, we take it as axiomatic that two functions

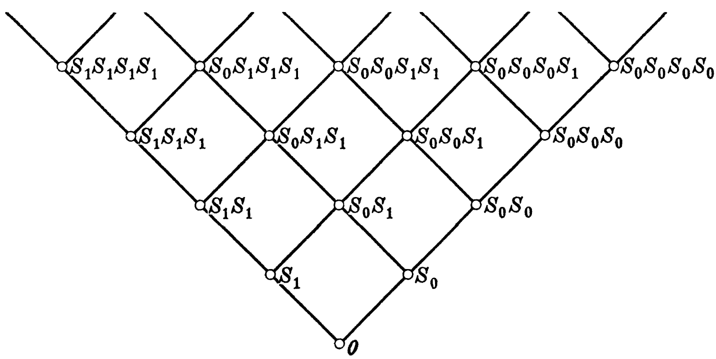

Fig. 2

with the same initial recursive equations (2.1) are identical, i.e. one and the same function.

3. Now we are able to develop partially ordered recursive arithmetic as a formal equation calculus. We prove first some theorems about the linear operations σ_ν.

(3.1) $$0+x = x\,.$$

PROOF. (In proving equations by uniqueness we shall always denote the left side by f and the right by φ).

$$f(0) = 0, \qquad f(S_\mu x) = 0+S_\mu x = S_\mu(0+x) = S_\mu f(x)\,.$$

The same initial equations are fulfilled by $Y(x)=x$.

(3.2) $$0\sigma_\nu S_\mu = S_{\mu+\nu}\,.$$

We recall that the addition of indices ν and μ is to be taken modulo n for finite n.

We write

(3.3) $$\sigma_\nu x \quad \text{for} \quad 0\sigma_\nu x\,.$$

(3.4) $$(S_\mu x)\sigma_\nu y = S_\mu(x\sigma_\nu y)\,.$$

PROOF. $f(x, 0)=S_\mu x$; $f(x, S_\tau y)=(S_\mu x)\sigma_\nu(S_\tau y)=S_{\tau+\nu}f(x, y)$; $\varphi(x, 0)=S_\mu x$; $\varphi(x, S_\tau y)=S_\mu(x\sigma_\nu S_\tau y)=S_\mu S_{\tau+\nu}(x\sigma_\nu y)=S_{\tau+\nu}S_\mu(x\sigma_\nu y)=S_{\tau+\nu}\varphi(x, y)$.

(3.5) $$x\sigma_\nu y = \sigma_\nu y+x\,.$$

PROOF. $f(x, 0)=x$; $f(x, S_\mu y)=S_{\mu+\nu}f(x, y)$; $\varphi(x, 0)=x$; $\varphi(x, S_\mu y)=\sigma_\nu S_\mu y+x=S_{\mu+\nu}(\sigma_\nu y)+x=S_{\mu+\nu}\varphi(x, y)$, by (3.4).

(3.6) $$(x+y)\sigma_\nu z = x+(y\sigma_\nu z)\,,$$

i.e. *addition is associative with all linear operations.*

PROOF. $f(x, y, 0)=x+y$; $f(x, y, S_\mu z)=S_{\mu+\nu}f$; $\varphi(x, y, 0)=x+y$; $\varphi(x, y, S_\mu z)=x+S_{\mu+\nu}(y\sigma_\nu z)=S_{\mu+\nu}\varphi$.

(3.7) $$(a\sigma_\tau x)\sigma_\mu y = (a\sigma_\mu y)\sigma_\tau x\,.$$

PROOF. $(a\sigma_\tau x)\sigma_\mu y=\sigma_\mu y+(a\sigma_\tau x)=(\sigma_\mu y+a)\sigma_\tau x=(a\sigma_\mu y)\sigma_\tau x$, by (3.5).

(3.8) $$a\sigma_\nu(b\sigma_\mu x) = (a\sigma_\nu b)\sigma_\mu(\sigma_\nu x)\,.$$

This is the associative law for our arithmetic.

PROOF. $f(0) = a\sigma_\nu b$; $f(S_\tau x) = a\sigma_\nu S_{\tau+\mu}(b\sigma_\mu x) = S_{\tau+\mu+\nu}f$; $\varphi(0) = a\sigma_\nu b$; $\varphi(S_\tau x) = (a\sigma_\nu b)\sigma_\mu(\sigma_\nu S_\tau x) = (a\sigma_\nu b)\sigma_\mu S_{\tau+\nu}(\sigma_\nu x) = S_{\tau+\nu+\mu}\varphi$.

