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ON THE ASYMPTOTIC DISTRIBUTION OF LINEAR COMBINATIONS OF INTERCHANGEABLE RANDOM VARIABLES

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1. Summary. Following Chernoff & Teicher [4], we call a sequence of random variables interchangeable if any finite subsequence has a joint distribution function which is symmetric with respect to its arguments.

After proving a lemma in Section 2, we shall demonstrate in Section 3 that, under certain conditions, a linear combination of interchangeable random variables is asymptotically normally distributed. Applications will be made in Section 4 to linear combinations of intervals obtained by random division of the unit interval. Blom [3] has shown that such combinations play an important rôle in the theory of order statistics. In Section 5 it is demonstrated that some previously known results concerning permutation variables may be obtained as special cases of the theory developed in this paper. In the last section, a central limit theorem due to Chernoff & Teicher [4] is generalized so as to apply to linear combinations of interchangeable variables.

2. A lemma. The following lemma will be used in the sequel. A special case is contained in a paper by Noether [9].

LEMMA. *Let $\nu_1, \ldots, \nu_m$ be given positive integers with the sum r, and let $\{c_{in}\}$, $(i=1, 2, \ldots, k_n;\ k_n \to \infty$ when $n \to \infty)$, be a double sequence of numbers such that for some $\alpha \geqq 0$*

$$(1)\qquad \sum_{i=1}^{k_n} c_{in}{}^{\varrho} = \begin{cases} 0 & \text{for } \varrho = 1\,, \\ 1 & \text{for } \varrho = 2\,, \\ o(n^{\alpha(1-\frac{1}{2}\varrho)}) & \text{for } \varrho = 3, 4, \ldots, r\,. \end{cases}$$

Further, set

$$s^{(n)}_{\nu_1 \ldots \nu_m} = \sum c_{j_1}{}^{\nu_1} c_{j_2}{}^{\nu_2} \ldots c_{j_m}{}^{\nu_m}\,,$$

where the sum contains all different terms which can be formed by taking subsequences $j_1, \ldots, j_m$ from the sequence $1, \ldots, k_n$ (the second subscript, n, of the c's is omitted). Then

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$$s^{(n)}_{\nu_1 \ldots \nu_m} \begin{cases} = O(1) & \text{for } m > \tfrac{1}{2}r , \\ \sim 1/(\tfrac{1}{2}r)! & \text{for } \nu_1 = \ldots = \nu_m = 2 , \\ = o(n^{\alpha(m-\frac{1}{2}r)}) & \text{otherwise} . \end{cases} \tag{2}$$

Before proving the lemma, we note that the number of terms in $s^{(n)}_{\nu_1 \ldots \nu_m}$ depends on the choice of the exponents ν_i. For example, if the ν_i's are all different, the number of terms is

$$k_n(k_n-1) \ldots (k_n-m+1) ,$$

and if they are all equal, the number is $\binom{k_n}{m}$.

In order to prove the lemma, rewrite $s^{(n)}_{\nu_1 \ldots \nu_m}$ as a sum of a finite number of products of the power sums

$$S_\varrho = \sum_{i=1}^{k_n} c_{in}{}^\varrho .$$

Since $s^{(n)}_{\nu_1 \ldots \nu_m}$ is symmetric with respect to the quantities c_{in}, this is always possible. We then obtain

$$s^{(n)}_{\nu_1 \ldots \nu_m} = \sum d_{\lambda_1 \ldots \lambda_\mu} S_{\lambda_1} S_{\lambda_2} \ldots S_{\lambda_\mu} , \tag{3}$$

where several of the λ_i's may be equal. In each term in this sum

$$\lambda_1 + \lambda_2 + \ldots + \lambda_\mu = r \tag{4}$$

is a partition of r which can be obtained from the partition

$$\nu_1 + \nu_2 + \ldots + \nu_m = r$$

by forming partial sums of the ν_i's in a suitable way. Evidently,

$$\mu \leqq m . \tag{5}$$

Note that, since $S_1 = 0$ by assumption, each $\lambda_i \geqq 2$.

