MATH. SCAND. 7 (1959), 337–346

# A CONDITION FOR $C$-SUMMABILITY OF NEGATIVE ORDER

A. F. ANDERSEN

**1.** The purpose of this note is to show that the general criterion for convergence of series, stated by Cauchy [6, p. 125], can be extended to a criterion for summability $(C, -\delta)$, where $0 \leqq \delta < 1$.

We use the notations

$$A_q^\alpha = \binom{q+\alpha}{q} \qquad \text{and} \qquad S_n^\alpha(u_\mu) = \sum_{\mu=0}^{n} u_\mu A_{n-\mu}^\alpha .$$

The Cesàro means of the order $-\delta$ belonging to the series $\Sigma u_\mu$ may then be denoted by $S_n^{-\delta}(u_\mu)/A_n^{-\delta}$ or simply by $S_n^{-\delta}/A_n^{-\delta}$, $n=0, 1, 2, \ldots$.

**2.** The result of the investigation is stated in the following

THEOREM 1. *A necessary and sufficient condition that the series* $\Sigma u_\mu$ *should be summable* $(C, -\delta)$, $0 \leqq \delta < 1$, *is that*

$$\sum_{\mu=\nu}^{n} u_\mu A_{n-\mu}^{-\delta} = o_\nu(1) n^{-\delta} . \tag{1}$$

The meaning of this condition is that, corresponding to a given number $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that the inequality

$$\left| \sum_{\mu=\nu}^{n} u_\mu A_{n-\mu}^{-\delta} \right| < \varepsilon n^{-\delta} \tag{1*}$$

is satisfied whenever $\nu > N$ and $n \geqq \nu$.

As in formula (1), the variable to which the symbol $o$ refers will be indicated by a subscript whenever any doubt may arise.

For $\delta = 0$ Theorem 1 coincides with the convergence criterion of Cauchy. Thus, in the proof of the theorem we may assume $0 < \delta < 1$.

*Necessity.* The substance of this part of the theorem is included in the following

Received May 20, 1959.

LEMMA 1. *If* $0<\delta<1$ *and*

$$S_n^{-\delta}(u_\mu) = o(n^{-\delta}), \tag{2}$$

*then corresponding to a given number* $\varepsilon>0$ *there exists a number* $M=M(\varepsilon)$ *such that the inequality*

$$\left|\sum_{\mu=0}^{\nu} u_\mu A_{n-\mu}^{-\delta}\right| < \tfrac{1}{2}\varepsilon A_n^{-\delta}$$

*is satisfied whenever* $\nu>M$ *and* $n\geqq\nu$.

To prove this lemma we apply an Abel transformation (cf. [5, (17) p. 7]) and obtain

$$\sum_{\mu=0}^{\nu} u_\mu A_{n-\mu}^{-\delta} = \sum_{\mu=0}^{\nu} S_\mu^{-\delta}(u_\lambda) \sum_{p=0}^{\nu-\mu} A_p^{\delta-2} A_{n-\mu-p}^{-\delta}.$$

Since $A_p^{\delta-2}<0$ for $p\geqq 1$, $A_q^{-\delta}>0$ for all values of $q$, we have for $n\geqq\nu$

$$\sum_{p=0}^{\nu-\mu} A_p^{\delta-2} A_{n-\mu-p}^{-\delta} \geqq \sum_{p=0}^{n-\mu} A_p^{\delta-2} A_{n-\mu-p}^{-\delta} = A_{n-\mu}^{-1} \geqq 0$$

and consequently

$$\left|\sum_{\mu=0}^{\nu} u_\mu A_{n-\mu}^{-\delta}\right| \leqq \sum_{\mu=0}^{\nu} |S_\mu^{-\delta}| \sum_{p=0}^{\nu-\mu} A_p^{\delta-2} A_{n-\mu-p}^{-\delta}. \tag{3}$$