We need still one theorem:

$$[a\sigma_\nu(b\sigma_\mu c)]\sigma_\mu x = (a\sigma_\nu b)\sigma_\mu(x\sigma_\nu c)\,. \tag{3.9}$$

PROOF. $f(0) = a\sigma_\nu(b\sigma_\mu c)$; $f(S_\tau x) = S_{\tau+\mu}f$; $\varphi(0) = (a\sigma_\nu b)\sigma_\mu(\sigma_\nu c) = a\sigma_\nu(b\sigma_\mu c)$, by (3.8); $\varphi(S_\tau x) = (a\sigma_\nu b)\sigma_\mu(S_\tau x\sigma_\nu c) = (a\sigma_\nu b)\sigma_\mu S_\tau(x\sigma_\nu c)$, by (3.4), so $\varphi(S_\tau x) = S_{\tau+\mu}\varphi$.

4. Multiplication is defined by

$$\begin{aligned} x\cdot 0 &= 0 \\ x\cdot S_\nu y &= (x\cdot y)\sigma_\nu x, \qquad \nu = 0, 1, \dots\,. \end{aligned} \tag{4.1}$$

We have to verify the fulfillment of (2.3′). By double application of (4.1) we have

$$\begin{aligned} x\cdot S_\mu S_\nu y &= [(x\cdot y)\sigma_\nu x]\sigma_\mu x\,, \\ x\cdot S_\nu S_\mu y &= [(x\cdot y)\sigma_\mu x]\sigma_\nu x\,. \end{aligned} \tag{4.2}$$

By (3.7) $(a\sigma_\nu x)\sigma_\mu x = (a\sigma_\mu x)\sigma_\nu x$, so the right hand sides of equations (4.2) are equal.

We point out that it is possible (and, for the definition of exponentiation, necessary) to introduce n (or ω) multiplicative operations $x \overset{*}{_\nu} y$ by

$$\begin{aligned} x \overset{*}{_\nu} 0 &= 0 \\ x \overset{*}{_\nu} S_\mu y &= (x \overset{*}{_\nu} y)\sigma_{\mu+\nu} x\,, \end{aligned} \tag{4.1′}$$

the addition of indices μ and ν being taken modulo n for finite n. Multiplication $x\cdot y$ is the operation $x \overset{*}{_0} y$.

In this article we shall not need other multiplicative operations than $x\cdot y$, so we put the others aside.

$$0\cdot x = 0\,. \tag{4.3}$$

$$(S_\mu x)\cdot y = (x\cdot y)\sigma_\mu y\,. \tag{4.4}$$

PROOF. $f(x, 0) = 0$; $f(x, S_\tau y) = S_\mu x\cdot S_\tau y = [(S_\mu x)\cdot y]\sigma_\tau S_\mu x = f(x, y)\sigma_\tau S_\mu x$; $\varphi(x, 0) = 0$; $\varphi(x, S_\tau y) = (x\cdot S_\tau y)\sigma_\mu S_\tau y = [(x\cdot y)\sigma_\tau x]\sigma_\mu S_\tau y = S_{\tau+\mu}\{[(x\cdot y)\sigma_\tau x]\sigma_\mu y\} = S_{\tau+\mu}\{([x\cdot y]\sigma_\mu y)\sigma_\tau x\} = [(x\cdot y)\sigma_\mu y]\sigma_\tau S_\mu x = \varphi(x, y)\sigma_\tau S_\mu x$, by (3.7).

$$x\cdot y = y\cdot x \tag{4.5}$$

i.e. *multiplication is commutative.*

PROOF. $f(x, 0)=0$; $f(x, S_\mu y)=f(x, y)\sigma_\mu x$; $\varphi(x, 0)=0$; $\varphi(x, S_\mu y)=S_\mu y\cdot x$ $=(y\cdot x)\sigma_\mu x=\varphi(x, y)\sigma_\mu x$, by (4.4).

$$(x\sigma_\nu y)\cdot z = (x\cdot z)\sigma_\nu(y\cdot z) , \tag{4.6}$$

i.e. *multiplication is distributive in regard to all linear operations.*

PROOF. $f(x, y, 0)=0$; $f(x, y, S_\tau z)=f\sigma_\tau(x\sigma_\nu y)$; $\varphi(0)=0$; $\varphi(S_\tau z)=$ $(x\cdot S_\tau z)\sigma_\nu(y\cdot S_\tau z) = (x\cdot z\sigma_\tau x)\sigma_\nu(y\cdot z\sigma_\tau y) = \{(x\cdot z)\sigma_\nu[(y\cdot z)\sigma_\tau y]\}\sigma_\tau x = [(x\cdot z)\sigma_\nu$ $(y\cdot z)]\sigma_\tau(x\sigma_\nu y)=\varphi\sigma_\tau(x\sigma_\nu y)$, by (3.7) and (3.9).

