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# A NOTE ON THE CONVOLUTION OF REGULAR MEASURES

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Throughout mathematical literature there occur two popular ways of defining the convolution of Borel measures $\mu$ and $\nu$ on a locally compact Hausdorff group $G$. The first definition [1] [7] is made by appealing to the Riesz representation theorem and letting $\mu * \nu$ be that unique regular Borel measure on $G$ such that

$$(1) \qquad \int_G f(z)\, d\mu * \nu(z) = \int_G \int_G f(xy)\, d\mu(x)\, d\nu(y)$$

for all continuous functions $f$ on $G$ which vanish at infinity. The second [4] [6] defines $\mu * \nu$ explicitly by the formula

$$(2) \qquad \mu * \nu(E) = \int_G \mu(Ex^{-1})\, d\nu(x)$$

for each Borel subset $E$ of $G$. These two definitions need not be equivalent unless $\mu$ and $\nu$ are regular since the latter definition may not yield a regular measure. The main purpose of this paper is to prove that these two definitions are equivalent in the case that both $\mu$ and $\nu$ are regular measures. This fact and others are derived from our main theorem on the regularity of certain measures.

**1. Preliminaries.** The class of *Borel sets* in a locally compact Hausdorff space $X$ is the smallest $\sigma$-algebra $\mathscr{B}(X)$ of subsets of $X$ that contains all open subsets of $X$. A *complex-valued countably additive Borel measure* $\mu$ on $X$ is any complex-valued function $\mu$ having $\mathscr{B}(X)$ as its domain such that $\mu(0)=0$ and $\mu\left(\bigcup_{n=1}^{\infty} E_n\right)=\sum_{n=1}^{\infty}\mu(E_n)$ for each sequence $E_1, E_2, \ldots$ of pairwise disjoint Borel subsets of $X$. The *total variation* of $\mu$ is the non-negative, real-valued, countably additive Borel measure $|\mu|$ on $X$ defined as follows [3]:

$$|\mu|(E) = \sup \sum_{j=1}^{n} |\mu(E_j)|$$

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where the supremum is taken over all partitions of $E$ into a finite number of Borel sets $E_1, \ldots, E_n$. The measure $\mu$ is said to be *inner regular* (*outer regular*) if for each Borel set $E$ and each positive number $\varepsilon$ there exists a compact set $C \subset E$ (an open set $J \supset E$) such that

$$|\mu|(E \backslash C) < \varepsilon \qquad \big(|\mu|(J \backslash E) < \varepsilon\big).$$

It is said to be *regular* if it is both inner regular and outer regular.

Let $\mathscr{C}_0(X)$ denote the set of all complex-valued continuous functions on $X$ which vanish at infinity. Make $\mathscr{C}_0(X)$ into a Banach space in the usual way by defining the linear operations pointwise and using the uniform norm

$$\|f\| = \max\{|f(x)| : x \in X\}.$$

Let $\mathscr{M}(X)$ denote the set of all complex-valued, countably additive, regular Borel measures on $X$. Make $\mathscr{M}(X)$ into a Banach space by defining linear operations setwise and defining $\|\mu\| = |\mu|(X)$. According to the Riesz representation theorem [2, pp. 247–248] [3] [5, p. 137] $\mathscr{M}(X)$ is isomorphically isometric to the first conjugate space $\mathscr{C}_0(X)^*$ of $\mathscr{C}_0(X)$ under the mapping $\mu \to M$ where

$$M(f) = \int_X f(x)\, d\mu(x), \qquad f \in \mathscr{C}_0(X). \tag{3}$$

**2. The main theorem.** Let $X$ and $Y$ be locally compact Hausdorff spaces and suppose that $\Phi$ is a continuous mapping of $X$ into $Y$. It follows easily from our definition of Borel sets that the mapping: $E \to \Phi^{-1}(E)$ is a mapping of $\mathscr{B}(Y)$ into $\mathscr{B}(X)$. Thus if $\mu \in \mathscr{M}(G)$, then we may define $\lambda$ on $\mathscr{B}(Y)$ by setting

$$\lambda(E) = \mu\big(\Phi^{-1}(E)\big), \qquad E \in \mathscr{B}(Y).$$

It is clear that $\lambda$ is a complex-valued countably additive Borel measure on $Y$. It is called the *measure induced by $\mu$ and $\Phi$*.