Let a number $\varepsilon>0$ be given. According to the assumption (2) there exists a number $P$ such that

$$|S_\mu^{-\delta}| < \tfrac{1}{4}\varepsilon A_\mu^{-\delta} \qquad \text{for} \quad \mu > P.$$

Restricting $\nu$ to values *greater than* $P$, we can divide the sum occurring on the right side of (3) into two sums,

$$\sum_{\mu=0}^{P} \quad \text{and} \quad \sum_{\mu=P+1}^{\nu}.$$

For all values of $n$ and $\nu$ under consideration, i.e. for $n\geqq\nu>P$, we find for the second sum

$$\sum_{\mu=P+1}^{\nu} |S_\mu^{-\delta}| \sum_{p=0}^{\nu-\mu} A_p^{\delta-2} A_{n-\mu-p}^{-\delta} \leqq \tfrac{1}{4}\varepsilon \sum_{\mu=0}^{\nu} A_\mu^{-\delta} \sum_{p=0}^{\nu-\mu} A_p^{\delta-2} A_{n-\mu-p}^{-\delta} \tag{4}$$

$$= \tfrac{1}{4}\varepsilon \sum_{\mu=0}^{\nu} A_\mu^{-1} A_{n-\mu}^{-\delta} = \tfrac{1}{4}\varepsilon A_n^{-\delta},$$

where the evaluation of the repeated sum may be verified by the Abel transformation used above.

In order to estimate the sum $\Sigma_{\mu=0}^{P}$ we first notice that the binomial coefficients $A_q^{-\delta}$, $q=0, 1, 2, \ldots$, are *decreasing*, and that the coefficients $A_p^{\delta-2}$ are *negative for* $p>0$. Thus for $0<p\leqq \nu-\mu$ it holds that

$$A_{n-\mu-p}^{-\delta} > A_{n-\mu}^{-\delta},$$
$$A_p^{\delta-2}A_{n-\mu-p}^{-\delta} < A_p^{\delta-2}A_{n-\mu}^{-\delta}$$

so that

$$\sum_{p=0}^{\nu-\mu} A_p^{\delta-2}A_{n-\mu-p}^{-\delta} \leqq \sum_{p=0}^{\nu-\mu} A_p^{\delta-2}A_{n-\mu}^{-\delta} = A_{\nu-\mu}^{\delta-1}A_{n-\mu}^{-\delta} \leqq A_{\nu-P}^{\delta-1}A_{n-P}^{-\delta}.$$

Assuming again that $n\geqq\nu>P$, we have

$$\frac{A_n^{-\delta}}{A_{n-P}^{-\delta}} = \frac{(n-\delta)(n-1-\delta)\ldots(n-P+1-\delta)}{n(n-1)\ldots(n-P+1)} > \left(1-\frac{\delta}{n-P+1}\right)^P > (1-\delta)^P,$$
$$A_{n-P}^{-\delta} < (1-\delta)^{-P}A_n^{-\delta},$$

and hence

$$\sum_{p=0}^{\nu-\mu} A_p^{\delta-2}A_{n-\mu-p}^{-\delta} < (1-\delta)^{-P}A_{\nu-P}^{\delta-1}A_n^{-\delta}.$$

Thus, if $k$ denotes a constant such that $|S_\mu^{-\delta}|<k$ for $0\leqq\mu\leqq P$, it holds that

$$\sum_{\mu=0}^{P} |S_\mu^{-\delta}| \sum_{p=0}^{\nu-\mu} A_p^{\delta-2}A_{n-\mu-p}^{-\delta} < (P+1)k(1-\delta)^{-P}A_{\nu-P}^{\delta-1}A_n^{-\delta}$$

and, since for a fixed value of $P$

$$A_{\nu-P}^{\delta-1}\to 0 \qquad \text{for} \quad \nu\to\infty,$$

it follows that there exists a number $M=M(\varepsilon)>P$ such that

$$\sum_{\mu=0}^{P} |S_\mu^{-\delta}| \sum_{p=0}^{\nu-\mu} A_p^{\delta-2}A_{n-\mu-p}^{-\delta} < \tfrac{1}{4}\varepsilon A_n^{-\delta}, \tag{5}$$

whenever $n\geqq\nu>M$.