5. We now introduce n (or ω) predecessors $P_\nu(x)$, $\nu=0, 1, \ldots,$ by:

$$\begin{aligned} P_\nu 0 &= 0 , \\ P_\nu S_\mu x &= \begin{cases} x & \text{for} \quad \nu=\mu , \\ S_\mu P_\nu x & \text{for} \quad \nu\neq\mu . \end{cases} \end{aligned} \tag{5.1}$$

We have

$$\begin{aligned} P_\nu S_\mu S_\tau x &= \begin{cases} S_\tau x & \text{for} \quad \nu=\mu , \\ S_\mu P_\nu S_\tau x & \text{for} \quad \nu\neq\mu \end{cases} \\ &= \begin{cases} S_\tau x & \text{for} \quad \nu=\mu , \\ S_\mu x & \text{for} \quad \nu=\tau , \\ S_\mu S_\tau P_\nu x & \text{for} \quad \nu\neq\mu \text{ and } \nu\neq\tau , \end{cases} \end{aligned}$$

and similarly

$$P_\nu S_\tau S_\mu x = \begin{cases} S_\tau x & \text{for} \quad \nu=\mu , \\ S_\mu x & \text{for} \quad \nu=\tau , \\ S_\tau S_\mu P_\nu x & \text{for} \quad \nu\neq\mu \text{ and } \nu\neq\tau . \end{cases}$$

So

$$P_\nu S_\mu S_\tau x = P_\nu S_\tau S_\mu x , \tag{5.2}$$

which proves (2.3′) for predecessor functions.

$$P_\nu P_\mu x = P_\mu P_\nu x . \tag{5.3}$$

PROOF. Both sides satisfy the initial equations

$$\begin{aligned} F(0) &= 0 \\ F(S_\tau x) &= \begin{cases} P_\nu(x) & \text{for} \quad \mu=\tau , \\ P_\mu(x) & \text{for} \quad \nu=\tau , \\ S_\tau F(x) & \text{for} \quad \nu\neq\tau \text{ and } \mu\neq\tau . \end{cases} \end{aligned}$$

With predecessor functions we now introduce the difference $x\dot{-}y$ by

$$x \dot{-} 0 = x\,, \quad x \dot{-} S_\nu y = P_\nu(x \dot{-} y)\,. \tag{5.4}$$

From (5.3) follows the fulfillment of (2.3′) for the function $x \dot{-} y$.

The following equations can be proved without difficulty by uniqueness.

$$x \dot{-} y = S_\nu x \dot{-} S_\nu y, \quad \nu = 0, \ldots, n-1 \text{ (or } \omega)\,. \tag{5.5}$$

$$x \dot{-} x = 0\,. \tag{5.6}$$

$$0 \dot{-} x = 0\,. \tag{5.7}$$

$$(x+y) \dot{-} y = x\,. \tag{5.8}$$

$$x \dot{-} (y+z) = (x \dot{-} y) \dot{-} z\,. \tag{5.9}$$

Easy consequences are

$$(x \dot{-} y) \dot{-} z = (x \dot{-} z) \dot{-} y\,. \tag{5.10}$$

$$(a+x) \dot{-} (b+x) = a \dot{-} b\,. \tag{5.11}$$

We point out that multiplication is not distributive with regard to difference.

6. At this stage we can introduce definition by double recursion. In our case it does take a different and somewhat more curious form than in ordinary recursive number theory.

DEFINITION 6.1. *A function $F(x, y)$ is said to be defined by double recursion if there are given $n+1$ (or ω) equations*

$$F(x, 0) = a(x), \quad F(x, S_\nu y) = b_\nu\big(x, y, F(P_\nu x, y)\big), \quad \nu = 0, 1, \ldots, \tag{6.1}$$

where P_ν is the ν-th predecessor function and $a(x)$, $b_\nu(x, y, z)$ are functions previously defined to satisfy

$$b_\nu\big(x, S_\tau y, b_\tau(P_\nu x, y, F(P_\tau P_\nu x, y))\big) = b_\tau\big(x, S_\nu y, b_\nu(P_\tau x, y, F(P_\nu P_\tau x, y))\big)\,, \tag{6.2}$$

for $\nu, \mu = 0, 1, \ldots$.

Naturally, (6.2) are conditions imposed by axioms (2.3).

We point out that the conditions (6.2) are always satisfied if

$$b_\nu\big(x, S_\tau y, b_\tau(P_\nu x, y, z)\big) = b_\tau\big(x, S_\nu y, b_\nu(P_\tau x, y, z)\big)\,. \tag{6.3}$$

In the applications all the functions $b_\nu(x, y, z)$ which we shall employ will satisfy (6.3).

The definition (6.1) will serve us to prove the fundamental relation

$$x+(y\dot{-}x) = y+(x\dot{-}y)\,.$$

First, we need some auxiliary formulae.