The term in (3) for which $\lambda_1 = \ldots = \lambda_\mu = 2$ is of special importance. For brevity we call it the S_2-term. As follows from the above description of the terms in (3), the S_2-term appears only when r is even and the ν_i's are either 1 or 2. Thus, in this special case

$$\begin{aligned} m &= \tfrac{1}{2}r \quad \text{for} \quad \nu_1 = \ldots = \nu_m = 2 , \\ m &> \tfrac{1}{2}r \quad \text{otherwise} . \end{aligned} \tag{6}$$

The coefficient $d_{2 \ldots 2}$ of the S_2-term can easily be determined. For the present proof it is sufficient to observe that

$$d_{2 \ldots 2} = 1/m! \quad \text{for} \quad \nu_1 = \ldots = \nu_m = 2 . \tag{7}$$

This follows e.g. from the fact that $1/d_{2\ldots 2}$ is the coefficient of $s^{(n)}_{2\ldots 2} = \Sigma c_{j_1}{}^2 c_{j_2}{}^2 \ldots c_{j_m}{}^2$ in the expansion of $(c_1{}^2 + \ldots + c_{k_n}{}^2)^m$, which coefficient is obviously $m!$.

There are two main situations depending upon whether r is odd or even.

(a) r is odd. In each term in (3), one at least of the indices $\lambda_1, \ldots, \lambda_\mu$ must be greater than 2. Hence by (1) any such term is $o(n^\beta)$, where

$$\beta = \alpha\mu - \tfrac{1}{2}\alpha(\lambda_1 + \ldots + \lambda_\mu)$$

is negative or at most zero. Moreover, by (4) and (5)

$$\beta \leqq \alpha(m - \tfrac{1}{2}r),$$

and thus

$$s^{(n)}_{\nu_1 \ldots \nu_m} = \begin{cases} o(1) & \text{for } m > \tfrac{1}{2}r, \\ o(n^{\alpha(m-\frac{1}{2}r)}) & \text{for } m \leqq \tfrac{1}{2}r. \end{cases} \tag{8}$$

(b) r is even. We have to distinguish between two subcases:

(i) The S_2-term does not appear in (3). Then (8) holds.

(ii) The S_2-term appears in (3). By (6) we have either $m > \tfrac{1}{2}r$ or $\nu_1 = \ldots = \nu_m = 2$. By the argument leading up to (8) we infer that all terms in (3) except the S_2-term are $o(1)$. Hence, since $S_2 = 1$ by assumption, we find

$$s^{(n)}_{\nu_1 \ldots \nu_m} \sim d_{2\ldots 2},$$

which by (7) becomes $1/m! = 1/(\tfrac{1}{2}r)!$ when $\nu_1 = \ldots = \nu_m = 2$. Summing up all these results, we obtain (2), and the lemma is proved.

It might be noted that we have proved a little more than the lemma states. In fact, when r is odd, $O(1)$ in the first line of (2) may be replaced by $o(1)$.

We also note that, if o is replaced by O in the last line of (1), then o should be replaced by O also in the last line of (2). This modification of the lemma will be used in Section 5.

Finally, it might be added that the third part of condition (1) is often unnecessarily restrictive. In fact, the proof is still valid if this condition is fulfilled only for those ϱ among the integers $3, 4, \ldots, r$ which correspond to S_ϱ's appearing in (3). Evidently, no detailed description of which ϱ's should be included in the condition can be given, since the answer depends upon the choice of exponents $\nu_1, \ldots, \nu_m$ in $s^{(n)}_{\nu_1 \ldots \nu_m}$.

3. Main theorem. Let

$$x_{1n}, \ldots, x_{k_n n} \qquad (n = 1, 2, \ldots; k_n \to \infty)$$

be a double sequence of random variables. For any fixed n the variables x_{in} are assumed to be interchangeable. Further, let

$$h_{1n}, \ldots, h_{k_n n}$$

be a double sequence of real numbers. Denote the means

$$\frac{1}{k_n} \sum_{i=1}^{k_n} h_{in} \quad \text{by} \quad \bar{h}_n \qquad (n = 1, 2, \ldots) .$$

In what follows the index n in x_{in}, h_{in} and $\bar{h}_n$ will sometimes be dropped, but it should always be borne in mind that these quantities may depend upon n.