The results (4) and (5), combined with (3) holding for $n\geqq\nu$, prove Lemma 1.

If $\sum_{\mu=0}^\infty u_\mu$ is summable $(C, -\delta)$ to the sum $U$, then the series $\sum_{\mu=0}^\infty u'_\mu$, where $u'_0=u_0-U$ and $u'_\mu=u_\mu$ for $\mu\geqq 1$, is summable $(C, -\delta)$ to the sum 0. Since for $\nu\geqq 1$

$$\sum_{\mu=\nu}^{n} u_\mu A_{n-\mu}^{-\delta} = \sum_{\mu=\nu}^{n} u'_\mu A_{n-\mu}^{-\delta} = \sum_{\mu=0}^{n} - \sum_{\mu=0}^{\nu-1},$$

we obtain, by application of Lemma 1, that

$$\left|\sum_{\mu=\nu}^{n} u_\mu A_{n-\mu}^{-\delta}\right| < \tfrac{1}{2}\varepsilon A_n^{-\delta} + \tfrac{1}{2}\varepsilon A_n^{-\delta} = \varepsilon A_n^{-\delta}$$

whenever $n\geqq\nu>M+1$, and this result is equivalent to (1*).

This completes the proof of the necessity part of Theorem 1, stating that the condition (1) is fulfilled if the series $\Sigma u_\mu$ is summable $(C, -\delta)$. If we take $n=\nu$, the condition (1) reduces to the well-known condition for summability $(C, -\delta)$

$$u_\nu = o(\nu^{-\delta}) \,, \tag{6}$$

corresponding to the condition $u_\nu = o(1)$ for convergence of $\Sigma u_\nu$.

*Sufficiency.* We begin by proving that the condition (1) implies the convergence of the series $\Sigma u_\mu$.

Writing

$$\sum_{\mu=\nu}^{n} u_\mu = \sum_{\mu=\nu}^{n} (u_\mu A_{n-\mu}^{-\delta}) \frac{1}{A_{n-\mu}^{-\delta}} \,,$$

we find, for every value of $n$ *greater than* $\nu$, by partial summation

$$\sum_{\mu=\nu}^{n} u_\mu = \sum_{\mu=\nu}^{n-1} \left( \sum_{p=\nu}^{\mu} u_p A_{n-p}^{-\delta} \right) \left( \frac{1}{A_{n-\mu}^{-\delta}} - \frac{1}{A_{n-\mu-1}^{-\delta}} \right) + \sum_{p=\nu}^{n} u_p A_{n-p}^{-\delta} \,. \tag{7}$$

Since the sums occurring in the main term of the transformation may be written

$$\sum_{p=\nu}^{\mu} u_p A_{n-p}^{-\delta} = \sum_{p=\nu}^{n} - \sum_{p=\mu+1}^{n} \,,$$

they can be estimated by application of the assumption (1), whereby we get

$$\sum_{p=\nu}^{\mu} u_p A_{n-p}^{-\delta} = o_\nu(1) n^{-\delta} + o_\mu(1) n^{-\delta} = o_\nu(1) n^{-\delta} \,.$$

The differences in the main term are all positive since $A_{n-\mu-1}^{-\delta} > A_{n-\mu}^{-\delta}$. We therefore obtain from (7)

$$\begin{aligned} \sum_{\mu=\nu}^{n} u_\mu &= o_\nu(1) n^{-\delta} \left( \frac{1}{A_{n-\nu}^{-\delta}} - \frac{1}{A_0^{-\delta}} \right) + o_\nu(1) n^{-\delta} \\ &= o_\nu(1) O(1) \left( \frac{n-\nu}{n} \right)^{\delta} + o_\nu(1) n^{-\delta} \\ &= o_\nu(1) O(1) + o_\nu(1) = o_\nu(1) \,, \end{aligned}$$

$\nu$ being less than $n$. For $n=\nu$ this result is trivially true (cf. (6)).