(6.4) *For* $\mu \neq \nu$, $\quad P_\mu b+(S_\nu\dot{-}x) = P_\mu\{b+(S_\nu\dot{-}x)\}\,.$

PROOF. $f(0)=P_\mu b+S_\nu=S_\nu P_\mu b,$

$$\begin{aligned}
f(S_\tau x) &= P_\mu b+(S_\nu\dot{-}S_\tau x)\\
&= \begin{cases} P_\mu b & \text{for} \quad \tau=\nu\,,\\ P_\mu b+[S_\nu\dot{-}(x+S_\tau)] & \text{for} \quad \tau\neq\nu\,,\end{cases}\\
&= \begin{cases} P_\mu b & \text{for} \quad \tau=\nu\,,\\ P_\mu b+[(S_\nu\dot{-}S_\tau)\dot{-}x] & \text{for} \quad \tau\neq\nu\,,\end{cases}\\
&= \begin{cases} P_\mu b & \text{for} \quad \tau=\nu\,,\\ P_\mu b+(S_\nu\dot{-}x) & \text{for} \quad \tau\neq\nu\,,\end{cases}\\
&= \begin{cases} P_\mu b & \text{for} \quad \tau=\nu\,,\\ f(x) & \text{for} \quad \tau\neq\nu\,, \quad \text{because, for } \tau\neq\nu\,,\end{cases}\\
S_\nu\dot{-}S_\tau &= P_\tau S_\nu 0 = S_\nu P_\tau 0 = S_\nu 0\,.\\
\varphi(0) &= P_\mu S_\nu b = S_\nu P_\mu b, \quad \text{because } \nu\neq\mu;\\
\varphi(S_\tau x) &= P_\mu\{b+(S_\nu\div S_\tau x)\}\\
&= \begin{cases} P_\mu b & \text{for} \quad \tau=\nu\,,\\ P_\mu\{b+(S\dot{-}x)\} & \text{for} \quad \tau\neq\nu\,,\end{cases}\\
&= \begin{cases} P_\mu b & \text{for} \quad \tau=\nu\,,\\ \varphi(x) & \text{for} \quad \tau\neq\nu\,.\end{cases}
\end{aligned}$$

(6.5) $$S_\nu a\dot{-}x = (a\dot{-}P_\nu x)+(S_\nu\dot{-}x)\,.$$

PROOF. $f(0)=S_\nu a;$

$$\begin{aligned}
f(S_\tau x) &= S_\nu a\dot{-}S_\tau x\\
&= \begin{cases} a\dot{-}x & \text{for} \quad \tau=\nu,\\ P_\tau f(x) & \text{for} \quad \tau\neq\nu\,.\end{cases}\\
\varphi(0) &= a+S_\nu = S_\nu a\,,\\
\varphi(S_\tau x) &= (a\dot{-}P_\nu S_\tau x)+(S_\nu\dot{-}S_\tau x)\,,\\
&= \begin{cases} a\dot{-}x & \text{for} \quad \tau=\nu\,,\\ P_\tau(a\dot{-}P_\nu x)+(S_\nu\dot{-}x) & \text{for} \quad \tau\neq\nu\,,\end{cases}\\
&= \begin{cases} a\dot{-}x & \text{for} \quad \tau=\nu\,,\\ P_\tau\varphi(x) & \text{for} \quad \tau\neq\nu, \quad \text{by (6.4).}\end{cases}
\end{aligned}$$

(6.6) $$x+(S_\nu\dot{-}x) = S_\nu+(x\dot{-}S_\nu)\,.$$

PROOF. $f(0)=S_\nu$;

$$f(S_\tau x) = S_\tau x+(S_\nu \dot- S_\tau x)$$
$$= \begin{cases} S_\tau x & \text{for} \quad \tau = \nu\,, \\ S_\tau x+(S_\nu \dot- x) & \text{for} \quad \tau \neq \nu \end{cases}$$
$$= \begin{cases} S_\tau x & \text{for} \quad \tau = \nu\,, \\ S_\tau f(x) & \text{for} \quad \tau \neq \nu\,; \end{cases}$$
$$\varphi(0) = S_\nu\,,$$
$$\varphi(S_\tau x) = S_\nu+(S_\tau x \dot- S_\nu)$$
$$= \begin{cases} S_\nu+x & \text{for} \quad \tau = \nu\,, \\ S_\nu+P_\nu S_\tau x & \text{for} \quad \tau \neq \nu\,, \end{cases}$$
$$= \begin{cases} S_\nu x & \text{for} \quad \tau = \nu\,, \\ S_\nu+S_\tau P_\nu x & \text{for} \quad \tau \neq \nu\,, \end{cases}$$
$$= \begin{cases} S_\tau x & \text{for} \quad \tau = \nu\,, \\ S_\tau \varphi(x) & \text{for} \quad \tau \neq \nu\,. \end{cases}$$

(6.7) $$x+(y \dot- x) = y+(x \dot- y)\,.$$

PROOF. By double recursion. $f(x, 0)=x$; $f(x, S_\nu y)=x+(S_\nu y \dot- x)=x+(y \dot- P_\nu x)+(S_\nu \dot- x)$, by (6.5). So, $f(x, S_\nu y)=x+(S_\nu \dot- x)+(y \dot- P_\nu x)=S_\nu+(x \dot- S_\nu)+(y \dot- P_\nu x)$ by (6.6). At last, $f(x, S_\nu y)=S_\nu+P_\nu x+(y \dot- P_\nu x)=S_\nu f(P_\nu x, y)$.