THEOREM 1. *Let $X$, $Y$, $\Phi$, $\mu$ and $\lambda$ be as above. Then $\lambda \in \mathscr{M}(Y)$, that is $\lambda$ is a regular measure.*

PROOF. Let $E \in \mathscr{B}(Y)$ and $\varepsilon > 0$. Set $E_0 = \Phi^{-1}(E)$. Since $\mu$ is regular, there exists a compact set $A_0 \subset E_0$ such that $|\mu|(E_0 \backslash A_0) < \varepsilon$. Now set

$$C = \Phi(A_0) \qquad \text{and} \qquad C_0 = \Phi^{-1}(C).$$

Then $C$ is compact, $C \subset E$, $C_0$ is closed, $A_0 \subset C_0 \subset E_0$ and $|\mu|(E_0 \backslash C_0) < \varepsilon$. Thus

$$\begin{aligned} |\lambda|(E\backslash C) &= \sup \sum_{j=1}^{n} |\lambda(E_j)| \\ &= \sup \sum_{j=1}^{n} \left|\mu\left(\Phi^{-1}(E_j)\right)\right| \\ &\leqq |\mu|\left(\Phi^{-1}(E\backslash C)\right) = |\mu|(E_0\backslash C_0) < \varepsilon \end{aligned}$$

where the two suprema are taken over the family of all partitions of $E\backslash C$ into a finite number of Borel sets $E_1, \ldots, E_n$. Thus $\lambda$ is inner regular. Since $|\lambda|$ is a finite measure, the proof is completed by complementation.

COROLLARY 1. *Suppose that $X$, $Y$ and $Z$ are locally compact Hausdorff spaces and that $\Phi$ is a continuous mapping of $X \times Y$ into $Z$. Let $\mu \in \mathscr{M}(X)$ and $\nu \in \mathscr{M}(Y)$. Then the measure $\lambda$ induced on $Z$ by $\mu \times \nu$ and $\Phi$ is in $\mathscr{M}(Z)$.*

PROOF. This follows from Theorem 1 and the fact that the product of two regular measures is a regular measure.

REMARK. If, in the above corollary, $X = Y = Z$, then the mapping $(\mu, \nu) \to \lambda$ is a binary operation on $\mathscr{M}(X)$. Of course this operation need not be associative. In fact it is associative if and only if the binary operation $\Phi$ is associative on $X$.

**3. The measure algebra $\mathscr{M}(G)$.** Let $G$ be a locally compact Hausdorff semigroup, i.e. $G$ is a locally compact Hausdorff space with a jointly continuous, associative, binary operation $p$: $G \times G \to G$. We write $p(x, y) = xy$.

PROPOSITION 1. *Let $\mu$ and $\nu$ be in $\mathscr{M}(G)$ and define the functional $L$ on $\mathscr{C}_0(G)$ by the formula*

$$L(f) = \int_G \int_G f(xy)\, d\mu(x)\, d\nu(y)\,.$$

*Then*

$$L \in \mathscr{C}_0(G)^* \qquad \text{and} \qquad \|L\| \leqq \|\mu\| \cdot \|\nu\|\,.$$

DEFINITION 1. *Let $\mu$, $\nu$ and $L$ be as in Proposition 1 and let $\mu * \nu$ denote the unique member of $\mathscr{M}(G)$ such that*

$$L(f) = \int_G f\, d\mu*\nu, \qquad f \in \mathscr{C}_0(G)\,.$$

*Then $\mu * \nu$ is called the convolution of $\mu$ with $\nu$.*

It is easily shown that $\mathscr{M}(G)$ with convolution as multiplication is a Banach algebra. It follows from Fubini's theorem that $\mathscr{M}(G)$ is commutative, if $G$ is commutative.