In proving that the condition (1) further implies the summability $(C, -\delta)$ of $\Sigma u_\mu$, we suppose, as we may, that the sum $\Sigma u_\nu$ is zero. For

$$s_\mu = u_0 + u_1 + \ldots + u_\mu$$

we then have

$$|s_\mu| < k \quad \text{for all values of } \mu,$$

where $k$ is a suitable constant, and to a given $\varepsilon > 0$ there corresponds a number $Q = Q(\varepsilon)$ such that

$$|s_\mu| < \varepsilon \qquad \text{for} \quad \mu > Q\,.$$

We put

$$S_n^{-\delta}(u_\mu) = \sum_{\mu=0}^{n} u_\mu A_{n-\mu}^{-\delta} = \sum_{\mu=0}^{m} + \sum_{\mu=m+1}^{n} = \alpha_n + \beta_n\,,$$

where $m = [\frac{1}{2}n]$, and estimate each of the two sums separately.

For $\alpha_n$ we find, by partial summation,

$$\alpha_n = \sum_{\mu=0}^{m} u_\mu A_{n-\mu}^{-\delta} = \sum_{\mu=0}^{m-1} s_\mu A_{n-\mu}^{-\delta-1} + s_m A_{n-m}^{-\delta}$$

and hence

$$\begin{aligned} |\alpha_n| &\leqq (Q+1)k\,|A_{n-Q}^{-\delta-1}| + \varepsilon(m-Q)\,|A_{n-m+1}^{-\delta-1}| + o_n(1)n^{-\delta} \\ &\leqq o_n(1)n^{-\delta} + \varepsilon K n^{-\delta} + o_n(1)n^{-\delta}\,, \end{aligned}$$

where $K$ is a suitable constant (depending only on $\delta$). This shows that there exists a number $N$ such that

$$|\alpha_n| \leqq \varepsilon(K+2)n^{-\delta} \qquad \text{for} \quad n > N\,.$$

For $\beta_n$ we find, by applying the assumption (1),

$$\beta_m = \sum_{\mu=m+1}^{n} u_\mu A_{n-\mu}^{-\delta} = o_m(1)n^{-\delta} = o(n^{-\delta})\,.$$

From these results it appears that

$$S_n^{-\delta}(u_\mu) = o(n^{-\delta})\,,$$

and this completes the proof of Theorem 1.

**3.** With Theorem 1 at our disposal the following well-known theorem [1, p. 31] can be proved for $0 < \delta < 1$ in the same simple manner as for $\delta = 0$ (the case of convergence).

THEOREM 2. *The series $\Sigma u_\nu$ is summable $(C, -\delta)$, $0 \leqq \delta < 1$, if $\Sigma |u_\nu|$ is summable $(C, -\delta)$.*

To give an application of Theorem 1 leading to new results, we shall prove a theorem concerning the order of magnitude of the differences $\Delta^{-r} a_\nu$, where $r < 1$.

These differences are defined by

(8) $$\Delta^{-r} a_\nu = \sum_{\mu=0}^{\infty} A_\mu^{r-1} a_{\nu+\mu}, \qquad \nu = 0, 1, 2, \ldots,$$

whenever the series on the right converge. If one of these series,

(9) $$\sum_{\mu=0}^{\infty} A_\mu^{r-1} a_\mu$$

for instance, is convergent, then all of the series are convergent. Thus, all of the differences (8) exist if but one of them exists.