$\varphi(x, 0)=x$; $\varphi(x, S_\nu y)=S_\nu y+(x \dot- S_\nu y)=S_\nu y+(P_\nu x \dot- y)=S_\nu \varphi(P_\nu x, y)$.

So f and φ fulfill the same initial equations $f(x, 0)=x$, $f(x, S_\nu y)=S_\nu F(P_\nu x, y)$ for double recursion, and (6.3) are trivially satisfied.

7. We can introduce a partial ordering in our arithmetic by exactly the same equation which serves in ordinary recursive number theory for the definition of the "less than" relation:

(7.1) $a \leqq b$ *is written for* $a = b \dot- (b \dot- a)$.

Exactly in the same manner as there we can prove:

(7.2) *From* $a \leqq b$ *follows* $a \dot- b = 0$ *and* $b = a+(b \dot- a)$.

(7.3) *From* $a \leqq b$ *and* $b \leqq c$ *follows* $a \leqq c$.

(7.4) $$a \leqq a+x\,.$$

(7.5) *From* $a \leqq b$ *follows* $a+x \leqq b+x$.

We emphasize that "From ... follows" is to be read: "If there is a proof of the first equation then one can also prove the second equation".

8. With the formalism we have introduced we are now able to build up our arithmetic along lines close to those of ordinary recursive number theory. Naturally, our arithmetic is poorer than recursive number theory. Many equations of recursive number theory do not hold in our arithmetic. In particular we do not have the relation $x\cdot(1\dot{-}x)=0$ which is decisive for classical recursive number theory, as exposed by R. L. Goodstein in [1]. But we shall find a way to replace this relation.

We introduce now the first proof schemata. Their sense is so well known that further explanations are unnecessary.

$$(8.1)\qquad \frac{x+y = 0}{x = 0}.$$

PROOF. By (5.8).

To make our formulae like the formulae of ordinary recursive number theory we shall write 1 for the numeral $S_0 0$. We recall that $x\cdot 1 = 1\cdot x = x$.

$$(8.2)\qquad \frac{\{x+(1\dot{-}x)\}\cdot y = 0}{y = 0}.$$

PROOF. By (6.7) we have $x+(1\dot{-}x)=1+(x\dot{-}1)$, so the first equation is $\{1+(x\dot{-}1)\}y=0$, that is, $y+(x\dot{-}1)y=0$. By (8.1) it follows that $y=0$.

$$(8.3)\qquad \frac{\begin{array}{c} x\cdot y = 0 \\ (1\dot{-}y)\cdot z = 0 \end{array}}{x\cdot z = 0}.$$

PROOF. The same as in Goodstein [1, Ex. 2.36].

$$(8.4)\qquad \frac{\begin{array}{c} (1\dot{-}x)\cdot y = 0 \\ (1\dot{-}y)\cdot z = 0 \end{array}}{(1\dot{-}x)\cdot z = 0}.$$

PROOF. By (8.3).

(8.5) *Let* $|x, y| = (x\dot{-}y)+(y\dot{-}x)$. *Then the proof schemata*

$$\frac{|x, y| = 0}{x = y} \quad \textit{and} \quad \frac{x = y}{|x, y| = 0}$$

both hold.

PROOF. The second schema is obvious. We prove the first one. From

$(x \dot{-} y)+(y \dot{-} x)=0$ we have by (8.1) $x \dot{-} y=0$ and $y \dot{-} x=0$. Putting these two equations into (6.7) we get $x=y$.

(8.6)
$$\frac{\begin{array}{c} f(0) = g(0) \\ f(S_\mu x) = g(S_\mu x) \end{array}}{f(x) = g(x)\,.}$$

9. We introduce now the function $\alpha(x)$ by

(9.1)
$$\alpha(0) = 0\,, \quad \alpha(S_\mu x) = 1\,.$$

(2.3″) are trivially fulfilled.

We quote some relations involving $\alpha(x)$.