Definition 2. *Let $\mu$ and $\nu$ be in $\mathscr{M}(G)$. Let $\mu\cdot\nu$ denote the measure in $\mathscr{M}(G)$ induced by the product measure $\mu\times\nu$ and the semigroup operation $p$* (as in Corollary 1).

Thus for $E\in\mathscr{B}(G)$, letting $\psi_E$ denote the characteristic function of $E$, we have

$$\mu\cdot\nu(E) = \mu\times\nu\big(p^{-1}(E)\big) = \int_G\int_G \psi_E(xy)\,d\mu(x)\,d\nu(y)\,. \tag{4}$$

Theorem 2. *Let $\mu$ and $\nu$ be in $\mathscr{M}(G)$. Then $\mu\cdot\nu=\mu*\nu$.*

Proof. First suppose that $\mu$ and $\nu$ are non-negative and real-valued. Let $C$ be an arbitrary compact subset of $G$ and let $\varepsilon>0$ be given. Since $\mu*\nu$ and $\mu\cdot\nu$ are regular, non-negative, real-valued measures there exists an open set $J$ such that $C\subset J$, $\mu*\nu(J)<\mu*\nu(C)+\varepsilon$ and $\mu\cdot\nu(J)<\mu\cdot\nu(C)+\varepsilon$. Let $f$ be a continuous function from $G$ into $[0, 1]$ such that $f(x)=1$ for all $x\in C$ and $f(x)=0$ for all $x\in G\backslash J$. Then

$$\begin{aligned}
\mu*\nu(C) &= \int_G \psi_C(z)\,d\mu*\nu(z)\\
&\leqq \int_G f(z)\,d\mu*\nu(z)\\
&= \int_G\int_G f(xy)\,d\mu(x)\,d\nu(y)\\
&\leqq \int_G\int_G \psi_J(xy)\,d\mu(x)\,d\nu(y)\\
&= \mu\cdot\nu(J) < \mu\cdot\nu(C) + \varepsilon\,.
\end{aligned}$$

Since $\varepsilon$ is arbitrary we conclude that $\mu*\nu(C)\leqq\mu\cdot\nu(C)$. On the other hand

$$\begin{aligned}
\mu\cdot\nu(C) &= \int_G\int_G \psi_C(xy)\,d\mu(x)\,d\nu(y)\\
&\leqq \int_G\int_G f(xy)\,d\mu(x)\,d\nu(y)\\
&= \int_G f(z)\,d\mu*\nu(z)\\
&\leqq \int_G \psi_J(z)\,d\mu*\nu(z)\\
&= \mu*\nu(J) < \mu*\nu(C) + \varepsilon\,.
\end{aligned}$$

Therefore $\mu\cdot\nu(C)=\mu*\nu(C)$. This proves that $\mu\cdot\nu$ and $\mu*\nu$ agree on all compact sets. Thus, for arbitrary $E\in\mathscr{B}(G)$, we have

$$\begin{aligned}\mu\cdot\nu(E) &= \sup\{\mu\cdot\nu(C)\colon C \text{ is compact, } C\subset E\}\\ &= \sup\{\mu*\nu(C)\colon C \text{ is compact, } C\subset E\}\\ &= \mu*\nu(E)\,.\end{aligned}$$

This completes the proof in the case that $\mu$ and $\nu$ are non-negative and real-valued. The second case, in which $\mu$ and $\nu$ are arbitrary, follows immediately from the first case and the Jordan decomposition theorem [2, p. 123].

In view of this theorem we shall now abandon the notation $\mu\cdot\nu$ in favor of the more standard notation $\mu*\nu$.