It has been proved [4, p. 34] that the convergence of the series (9) implies that

$$\Delta^{-r} a_\nu = o(1), \qquad \text{if} \quad r \geqq 1,$$

and

$$\Delta^{-r} a_\nu = o(\nu^{1-r}), \qquad \text{if} \quad 0 < r < 1.$$

Recently, Kuttner [7] has shown that the latter result remains true for all negative non-integral values of $r$ and that, in these cases, no better result concerning the order of magnitude can be inferred from the mere existence of the differences.

It now turns out that for $r < 1$ the order of magnitude will be lowered if we change the requirement to the series (9) from convergence to summability of a negative order greater than $-1$. If $r > 0$, we obtain the same result as for $r \geqq 1$ if we require that (9) should be summable of an order less than $r-1$. We confine ourselves to stating and proving the results for $0 < r < 1$. For $r < 0$ no new significant feature does emerge.

THEOREM 3. *If* $0 < r < 1$ *and the series* (9) *is summable* $(C, -\delta)$, *where* $0 < \delta < 1$, *then*

$$\Delta^{-r} a_\nu = \begin{cases} o(\nu^{1-r-\delta}) & \text{for} \quad 0 < \delta < 1-r, \\ o(\log \nu) & \text{for} \quad \delta = 1-r, \\ o(1) & \text{for} \quad 1-r < \delta < 1. \end{cases}$$

To prove this theorem we use the following lemma, proved elsewhere ([3, Theorem Ib, p. 332–34]).

LEMMA 2. *If* $0 < \delta < 1$ *and if*

1) $\sum_{\mu=0}^{\infty} u_\mu$ *is summable* $(C, -\delta)$ *to the sum* $U$,

2) $\varepsilon_\mu$ *is bounded below*,

3) $\sum_{\mu=0}^{\infty} A_\mu^{-\delta} \Delta^{-\delta+1} \varepsilon_\mu$ *is absolutely convergent to the sum* $e$,

*then the series* $\sum u_\mu \varepsilon_\mu$ *is convergent and the sum is determined by the formula*

(10) $$\sum_{\mu=0}^{\infty} u_\mu \varepsilon_\mu = \sum_{\mu=0}^{\infty} S_\mu^{-\delta}(u_p) \Delta^{-\delta+1} \varepsilon_\mu + (\varepsilon_0 - e) U.$$

The series

$$\Delta^{-r} a_\nu = \sum_{\mu=0}^{\infty} A_\mu^{r-1} a_{\nu+\mu}$$

may be obtained by multiplying the series

$$\sum_{\mu=0}^{\infty} u_\mu = \sum_{\mu=0}^{\infty} A_{\nu+\mu}^{r-1} a_{\nu+\mu}$$

term by term by the factors

$$\varepsilon_\mu = A_\mu^{r-1} / A_{\nu+\mu}^{r-1} . \tag{11}$$

For every fixed value of $\nu$ the series $\Sigma_{\mu=0}^{\infty} u_\mu$ *is summable* $(C, -\delta)$ to a sum which may be denoted by $U_\nu$, and the factors $\varepsilon_\mu$ are *positive* for all values of $\mu$. For $\nu \geqq 1$ these factors are even *decreasing* to *the limit* 1. Obviously, we only need to consider large values of $\nu$.

Since $\varepsilon_\mu \to 1$, we have $\varepsilon_\mu = O(1)$, and this implies that the difference transformation

$$\Delta^{-\delta+1} \varepsilon_\mu = \Delta^{-\delta}(\Delta^1 \varepsilon_\mu) = \sum_{\lambda=0}^{\infty} A_\lambda^{\delta-1} \Delta^1 \varepsilon_{\mu+\lambda}$$

is valid for all values of $\mu$ (see for instance [2, p. 20–21]), and hence that

$$\sum_{\mu=0}^{\infty} A_\mu^{-\delta} \Delta^{-\delta+1} \varepsilon_\mu = \sum_{\mu=0}^{\infty} A_\mu^{-\delta} \sum_{\lambda=0}^{\infty} A_\lambda^{\delta-1} \Delta^1 \varepsilon_{\mu+\lambda} . \tag{12}$$