Several fundamental properties of interchangeable random variables have been derived by Andersen [1] [2] and Chernoff & Teicher [4]. An important consequence of the definition is that, for any given n, the variables x_i have the same marginal distribution. We shall suppose that their common mean μ and variance μ_2 exist and are finite for any n (note that μ and μ_2 may be functions of n, which may or may not remain bounded when n tends to infinity).

More generally, we shall assume that, for any $n = 1, 2, \ldots$, the mixed central moment

$$\mu_{\nu_1 \ldots \nu_m} = E[(x_{j_1} - \mu)^{\nu_1} (x_{j_2} - \mu)^{\nu_2} \ldots (x_{j_m} - \mu)^{\nu_m}]$$

exists for any positive integers $\nu_1, \ldots, \nu_m$ and any $m \leqq k_n$. Any such moment has the same value irrespective of which m variables are considered among $x_1, \ldots, x_{k_n}$.

We shall investigate the asymptotic behaviour of a linear combination of the x_i's with the h_i's as coefficients. The discussion will be limited to the following two situations. Either the sum of the coefficients is zero for any n, or the sum of the variables is non-random for any n (or both).

Let as usual $E(\ )$ and $\mathrm{var}(\ )$ denote the mean and variance, respectively, of the random variable within parentheses.

We shall prove the following theorem.

THEOREM 1. *Let*

$$T_n = \sum_{i=1}^{k_n} h_{in} x_{in}$$

be a linear combination of interchangeable random variables with finite moments of all orders (and a variance which remains bounded away from zero when $n \to \infty$). Suppose that, for any $n = 1, 2, \ldots$, either

(A 1) $$\sum_{i=1}^{k_n} h_{in} = 0$$

or

(A 2) $$\sum_{i=1}^{k_n} x_{in} = C_n ,$$

where the C_n are given non-random quantities. If a quantity $\alpha \geqq 0$ can be found such that

(B) $$\frac{\sum_{i=1}^{k_n} (h_{in} - \bar{h}_n)^r}{\left[\sum_{i=1}^{k_n} (h_{in} - \bar{h}_n)^2\right]^{\frac{1}{2}r}} = o(n^{\alpha(1-\frac{1}{2}r)}) \quad \text{for} \quad r = 3, 4, \ldots ,$$

and if, for any given positive integers $\nu_1, \ldots, \nu_m$ with sum $r \geqq 2$,

(C) $$\frac{\mu_{\nu_1 \ldots \nu_m}}{\mu_2^{\frac{1}{2}r}} \begin{cases} = o(1) & \text{for} \quad m > \frac{1}{2}r \\ \sim 1 & \text{for} \quad \nu_1 = \ldots = \nu_m = 2 \\ = O(n^{\alpha(\frac{1}{2}r-m)}) & \text{otherwise,} \end{cases}$$

then

$$T_n^0 = [T_n - E(T_n)]/[\text{var}(T_n)]^{\frac{1}{2}}$$

is asymptotically normally distributed with mean 0 *and variance* 1.

Before proving the theorem, we note that condition (C) implies *inter alia* that the correlation coefficient μ_{11}/μ_2 of any two variables tends to zero when n tends to infinity.

We may without loss of generality assume that condition (A 1) always holds, and hence that $T_n^0 = T_n/[\text{var}(T_n)]^{\frac{1}{2}}$. For if condition (A 2) holds good, $T_n - E(T_n)$ remains unchanged if h_i is replaced by $h_i - \bar{h}$.

We shall begin the proof by determining the variance of T_n. We have

$$\begin{aligned} \text{var}(T_n) = E(T_n^2) &= \mu_2 \sum h_i^2 + \mu_{11} \sum_{i \neq j} h_i h_j \\ &= \mu_2 \sum h_i^2 + \mu_{11} \left[(\sum h_i)^2 - \sum h_i^2 \right] \\ &= (\mu_2 - \mu_{11}) \sum h_i^2 . \end{aligned}$$