(9.2) $$[1 \dot{-} \alpha(x)]\cdot[1 \dot{-} \alpha(x)] = 1 \dot{-} \alpha(x)\,.$$

(9.3) $$[1 \dot{-} \alpha(x)\alpha(y)]\cdot\alpha(x)\cdot\alpha(y) = 0\,.$$

(9.4) $$\alpha(x\cdot y) = \alpha(x)\cdot\alpha(y)\,.$$

(9.5) $$1 \dot{-} \alpha(x\sigma_\nu y) = \{1 \dot{-} \alpha(x)\}\cdot\{1 \dot{-} \alpha(y)\}\,.$$

(9.6) $$1 \dot{-} [\alpha(x)+\alpha(y)] = [1 \dot{-} \alpha(x)]\cdot[1 \dot{-} \alpha(y)]\,.$$

(9.7) $$\{1 \dot{-} \alpha(x \dot{-} y)\}\cdot\{1 \dot{-} \alpha(|x, y|)\} = 1 \dot{-} \alpha(|x, y|)\,.$$

PROOF. By double recursion, using the fact that

$$\begin{aligned} |x, S_\mu y| &= (x \dot{-} S_\mu y)+(S_\mu y \dot{-} x) \\ &= (P_\mu x \dot{-} y)+(S_\mu y \dot{-} x) \\ &= (P_\mu x \dot{-} y)+(y \dot{-} P_\mu x)+(S_\mu \dot{-} x)\,, \text{ by (6.5).} \end{aligned}$$

(9.8) $$\begin{aligned} &\{1 \dot{-} [1 \dot{-} \alpha(x)\alpha(y)]\}\cdot\{[1 \dot{-} \alpha(x)]+[1 \dot{-} \alpha(y)]\}+ \\ &\quad +\{1 \dot{-} [(1 \dot{-} \alpha(x))+(1 \dot{-} \alpha(y))]\}\{1 \dot{-} \alpha(x)\alpha(y)\} = 0\,. \end{aligned}$$

(9.9) $$1 \dot{-} (1 \dot{-} \alpha(x)) = \alpha(x)\,.$$

(9.10) $$1 \dot{-} \{\alpha(x)+\alpha(x)\alpha(y)\} = 1 \dot{-} \alpha(x)$$

(9.11) $$\begin{aligned} &\{1 \dot{-} [\alpha(x)+\alpha(y)\alpha(z)]\}\{\alpha(x)+\alpha(y)\}\cdot\{\alpha(x)+\alpha(z)\}+ \\ &\quad +\{1 \dot{-} [\alpha(x)+\alpha(y)][\alpha(x)+\alpha(z)]\}\cdot\{\alpha(x)+\alpha(y)\alpha(z)\} = 0\,. \end{aligned}$$

(9.12) $$\{1 \dot{-} \alpha(|x, y|)\}\cdot\alpha[f(x)] = \{1 \dot{-} \alpha(|x, y|)\}\cdot\alpha[f(y)]\,.$$

PROOF. A slight modification of the proof of the equation 2.63 in Goodstein [1].

(9.13) $$\{1 \dot{-} \alpha(|x, y|)\} \cdot \{1 \dot{-} \alpha[f(x)]\} \cdot \alpha[f(y)] = 0 .$$

(9.14) $$\frac{\alpha[f(0)] = 0 \qquad \{1 \dot{-} \alpha[f(x)]\} \cdot \alpha[f(S_\mu x)] = 0, \quad \mu = 0, 1, \ldots}{\alpha[f(x)] = 0 .}$$

The proofs of (9.13) and (9.14) can be obtained also by known proofs (Goodstein [1, formula 2.68 and the proof in section 2.8]) by suitable generalisations.

10. The introduction of logical constants has to be carried out in a way slighty different from the one which is customary in ordinary recursive number theory. The difference is due to the fact that the equation $x \cdot (1 \dot{-} x) = 0$ does not hold in our arithmetic.

Therefore, we call the equation $a = b$ a *true proposition* if and only if the equation $\alpha(|a, b|) = 0$ is a provable equation. (The sense of "provable" is the same as in recursive number theory.) If $\alpha(|a, b|) = 1$ we call the equation $a = b$ a *false proposition*. Then every true proposition is a provable equation and conversely, and every proposition is true or false.

As is customary, propositions are denoted by $p, q, r, \ldots$, and their negations by $\bar{p}, \bar{q}, \bar{r}, \ldots$.