The differences $\Delta^1 \varepsilon_\mu$ being positive for all values of $\mu$, we realize that the terms of the series (12), in both of its forms, are *positive*. By rearrangement of the terms of the repeated series we obtain

$$\sum_{\mu=0}^{\infty} A_\mu^{-\delta} \sum_{\lambda=0}^{\infty} A_\lambda^{\delta-1} \Delta^1 \varepsilon_{\mu+\lambda} = \sum_{\mu=0}^{\infty} \Delta^1 \varepsilon_\mu$$

and, since the resulting series is convergent, it follows that the series (12) is *absolutely convergent* to the sum

$$e_\nu = \sum_{\mu=0}^{\infty} \Delta^1 \varepsilon_\mu = \varepsilon_0 - 1 = \frac{1}{A_\nu^{r-1}} - 1 = O(\nu^{1-r}) .$$

On account of these results we can determine the value of $\Delta^{-r} a_\nu$ by the sum-formula (10) of Lemma 2. We obtain

$$\Delta^{-r} a_\nu = \sum_{\mu=0}^{\infty} S_\mu^{-\delta}(u_p) \Delta^{-\delta+1} \varepsilon_\mu + 1 \cdot U_\nu . \tag{13}$$

Because of the convergence of the series (9) we have

$$U_\nu = \sum_{\mu=0}^{\infty} A_{\nu+\mu}^{r-1} a_{\nu+\mu} = \sum_{\mu=0}^{\infty} A_\mu^{r-1} a_\mu - \sum_{\mu=0}^{\nu-1} A_\mu^{r-1} a_\mu = o_\nu(1) . \tag{14}$$

To estimate the sum

$$V_\nu = \sum_{\mu=0}^{\infty} S_\mu^{-\delta}(u_p) \Delta^{-\delta+1}\varepsilon_\mu$$

we first consider the Cesàro sums occurring in the series. We find

$$S_\mu^{-\delta}(u_p) = \sum_{p=0}^{\mu} u_p A_{\mu-p}^{-\delta} = \sum_{p=0}^{\mu} (A_{\nu+p}^{r-1} a_{\nu+p}) A_{\mu-p}^{-\delta}$$
$$= \sum_{p=\nu}^{\nu+\mu} (A_p^{r-1} a_p) A_{\nu+\mu-p}^{-\delta}$$

and, using now that the series $\Sigma A_p^{r-1} a_p$ is summable $(C, -\delta)$, we obtain, by application of Theorem 1, that

$$S_\mu^{-\delta}(u_p) = o_\nu(1)(\nu+\mu)^{-\delta},$$

and this result gives

$$|V_\nu| \leqq o_\nu(1) \sum_{\mu=0}^{\infty} (\nu+\mu)^{-\delta} \Delta^{-\delta+1}\varepsilon_\mu, \tag{15}$$

the differences being positive for all values of $\mu$.

Then, having only to estimate the sum of the series occurring in (15), we write

$$\sum_{\mu=0}^{\infty} (\nu+\mu)^{-\delta} \Delta^{-\delta+1}\varepsilon_\mu = \sum_{\mu=0}^{\nu} + \sum_{\mu=\nu+1}^{\infty} = \gamma_\nu + \delta_\nu \tag{16}$$

and consider each part of the sum separately.

Beginning with $\delta_\nu$, we find

$$\delta_\nu = \sum_{\mu=\nu+1}^{\infty} (\nu+\mu)^{-\delta} \Delta^{-\delta+1}\varepsilon_\mu = \sum_{\mu=\nu+1}^{\infty} (1+\nu\mu^{-1})^{-\delta} \mu^{-\delta} \Delta^{-\delta+1}\varepsilon_\mu$$

and, observing that, for all values of $\mu > \nu$,

$$2^{-\delta} < (1+\nu\mu^{-1})^{-\delta} < 1,$$

we get

$$0 < \delta_\nu < K \sum_{\mu=\nu+1}^{\infty} A_\mu^{-\delta} \Delta^{-\delta+1}\varepsilon_\mu,$$