But, as said above, $\mu_{11}/\mu_2 \to 0$ when $n \to \infty$, and hence

$$\text{var}(T_n) \sim \mu_2 \sum h_i^2 .$$

Accordingly, the rth moment of T_n^0 around its mean satisfies

(9) $$E(T_n^0)^r \sim \frac{E[\sum h_i(x_i - \mu)]^r}{\mu_2^{\frac{1}{2}r} (\sum h_i^2)^{\frac{1}{2}r}} .$$

The numerator can be written

$$E\left[\sum_{i=1}^{k_n} h_i(x_i-\mu)\right]^r \tag{10}$$
$$= \sum \frac{r!}{\nu_1! \ldots \nu_m!} \sum{}' h_{j_1}^{\nu_1} \ldots h_{j_m}^{\nu_m} E[(x_{j_1}-\mu)^{\nu_1} \ldots (x_{j_m}-\mu)^{\nu_m}],$$

where Σ denotes summation over all distinct unordered sets of positive integers $\nu_1, \ldots, \nu_m$ with sum r, and Σ' contains all different terms which can be formed by taking subsequences $j_1, \ldots, j_m$ from $1, 2, \ldots, k_n$.

Since the variables are interchangeable, the factor $E[\ ]$ does not depend upon the particular variables chosen, and hence can be placed before the second summation sign. Using (9), we obtain

$$E(T_n^0)^r \sim \sum \frac{r!}{\nu_1! \ldots \nu_m!} \cdot \frac{\mu_{\nu_1 \ldots \nu_m}}{\mu_2^{\frac{1}{2}r}} \cdot H_{\nu_1 \ldots \nu_m}, \tag{11}$$

where

$$H_{\nu_1 \ldots \nu_m} = \frac{\sum' h_{j_1}^{\nu_1} \ldots h_{j_m}^{\nu_m}}{(\sum h_i^2)^{\frac{1}{2}r}}.$$

We now apply the lemma in Section 2 with

$$c_i = h_i/(\sum h_i^2)^{\frac{1}{2}}.$$

It follows from conditions (A 1) and (B) of the theorem that condition (1) of the lemma is fulfilled. Furthermore, $s^{(n)}_{\nu_1 \ldots \nu_m}$ is specialized to $H_{\nu_1 \ldots \nu_m}$, and we infer from the lemma that

$$H_{\nu_1 \ldots \nu_m} \begin{cases} = O(1) & \text{for} \quad m > \frac{1}{2}r, \\ \sim 1/(\frac{1}{2}r)! & \text{for} \quad \nu_1 = \ldots = \nu_m = 2, \\ = o(n^{\alpha(m-\frac{1}{2}r)}) & \text{otherwise.} \end{cases}$$

Combining this result with condition (C), we obtain

$$\frac{\mu_{\nu_1 \ldots \nu_m}}{\mu_2^{\frac{1}{2}r}} \cdot H_{\nu_1 \ldots \nu_m} \begin{cases} \sim 1/(\frac{1}{2}r)! & \text{for } \nu_1 = \ldots = \nu_m = 2, \\ = o(1) & \text{otherwise.} \end{cases}$$

Consequently, by (11) any odd moment of T_n^0 tends to zero as $n \to \infty$. Further, when r is even,

$$E(T_n^0)^r \to \frac{r!}{2^{\frac{1}{2}r}(\frac{1}{2}r)!} = (r-1)(r-3) \ldots 1.$$

Thus T_n^0 has in the limit the same moments as a standardized normal variable. Since the normal distribution is uniquely determined by its moments, T_n^0 is asymptotically normally distributed (cf. Cramér [5, p. 176]). This proves the theorem.

REMARK 1. *The conclusion of the theorem remains true if the symbols o and O appearing in condition* (B) *and in the last line of condition* (C), *respectively, are interchanged.*

When proving this statement, we use the modification of the lemma mentioned in the last but one paragraph of Section 2. In all other respects the proof is unchanged.

REMARK 2. *The conclusion of the theorem remains true if, in the right member of conditions* (B) *and* (C), *n is replaced by any function l_n of n such that $l_n \to \infty$ when $n \to \infty$.*

This is self-evident.