COROLLARY 2. *Let $G$ be a locally compact Hausdorff group and suppose that $\mu$ and $\nu$ are in $\mathscr{M}(G)$. Then*

$$\mu*\nu(E) = \int_G \mu(Ey^{-1})\,d\nu(y) = \int_G \nu(x^{-1}E)\,d\mu(x)$$

*for each $E\in\mathscr{B}(G)$.*

PROOF. This follows from Theorem 2 and the formulas

$$\int_G \psi_E(xy)\,d\mu(x) = \mu(Ey^{-1}), \qquad \int_G \psi_E(xy)\,d\nu(y) = \nu(x^{-1}E)\,.$$

**4. The center of $\mathscr{M}(G)$.** Let $G$ be as in Section 3. For $x\in G$ let $\varepsilon_x$ be the measure defined as follows:

$$\varepsilon_x(E) = \begin{cases} 0 & \text{if } x\notin E\,,\\ 1 & \text{if } x\in E\,,\end{cases} \qquad E\in\mathscr{B}(G)\,.$$

The measure $\varepsilon_x$ is called the *unit point mass at $x$*. It is the measure corresponding to the linear functional $f\to f(x)$ on $\mathscr{C}_0(G)$.

An easy computation shows that $\varepsilon_x*\varepsilon_y=\varepsilon_{xy}$ for all $x$ and $y$ in $G$. Thus the mapping $x\to\varepsilon_x$ is an algebraic isomorphism of $G$ into $\mathscr{M}(G)$. It is a homeomorphism if $\mathscr{M}(G)$ $\left(=\mathscr{C}_0(G)^*\right)$ is supplied with the weak* topology. Hence $\mathscr{M}(G)$ is commutative if and only if $G$ is commutative.

We also observe that $\varepsilon_e$ is a two-sided identity for $\mathscr{M}(G)$ if and only if $e$ is a two-sided identity for $G$.

The *center* of $\mathscr{M}(G)$ is the set of all $\mu\in\mathscr{M}(G)$ such that $\mu*\nu=\nu*\mu$ for each $\nu\in\mathscr{M}(G)$.

THEOREM 3. *Let $\mu \in \mathscr{M}(G)$. Then $\mu$ is in the center of $\mathscr{M}(G)$ if and only if $\mu * \varepsilon_x = \varepsilon_x * \mu$ for each $x \in G$.*

PROOF. The necessity of the condition is obvious. To prove its sufficiency let $\nu \in \mathscr{M}(G)$ and $E \in \mathscr{B}(G)$. Then

$$\begin{aligned} \mu*\nu(E) &= \int_G \int_G \psi_E(xy)\, d\mu(x)\, d\nu(y) \\ &= \int_G \int_G \psi_E(x)\, d\mu*\varepsilon_y(x)\, d\nu(y) \\ &= \int_G \int_G \psi_E(x)\, d\varepsilon_y*\mu(x)\, d\nu(y) \\ &= \int_G \int_G \psi_E(yx)\, d\mu(x)\, d\nu(y) = \nu*\mu(E) . \end{aligned}$$

Thus $\mu * \nu = \nu * \mu$ and the theorem is proved.

COROLLARY 3. *Let $G$ be a locally compact Hausdorff group and let $\mu$ be in $\mathscr{M}(G)$. Then the following four propositions are equivalent:*

(a) *$\mu$ is in the center of $\mathscr{M}(G)$;*
(b) $\mu * \varepsilon_x = \varepsilon_x * \mu$ *for each $x \in G$;*
(c) $\mu(xEx^{-1}) = \mu(E)$ *for $x \in G$, $E \in \mathscr{B}(G)$;*
(d) $\mu(Ex) = \mu(xE)$ *for $x \in G$, $E \in \mathscr{B}(G)$.*

PROOF. The equivalence of (a) and (b) follows from Theorem 3. Propositions (c) and (d) are obviously equivalent. The formulas $\mu * \varepsilon_x(E) = \mu(Ex^{-1})$ and $\varepsilon_x * \mu(E) = \mu(x^{-1}E)$, which follow from Corollary 2, imply the equivalence of (b) and (d). This completes the proof.

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