Let p be the proposition $a = b$ and q the proposition $c = d$. Then

$p \,\&\, q$	is	$\alpha(\|a, b\|) + \alpha(\|c, d\|) = 0$,
$p \vee q$	is	$\alpha(\|a, b\|) \cdot \alpha(\|c, d\|) = 0$,
$p \rightarrow q$	is	$\{1 \dot{-} \alpha(\|a, b\|)\} \cdot \alpha(\|c, d\|) = 0$,
$p \longleftrightarrow q$	is	$\{1 \dot{-} \alpha(\|a, b\|)\} \cdot \alpha(\|c, d\|) + \{1 \dot{-} \alpha(\|c, d\|)\} \cdot \alpha(\|a, b\|) = 0$,
$\bar{p}$	is	$1 \dot{-} \alpha(\|a, b\|) = 0$,
$\overline{p \,\&\, q}$	is	$1 \dot{-} \{\alpha(\|a, b\|) + \alpha(\|c, d\|)\} = 0$,
$\overline{p \vee q}$	is	$1 \dot{-} \alpha(\|a, b\|) \cdot \alpha(\|c, d\|) = 0$,
$\overline{p \rightarrow q}$	is	$1 \dot{-} \{1 \dot{-} \alpha(\|a, b\|)\} \cdot \alpha(\|c, d\|) = 0$,
$\overline{p \longleftrightarrow q}$	is	$1 \dot{-} \{[1 \dot{-} \alpha(\|a, b\|)] \cdot \alpha(\|c, d\|) + [1 \dot{-} \alpha(\|c, d\|)]\alpha(\|a, b)\} = 0.$

From (9.6) it follows that $\overline{p \& q}$ and $\bar{p} \vee \bar{q}$ are the same proposition. From $\{1 \dot{-} \alpha(|a, b|)\} \cdot \alpha(|a, b|) = 0$ it follows that $p \longleftrightarrow p$, so

$$\overline{p \,\&\, q} \longleftrightarrow \bar{p} \vee \bar{q} .$$

From (9.8) we have

$$\overline{p \vee q} \longleftrightarrow \bar{p} \,\&\, \bar{q} ,$$

and from (9.9) follows

$$\bar{\bar{p}} \longleftrightarrow p\,.$$

By the distributivity of multiplication we have

$$p \vee (q \;\&\; r) \longleftrightarrow (p \vee q) \;\&\; (p \vee r)$$

and by (9.11)

$$p \;\&\; (q \vee r) \longleftrightarrow (p \;\&\; q) \vee (p \;\&\; r)\,.$$

We shall not develop all the details of the introduction of logical symbols, because one can now, using the formulae of section 9, proceed very nearly as in ordinary recursive number theory. Thus, without loss of generality we can suppose any proposition to have the form $\alpha(c)=0$.

Allowing $a, b, c, d, \ldots$ to be functions of x, that is $a=a(x)$, $b=b(x)$, $\ldots$, we introduce propositional functions $p(x)$, $q(x)$, $\ldots$.

By (9.13) we have

$$(10.1) \qquad (x = y) \rightarrow \{p(x) \rightarrow p(y)\}\,,$$

and by (9.14) the schema of induction:

$$\frac{\begin{array}{l} p(0) \\ p(x) \rightarrow p(S_\mu x)\,, \quad \mu = 0, 1, \ldots \end{array}}{p(x)\,.}$$

As an exemple we note the proposition

$$(x \cdot y = 0) \rightarrow (x = 0) \vee (y = 0)\,,$$

i.e. the equation

$$\{1 \dot{-} \alpha(|x \cdot y, 0|)\} \cdot \alpha(|x, 0|) \cdot \alpha(|y, 0|) = 0\,,$$

which is the proved equation (9.3).

The following exemples are due to R. L. Goodstein.

By 10.1 we have

$$P_\mu(1 \dot{-} x) = 1 \dot{-} x \rightarrow P_\nu P_\mu(1 \dot{-} x) = P_\nu(1 \dot{-} x)\,.$$

But

$$P_\nu P_\mu(1 \dot{-} x) = P_\mu P_\nu(1 \dot{-} x) = P_\mu(1 \dot{-} S_\nu x)\,,$$

$$P_\nu(1 \dot{-} x) = 1 \dot{-} S_\nu x\,,$$

and hence, writing $p(x)$ for $P_\mu(1 \dot{-} x) = 1 \dot{-} x$ when $\mu > 0$, we have $p(x) \rightarrow p(S_\nu x)$. Also $p(0)$ holds, because, for $\mu \neq 0$, $P_\mu(1 \dot{-} 0) = P_\mu(1) = 1 \dot{-} 0$. So, by the schema of induction, $p(x)$ holds, i.e. the equation

$$(10.2) \qquad P_\mu(1 \dot{-} x) = 1 \dot{-} x \quad \text{for} \quad \mu > 0$$

is a proved equation.

We prove now

(10.3) $$(1 \dot{-} x)\cdot(1 \dot{-} x) = 1 \dot{-} x .$$

PROOF. $f(0)=1, f(S_0 x)=0$, and for $\mu > 0$ by (10.2)

$$f(S_\mu x) = (1 \dot{-} S_\mu x)\cdot(1 \dot{-} S_\mu x) = [P_\mu(1 \dot{-} x)]^2 = f(x);$$

$\varphi(0)=1$, $\varphi(S_0 x)=0$, and for $\mu > 0$ $\varphi(S_\mu x)=P_\mu(1 \dot{-} x)=1 \dot{-} x=\varphi(x)$.