REMARK 3. *Suppose that the marginal distribution of x_{in} tends to a limiting distribution with finite central moments $\mu_{\nu\infty}$ of all orders and positive variance, and, furthermore, that any mixed central moment $\mu_{\nu_1 \ldots \nu_m}$ of the variables x_{in} tends to the product $\mu_{\nu_1\infty}\mu_{\nu_2\infty}\cdots\mu_{\nu_m\infty}$. If, in addition, condition* (B) *is satisfied for $\alpha = 0$, then the conclusion of the theorem holds true.*

The truth of this remark is seen as follows. When $n \to \infty$ we have by assumption

$$\frac{\mu_{\nu_1 \ldots \nu_m}}{\mu_2^{\frac{1}{2}r}} \to \frac{\mu_{\nu_1\infty} \cdots \mu_{\nu_m\infty}}{\mu_{2\infty}^{\frac{1}{2}r}} .$$

Since $\mu_{\nu_i\infty} = O(1)$ and $\mu_{1\infty} = 0$, it follows that condition (C) is satisfied for $\alpha = 0$.

REMARK 4. *When Theorem* 1 *is used in the special case when $\alpha = 0$, a sufficient condition for the validity of condition* (B) *is that*

$$\frac{\sum_{i=1}^{k_n} |h_{in} - \bar{h}_n|^3}{\left[\sum_{i=1}^{k_n} (h_{in} - \bar{h}_n)^2\right]^{3/2}} = o(1) . \tag{12}$$

The truth of this assertion is easily seen.

4. Application to ordered uniformly distributed variables. Divide the unit interval (0, 1) into $n+1$ parts by taking n points at random in the interval. Denote the lengths of the parts by $\delta_1, \ldots, \delta_{n+1}$. Consider the linear combination

$$Z_n = n \sum_{i=1}^{n+1} h_i\big(\delta_i - (n+1)^{-1}\big) .$$

We shall prove the following result, which, in a slightly different form, was used (but not conclusively proved) by Blom [3, pp. 96 and 175]:

If

$$\frac{\sum_{i=1}^{n+1} (h_{in} - \bar{h}_n)^r}{\left[\sum_{i=1}^{n+1} (h_{in} - \bar{h}_n)^2\right]^{\frac{1}{2}r}} = o(1) \quad \textit{for} \quad r = 3, 4, \ldots ,$$

then Z_n *is asymptotically normally distributed with mean* 0 *and variance* $\sum_{i=1}^{n+1}(h_{in} - \bar{h}_n)^2$.

To prove this proposition, we observe first that the δ_i's are interchangeable and have a non-random sum. Their distribution has been studied by Blom [loc. cit., p. 40ff.], among others. We have

$$\begin{aligned} E(\delta_i) &= (n+1)^{-1}; \\ \operatorname{var}(\delta_i) &= n(n+1)^{-2}(n+2)^{-1}; \\ \operatorname{cov}(\delta_i, \delta_j) &= -(n+1)^{-2}(n+2)^{-1} \qquad (i \neq j) . \end{aligned}$$

Hence

$$\begin{aligned} E(Z_n) &= 0; \\ \operatorname{var}(Z_n) &= n^2(n+1)^{-1}(n+2)^{-1} \sum_{i=1}^{n+1} (h_{in} - \bar{h}_n)^2 \sim \sum_{i=1}^{n+1} (h_{in} - \bar{h}_n)^2 , \end{aligned}$$

as stated.

We now apply Theorem 1, Remark 3, with $x_{in} = n\delta_i$, $k_n = n+1$ and $\alpha = 0$. Condition (B) with $\alpha = 0$ is fulfilled by assumption. Furthermore, any variable δ_i has the frequency function $n(1-\delta_i)^{n-1}$. Hence, in the limit, x_{in} tends to a random variable x with the frequency function e^{-x} and central moments $\mu_{\nu\infty}$, where e.g. $\mu_{2\infty} = 1$ (cf. Blom, loc. cit., p. 59).