where $K$ is a suitable constant (depending only on $\delta$). By an evaluation similar to that of the series (12) we obtain

$$\sum_{\mu=\nu+1}^{\infty} A_\mu^{-\delta} \Delta^{-\delta+1}\varepsilon_\mu < \sum_{\mu=0}^{\infty} A_\mu^{-\delta} \Delta^{-\delta+1}\varepsilon_{\nu+1+\mu} = \sum_{\mu=0}^{\infty} \Delta^1 \varepsilon_{\nu+1+\mu}$$
$$= \varepsilon_{\nu+1} - 1 = \frac{A_{\nu+1}^{r-1}}{A_{2\nu+1}^{r-1}} - 1 < \frac{A_{\nu+1}^{r-1}}{A_{2\nu+1}^{r-1}}.$$

The sequence $A_{\nu+1}^{r-1}/A_{2\nu+1}^{r-1}$ being decreasing for $\nu \geqq 1$, we further find for all values of $\nu > 1$

$$\frac{A_{\nu+1}^{r-1}}{A_{2\nu+1}^{r-1}} < \frac{A_2^{r-1}}{A_3^{r-1}} = \frac{3}{r+2},$$

and it follows that

$$0 < \delta_\nu < \frac{3K}{r+2}. \tag{17}$$

To estimate the term $\gamma_\nu$ we first notice that the partial sums belonging to the series

$$\sum_{q=0}^{\infty} A_q^{\delta-2} \varepsilon_{\mu+q}$$

are *positive*, whatever the value of $\mu$ may be. This appears from the facts that the partial sums are decreasing, because $\varepsilon_\lambda > 0$ for all values of $\lambda$ and $A_q^{\delta-2} < 0$ for $q \geqq 1$, and that the series, which represents the difference $\Delta^{-\delta+1}\varepsilon_\mu$, has a positive sum.

We then put

$$\gamma_\nu = \gamma_\nu' + \gamma_\nu'', \tag{18}$$

where

$$\gamma_\nu' = \sum_{\mu=0}^{\nu} (\nu+\mu)^{-\delta} \sum_{q=0}^{\nu-\mu} A_q^{\delta-2} \varepsilon_{\mu+q}$$

and

$$\gamma_\nu'' = \sum_{\mu=0}^{\nu} (\nu+\mu)^{-\delta} \sum_{q=\nu-\mu+1}^{\infty} A_q^{\delta-2} \varepsilon_{\mu+q}.$$

Since for $0 \leqq \mu \leqq \nu$

$$2^{-\delta} \leqq (1+\mu\nu^{-1})^{-\delta} \leqq 1$$

and, accordingly,

$$(\nu+\mu)^{-\delta} = \nu^{-\delta}(1+\mu\nu^{-1})^{-\delta} \leqq \nu^{-\delta},$$

we obtain

$$0 < \gamma_\nu' < \nu^{-\delta} \sum_{\mu=0}^{\nu} \sum_{q=0}^{\nu-\mu} A_q^{\delta-2} \varepsilon_{\mu+q}$$

and

$$|\gamma_\nu''| \leqq \nu^{-\delta} \sum_{\mu=0}^{\nu} \sum_{q=\nu-\mu+1}^{\infty} |A_q^{\delta-2}| \varepsilon_{\mu+q}.$$

