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ON SOME EXTENSIONS OF THEOREMS OF FEJÉR

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1. Let

$$\Omega = (\omega_{n\nu}), \qquad n = 0, 1, 2, \ldots; 0 \leqq \nu \leqq n ,$$

be a triangular Toeplitz matrix of real or complex numbers, i.e., it satisfies the following three conditions:

(i) $$\omega_{n\nu} \to 0 \qquad \text{as} \quad n \to \infty$$

for every fixed ν;

(ii) $$\sum_{\nu=0}^{n} \omega_{n\nu} \to 1 \qquad \text{as} \quad n \to \infty;$$

and

(iii) $$\sum_{\nu=0}^{n} |\omega_{n\nu}| < M ,$$

where M is an absolute constant.

Given a sequence (s_n) or a series with partial sums s_n, we say that the sequence (s_n) or the series with partial sums s_n is summable (Ω) to the sum s, if

$$\sum_{\nu=0}^{n} \omega_{n\nu} s_\nu \to s \qquad \text{as} \quad n \to \infty .$$

2. Suppose that $f(x)$ is a Lebesgue integrable function and periodic with period 2π. Let

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series, and

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) = \sum B_n(x)$$

its derived series. Fejér [1] has established the following theorem.

THEOREM A. *If $f(x)$ is of bounded variation in $(0, 2\pi)$, then $(B_n(x))$ is summable (C, r) to $l(x) = \{f(x+0) - f(x-0)\}/\pi$ for every $r > 0$.*

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In this note, we shall extend the above result to a class of triangular Toeplitz matrices, which contains the (C, r) matrix as a special case.

THEOREM 1. *Let* (Ω) *be a triangular Toeplitz matrix which satisfies, in addition, the condition*

$$\text{(iv)} \qquad \sum_{\nu=0}^{n} |\Delta^2 \omega_{n\nu}| = o(1) \qquad (n \to \infty) ,$$

where

$$\Delta^2 \omega_{n\nu} = \Delta\omega_{n\nu} - \Delta\omega_{n(\nu+1)}, \qquad \Delta\omega_{n\nu} = \omega_{n\nu} - \omega_{n(\nu+1)} .$$

If $f(x)$ *is of bounded variation in* $(0, 2\pi)$, *then* $(B_n(x))$ *is summable* (Ω) *to the sum* $l(x)$.

There exists a large class of triangular Toeplitz matrices satisfying (iv); e.g., the regular Nörlund summability matrix $N(p_n)$, with $p_n \leqq p_{n+1}$ or $p_n \geqq p_{n+1}$ from a certain $n = n_0$ onwards, obviously satisfies (iv). If we take, for the Cesàro means of order α,

$$p_n = A_n^{\alpha-1} = \Gamma(n+\alpha)/\Gamma(n+1)\Gamma(\alpha) \qquad (\alpha > 0) ,$$

which is known to be monotone for $\alpha > 0$, then we see that our theorem contains Fejér's result as a special case. Moreover, if we take $p_n = 1/(n+1)$ for the harmonic means, then it follows from a known result of M. Riesz [3], that *a sequence which is summable by the harmonic means is also summable* (C, r) *for every* $r > 0$, *but not conversely*. Thus the present theorem is better than Fejér's.

3. The proof of the theorem is straightforward. If we write

$$\psi_x(t) = f(x+t) - f(x-t) ,$$

then

$$\begin{aligned} \sigma_n(x) &= \sum_{\nu=0}^{n} \omega_{n\nu} B_\nu(x) \\ &= \pi^{-1} \sum_{\nu=0}^{n} \omega_{n\nu} \int_0^\pi \psi_x(t)\, \nu \sin \nu t \, dt \\ &= l(x) \sum_{\nu=0}^{n} \omega_{n\nu} + \pi^{-1} \sum_{\nu=0}^{n} \omega_{n\nu} \int_0^\pi \cos \nu t \, d\psi_x(t) \\ &= l(x) + o(1) + \pi^{-1} \sum_{\nu=0}^{n} \omega_{n\nu} I_\nu \end{aligned}$$

by (ii). The function $\psi_x(t)$ is continuous at $t = 0$ and of bounded variation in $(0, \pi)$. Hence, for a given $\varepsilon > 0$, we can choose a positive number $\delta = \delta(\varepsilon)$ such that