More generally, any m intervals, for example $\delta_1, \ldots, \delta_m$, have the joint frequency function

$$n(n-1) \ldots (n-m+1)(1-\delta_1 - \ldots - \delta_m)^{n-m} .$$

Hence any mixed moment around zero is given by

$$\begin{aligned} E(\delta_1^{\nu_1}\delta_2^{\nu_2} \ldots \delta_m^{\nu_m}) &= n(n-1) \ldots (n-m+1) \cdot \\ &\cdot \int \ldots \int \delta_1^{\nu_1} \ldots \delta_m^{\nu_m}(1-\delta_1 - \ldots - \delta_m)^{n-m} d\delta_1 \ldots d\delta_m \qquad (13) \\ &= \frac{n!\, \nu_1! \ldots \nu_m!}{(n + \sum \nu_i)!} . \end{aligned}$$

It follows that

$$E(x_{1n}^{\nu_1} \dots x_{mn}^{\nu_m}) \to \nu_1! \dots \nu_m!$$

when $n \to \infty$. Since $\nu_i!$ is the ν_ith moment around zero of the exponential variable x, this implies that we can write

$$E(x_{1n}^{\nu_1} \dots x_{mn}^{\nu_m}) \to E(x^{\nu_1}) \dots E(x^{\nu_m}) .$$

Finally, it follows from this relation that also the mixed central moment $\mu_{\nu_1 \dots \nu_m}$ of the variables x_{in} tends to a corresponding product $\mu_{\nu_1 \infty} \dots \mu_{\nu_m \infty}$ of the central moments of the limiting variable.

It is seen from all these considerations that we are entitled to apply Remark 3 following Theorem 1, and the proposition is proved.

5. Application to permutation variables. Linear combinations of permutation variables have been investigated by many authors, e.g. by Hotelling & Pabst [8], Wald & Wolfowitz [10], Noether [9], and Hoeffding [7]. A survey of the results obtained by these and other authors is given by Fraser [6, p. 235ff.]. We shall show that Noether's results may be obtained as a special case of Theorem 1.

Let the variables $x_1, \dots, x_n$ be generated by the $n!$ equally likely permutations of the given numbers $a_1, \dots, a_n$. The variables x_i are evidently interchangeable and have a non-random sum. Apply Theorem 1 with $k_n = n$ after interchanging the symbols o and O in condition (B) and the third part of condition (C) (which is allowed by Remark 1 to the theorem).

Condition (B) then assumes the form

$$\frac{\sum_{i=1}^{n} (h_i - \bar{h})^r}{\left[\sum_{i=1}^{n} (h_i - \bar{h})^2\right]^{\frac{1}{2}r}} = O(n^{\alpha(1-\frac{1}{2}r)}) \quad \text{for} \quad r = 3, 4, \dots , \tag{14}$$

which is a generalization of the condition W used by Noether. Further, the modified condition (C) holds for $0 \leqq \alpha \leqq 1$ if and only if

$$\frac{\sum_{i=1}^{n} (a_i - \bar{a})^r}{\left[\sum_{i=1}^{n} (a_i - \bar{a})^2\right]^{\frac{1}{2}r}} = o(n^{(1-\alpha)(1-\frac{1}{2}r)}) \quad \text{for} \quad r = 3, 4, \dots , \tag{15}$$

which is the corresponding generalization of the second condition introduced by Noether. To prove the last statement, take $m = 1$, $\nu_1 = r$, and $r = 3, 4, \dots$ in the third part of the modified condition (C). Since

$$\mu_r = \frac{1}{n} \sum_{i=1}^{n} (a_i - \bar{a})^r ,$$

we obtain

$$\frac{\frac{1}{n} \sum (a_i - \bar{a})^r}{\left[\frac{1}{n} \sum (a_i - \bar{a})^2\right]^{\frac{1}{2}r}} = o(n^{\alpha(\frac{1}{2}r-1)}) \quad \text{for} \quad r = 3, 4, \ldots ,$$

which is identical with (15).

Conversely, we shall demonstrate that if (15) holds good, then condition (C) is satisfied (with the modification referred to above). For this purpose, set

$$c_i = \frac{a_i - \bar{a}}{\sum (a_i - \bar{a})^2} . \tag{16}$$

Evidently

$$\frac{\mu_{\nu_1 \ldots \nu_m}}{\mu_2^{\frac{1}{2}r}} = \frac{n^{\frac{1}{2}r}}{\binom{n}{m}} \cdot \sum{}' c_{j_1}^{\ \nu_1} \ldots c_{j_m}^{\ \nu_m} \sim m!\, n^{\frac{1}{2}r-m} s^{(n)}_{\nu_1 \ldots \nu_m} . \tag{17}$$

Now apply the lemma in Section 2 with c_i given by (16) and α replaced by $1-\alpha$. Since $\alpha \leqq 1$, we know that $1-\alpha$ is non-negative as required in the lemma. Combining (2) with (17), we see that condition (C) is satisfied (with o replaced by O as described above).