Considering at first $\gamma_\nu''$, we find

$$\sum_{q=\nu-\mu+1}^{\infty} |A_q^{\delta-2}| \varepsilon_{\mu+q} = \sum_{q=\nu-\mu+1}^{\infty} (-A_q^{\delta-2}) \varepsilon_{\mu+q} < \varepsilon_{\nu+1} \sum_{q=\nu-\mu+1}^{\infty} (-A_q^{\delta-2})$$
$$= \varepsilon_{\nu+1}\left(-\sum_{q=0}^{\infty} A_q^{\delta-2} + \sum_{q=0}^{\nu-\mu} A_q^{\delta-2}\right) = \varepsilon_{\nu+1} A_{\nu-\mu}^{\delta-1}$$

and hence

$$|\gamma_\nu''| < \nu^{-\delta} \varepsilon_{\nu+1} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{\delta-1} = \nu^{-\delta} A_\nu^{\delta} \frac{A_{\nu+1}^{r-1}}{A_{2\nu+1}^{r-1}} = O(1). \tag{19}$$

Concerning $\gamma'_\nu$, we first notice that

$$\sum_{\mu=0}^{\nu}\sum_{q=0}^{\nu-\mu} A_q^{\delta-2}\varepsilon_{\mu+q} = \sum_{\mu=0}^{\nu} A_\mu^{\delta-1}\varepsilon_\mu = \sum_{\mu=0}^{\nu} A_\mu^{\delta-1}\frac{A_\mu^{r-1}}{A_{\nu+\mu}^{r-1}}$$

$$< k_1\nu^{1-r}\sum_{\mu=0}^{\nu} A_\mu^{\delta-1}A_\mu^{r-1} < k_1k_2\nu^{1-r}\sum_{\mu=0}^{\nu}(\mu+1)^{r+\delta-2},$$

where $k_1$ and $k_2$ are constants depending only on $r$ and $\delta$. It follows that

$$\gamma'_\nu < k_1k_2\nu^{1-r-\delta}\sum_{\mu=0}^{\nu}(\mu+1)^{r+\delta-2},$$

and it now appears that

$$(20)\quad\begin{cases}\gamma'_\nu = O(\nu^{1-r-\delta})O(1) = O(\nu^{1-r-\delta}) & \text{for}\quad 0<\delta<1-r,\\ \gamma'_\nu = O(\nu^{1-r-\delta})O(\log\nu) = O(\log\nu) & \text{for}\quad \delta = 1-r,\\ \gamma'_\nu = O(\nu^{1-r-\delta})O(\nu^{r+\delta-1}) = O(1) & \text{for}\quad 1-r<\delta<1.\end{cases}$$

Considering (13), (15), (16) and (18), Theorem 3 is proved by the results (14), (17), (19) and (20).

There are reasons for supposing that the result obtained in the limit case $\delta = 1-r$, viz. $\Delta^{-r}a_\nu = o(\log\nu)$, can be improved to the result $\Delta^{-r}a_\nu = o(1)$ holding in the cases $1-r<\delta<1$. However, in order to obtain this result, it seems necessary to elaborate a method of proof which utilizes more effectively the special character of series representing differences.

## REFERENCES

1. A. F. Andersen, *Sur la multiplication de séries absolument convergentes par des séries sommables par la méthode de Cesàro*, Mat.-Fys. Medd. Danske Vid. Selsk. 1, nr. 4 (1918), 39 pp.
2. A. F. Andersen, *Studier over Cesàro's Summabilitetsmetode*, København, 1921.
3. A. F. Andersen, *Über die Anwendung von Differenzen nicht ganzer Ordnung in der Reihentheorie*, C. R. 8ième Congr. Mathém. Scand. Stockholm 1934, 326–348.
4. A. F. Andersen, *Summation af ikke hel Orden*, Mat. Tidsskr. B 1946, 33–52.
5. A. F. Andersen, *On the extensions within the theory of Cesàro summability of a classical convergence theorem of Dedekind*, Proc. London Math. Soc. (3) 8 (1958), 1–52.
6. A. L. Cauchy, *Cours d'Analyse*, 1re partie, Paris 1821.
7. B. Kuttner, *On differences of fractional order*, Proc. London Math. Soc. (3) 7 (1957), 453–66.

THE TECHNICAL UNIVERSITY OF DENMARK, COPENHAGEN