We can now prove the equation

(10.4) $$(1 \dot{-} x)[1 \dot{-} (1 \dot{-} x)] = 0 ,$$

which is a substitute for the equation $x\cdot(1 \dot{-} x)=0$ of simple recursive number theory.

PROOF.
$$f(0)=0, f(S_0 x) = (S_0 \dot{-} S_0 x)[S_0 \dot{-} (S_0 \dot{-} S_0 x)] = 0 ,$$

and for $\mu > 0$

$$\begin{aligned} f(S_\mu x) &= P_\mu(1 \dot{-} x)\cdot\{1 \dot{-} P_\mu(1 \dot{-} x)\} \\ &= (1 \dot{-} x)\{1 \dot{-} (1 \dot{-} x)\} = f(x), \quad \text{by 10.2} . \end{aligned}$$

11. Another approach to the logical constants is also possible and, perhaps, more interesting.

As is well known, Heyting's propositional calculus is based upon eleven axioms (compare Heyting [2]):

A.1. $$p \supset p \wedge p ,$$

A.2. $$p \wedge q \supset q \wedge p ,$$

A.3. $$(p \supset q) \supset \big((p \wedge r) \supset (q \wedge r)\big) ,$$

A.4. $$\big((p \supset q) \wedge (q \supset r)\big) \supset (p \supset r) ,$$

A.5. $$q \supset (p \supset q) ,$$

A.6. $$\big(p \wedge (p \supset q)\big) \supset q ,$$

A.7. $$p \supset (p \vee q) ,$$

A.8. $$(p \vee q) \supset (q \vee p) ,$$

A.9. $$\big((p \supset r) \wedge (q \supset r)\big) \supset \big((p \vee q) \supset r\big) ,$$

A.10. $$\sim p \supset (p \supset q) ,$$

A.11. $$\big((p \supset q) \wedge (p \supset \sim q)\big) \supset \sim p ,$$

and upon modus ponens and the rule of substitution.

Let us call the *weak Heyting calculus* the calculus in which all the above axioms and rules are valid, except axiom A.11.

We note that it is possible to prove equations (10.2), (10.3) and (10.4) directly, without using the technique of section 10.

Let us now say that an equation $a=b$ is a true proposition if and only if the equation $|a, b|=0$ is provable, and a false proposition if and only if the equation $1 \dot{-} |a, b|=0$ is provable. Then a proposition may be neither true nor false (if, for instance, $|a, b|=S_1$).

Let p be the proposition $a=b$ and q the proposition $c=d$. Define

$p \wedge q$	to be	$\|a, b\|+\|c, d\| = 0$,
$p \vee q$	to be	$\|a, b\| \cdot \|c, d\| = 0$,
$p \supset q$	to be	$[1 \dot{-} \alpha(\|a, b\|)] \cdot \|c, d\| = 0$,
$\sim p$	to be	$1 \dot{-} \|a, b\| = 0$,
$\sim (p \wedge q)$	to be	$1 \dot{-} \{\|a, b\|+\|c, d\|\} = 0$,
$\sim (p \vee q)$	to be	$1 \dot{-} \|a, b\| \cdot \|c, d\| = 0$,
$\sim (p \supset q)$	to be	$1 \dot{-} [1 \dot{-} \alpha(\|a, b\|)] \cdot \|c, d\| = 0$,

and so on.

Then, because $x \cdot (1 \dot{-} x)=0$ does not hold, the law $p \vee \sim p$ of excluded middle does not hold, but the law $\sim p \vee \sim \sim p$ does hold (by (10.4)).

Also the implication $p \supset \sim \sim p$ does hold, but not $\sim \sim p \supset p$.

We can show that all the propositions A.1–A.10. are true propositions, but not A.11. Also modus ponens holds.

So, we have: the logical calculus introduced by this interpretation is the weak Heyting logical calculus (more exactly, a larger calculus without A. 11.).

We shall develop this approach in more detail in another article.

12. The preceding account will perhaps suffice to give a picture of partially ordered recursive arithmetics. They are a direct generalisation of recursive number theory.

Naturally, many questions remain open. For instance, we are not able to define analogues of the functions $\Sigma_f(x)$ and $\Pi_f(x)$ and so lack the limited quantifiers $A_x^n(f(x)=0)$ and $E_x^n(f(x)=0)$ (in the notation of Goodstein [1]). So we can form only propositions with free variables.

We note that it is possible to develop an effective algorithm for calculation in our arithmetics, the number n of successors being finite.

The author expresses his acknowledgment to R. L. Goodstein for helpful criticism.

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