Summing up, we have proved the following result, which is somewhat more general than Noether's theorem given in [9]:

If the coefficients h_i and the numbers a_i satisfy (14) *and* (15), *respectively, with* $0 \leqq \alpha \leqq 1$, *then* $\sum_{i=1}^{n} h_i x_i$ *is asymptotically normally distributed.*

A still more general result was obtained by Hoeffding [7, Theorem 4].

6. A further theorem. We shall prove a theorem valid for interchangeable random variables with a non-random sum, which is closely related to a result due to Chernoff & Teicher [4]. This theorem may sometimes be more convenient to apply than Theorem 1.

Without loss of generality it may be assumed that, for any $n = 1, 2, \ldots$,

$$\sum_{i=1}^{k_n} x_{in} = 0; \quad \operatorname{var}(x_{in}) = 1 \qquad (i = 1, \ldots, k_n) . \tag{18}$$

By "$x_n = o(n)$ in probability" we mean that x_n/n tends to zero in probability.

THEOREM 2. *Let*

$$T_n = \sum_{i=1}^{k_n} h_{in} x_{in}$$

be a linear combination of interchangeable variables which satisfy (18). *If, for some* α *in the interval* $0 \leqq \alpha \leqq 1$,

$$\frac{\sum_{i=1}^{k_n} (h_{in} - \bar{h}_n)^r}{\left[\sum_{i=1}^{k_n} (h_{in} - \bar{h}_n)^2\right]^{\frac{1}{2}r}} = O(k_n^{\alpha(1-\frac{1}{2}r)}) \quad \textit{for} \quad r = 3, 4, \ldots ,$$

and if the relations

$$\max_{1 \leqq i \leqq k_n} |x_{in}| = o(k_n^{\frac{1}{2}\alpha}) \quad \textit{and} \quad \frac{1}{k_n} \sum_{i=1}^{k_n} x_{in}^2 \to 1$$

hold in probability, then $T_n/[\mathrm{var}(T_n)]^{\frac{1}{2}}$ *is asymptotically normally distributed with mean* 0 *and variance* 1.

The theorem can be proved by a slight extension of the method used in [4]. We shall give only a few hints of the proof. Without any essential loss of generality it may be assumed that $k_n = n$.

First, replace the variables $x_1, \ldots, x_n$ by *fixed* numbers $a_1, \ldots, a_n$ which satisfy the relations

$$(19) \qquad \sum_{i=1}^{n} a_i = 0, \qquad \max_{1 \leqq i \leqq n} |a_i| = o(n^{\frac{1}{2}\alpha}), \qquad \frac{1}{n} \sum_{i=1}^{n} a_i^2 \to 1 .$$

Consider the random variable

$$T_n' = \sum_{i=1}^{n} h_i y_i$$

generated by the $n!$ equally likely permutations of the numbers a_i. Now apply Noether's theorem in the generalized form given in Section 5. Condition (14) is satisfied by assumption. Condition (15) also holds, since by (19) for $r = 3, 4, \ldots$

$$\frac{|\sum a_i^r|}{(\sum a_i^2)^{\frac{1}{2}r}} \leqq \frac{\max_{1 \leqq i \leqq n} |a_i|^{r-2}}{(\sum a_i^2)^{\frac{1}{2}r-1}} = o(n^{(1-\alpha)(1-\frac{1}{2}r)}) .$$

Consequently, after suitable standardization, T_n' is asymptotically normally distributed with mean 0 and variance 1.

Secondly, it is proved exactly in the same way as in [4, pp. 122–123] that T_n has the same limiting distribution as T_n'. This proves Theorem 2.

Undoubtedly, Theorem 2 can be given a more general formulation, for instance by using Hoeffding's result referred to at the end of Section 5. However, it seems that the present result is general enough for several purposes.

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