$$\int_0^\delta |d\psi_x(t)| < \varepsilon .$$

Thus splitting

$$I_\nu = \int_0^\delta + \int_\delta^\pi = I_{1\nu} + I_{2\nu} ,$$

we obtain

$$\left| \sum_{\nu=0}^n \omega_{n\nu} I_{1\nu} \right| \leqq \sum_{\nu=0}^n |\omega_{n\nu}| \int_0^\delta |d\psi_x(t)| < M\varepsilon$$

by (iii). On summing $\Sigma_{\nu=0}^n \omega_{n\nu} I_{2\nu}$ by parts twice, we see that

$$\begin{aligned} \left| \sum_{\nu=0}^n \omega_{n\nu} I_{2\nu} \right| &= \left| \int_\delta^\pi \left\{ \sum_{\nu=0}^n \Delta\omega_{n\nu} \frac{\sin(\nu+\frac{1}{2})t}{2\sin(t/2)} + \tfrac{1}{2}\omega_{n0} \right\} d\psi_x(t) \right| \\ &= 2 \left| \int_\delta^\pi \left\{ \sum_{\nu=0}^n \Delta^2\omega_{n\nu} \left(\frac{\sin((\nu+1)t/2)}{2\sin(t/2)} \right)^2 + \tfrac{1}{4}\omega_{n0} \right\} d\psi_x(t) \right| \\ &\leqq 2 \sum_{\nu=0}^n |\Delta^2\omega_{n\nu}| \int_\delta^\pi \left(\frac{\sin((\nu+1)t/2)}{2\sin(t/2)} \right)^2 |d\psi_x(t)| + \tfrac{1}{2}|\omega_{n0}| \int_\delta^\pi |d\psi_x(t)| \\ &< \frac{V}{2\sin^2(\delta/2)} \sum_{\nu=0}^n |\Delta^2\omega_{n\nu}| + \tfrac{1}{2}V\varepsilon \end{aligned}$$

for $n \geqq n^*(\varepsilon)$, where V is the total variation of $\psi_x(t)$ in $(0, \pi)$, since by (i), $|\omega_{n0}| < \varepsilon$ eventually. It follows from (iv) that

$$\sum_{\nu=0}^n \omega_{n\nu} I_{2\nu} = o(1)$$

as $n \to \infty$. This proves Theorem 1.

We remark that, if we would apply summation by parts only once in the above proof, then

$$\text{(iv}'\text{)} \qquad \sum_{\nu=0}^n |\Delta\omega_{n\nu}| = o(1) \qquad (n \to \infty)$$

would give the same conclusion for summability (Ω) of the sequence $(B_n(x))$. But, in view of the relation

$$\sum_{\nu=0}^n |\Delta^2\omega_{n\nu}| = \sum_{\nu=0}^n |\Delta\omega_{n\nu} - \Delta\omega_{n\nu+1}| \leqq 2\sum_{\nu=0}^n |\Delta\omega_{n\nu}| ,$$

(iv) is plainly a weaker condition than (iv′).

4. The following theorem is also due to Fejér [3].

THEOREM B. *If* $l(x)$ *exists and is finite at* x, *then* $(B_n(x))$ *is summable* (C, r) *to* $l(x)$ *for every* $r > 1$.

This theorem can be extended to Borel's method of summation. A sequence (s_n), or a series with partial sums s_n, is said to be summable by Borel's method of summation, or summable (B) to the sum s if

$$e^{-x} \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu!} s_\nu \to s \qquad \text{as} \qquad x \to \infty .$$

We establish

THEOREM 2. *If* $l(x)$ *exists and is finite at* x, *then* $(B_n(x))$ *is summable* (B) *to* $l(x)$.

This theorem can be derived from Theorem B and the following well-known theorem due to Hardy and Littlewood [2, pp. 2–3, especially Theorem 5], which is stated in the following form:

LEMMA. *A sequence* (S_n), *or a series with partial sums* s_n, *for which* $s_n = o(n^{k-(r-1)/2})$, *cannot be summable* (B) *unless it is summable* $(C, k+r)$, k *and* r *being non-negative integers.*

Since $f(x)$ is integrable, $B_n(x) = o(n)$ by Riemann–Lebesgue's theorem. Moreover, from Theorem B, $(B_n(x))$ is summable $(C, 2)$. Thus, by taking $k, r = 1$ in Hardy–Littlewood's theorem, we establish Theorem 2.

REFERENCES

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3. M. Riesz, *Sur l'équivalence de certaines méthodes de sommation*, Proc. London Math. Soc. (2) 22 (1924), 412–419.

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