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ON DIRECT DECOMPOSITION OF TORSION FREE ABELIAN GROUPS

BJARNI JÓNSSON

Introduction. It is known (cf. [3]) that there exist torsion free Abelian groups of finite rank which do not have the unique factorization property. The principal purpose of this note is to establish for this class of groups a weaker form of this property. Roughly speaking, the modification consists in replacing the relation of isomorphism by an equivalence relation which holds between two groups if and only if each of them is isomorphic to a subgroup of the other. Of course this will have to be accompanied by a corresponding change in the notion of an indecomposable group. It also gives rise to a congruence relation over the lattice L of all subgroups of a torsion free Abelian group A of finite rank, two members of L being indentified if and only if their sum and their intersection are equivalent. In fact, this turns out to be precisely the congruence relation which we obtain by collapsing all finite dimensional quotients in L. Our problem therefore reduces to showing that in the resulting quotient lattice any two representations of an element as a sum of linearly independent directly irreducible elements are equivalent.

Since this lattice need not be finite dimensional, we cannot apply Ore's theorem [2, Theorem 1, p. 262]. However, our lattice does possess an isotone integer-valued valuation induced by the ranks of the subgroups of A. The only significant property of the dimension function that is missing is the property of being positive, since an element may have the same value as some of its proper parts. Instead we have here two weaker properties: the zero element is the only element whose value is zero, and no element is projective to a proper part of itself. As we shall show in the next section, Birkhoff's proof of Ore's theorem [1, p. 94] can be modified to apply to all lattices having these properties.

Our principal result, Theorem 2.6, was announced in [4], but the original proof, which has never been published, was quite different from

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the one given here. So far as we know, the generalized form of Ore's theorem, our Theorem 1.4, is new.

1. A generalization of Ore's theorem. We shall use + and · for the lattice operations of binary addition and multiplication, and Σ and Π for the corresponding operations on finite sequences. In a lattice with a zero element 0, we write $a \dot{+} b$ for $a+b$ in case $ab=0$, and

$$\dot{\sum_{i<n}} a_i = \sum_{i<n} a_i$$

in case the elements a_i are independent. If $b \leq a$, we write a/b for the quotient, or sublattice, consiting of all elements between a and b. We write a/b tr c/d if the quotients a/b and c/d are transposes of each other. The relations of perspectivity and of projectivity, between quotients or between elements, will be denoted, respectively, by $\sim$ and by $\approx$.

DEFINITION 1.1. *In a lattice M with a zero element* 0, *an element u is said to be directly irreducible provided $u \neq 0$ and, for all $a,b \in M$, the condition $u = a \dot{+} b$ implies that either $a=0$ or $b=0$.*

DEFINITION 1.2. *In a lattice M with a zero element* 0, *an element u is said to have the double exchange property provided for all a, b, $c_0, c_1, \ldots, c_n \in M$ with a directly irreducible, the condition*

$$u = a \dot{+} b = \dot{\sum_{j \leq n}} c_j$$

implies that there exist $q \leq n$ and elements $x,y \in M$ such that

$$c_q = x \dot{+} y \qquad \text{and} \qquad u = x \dot{+} b = a \dot{+} y \dot{+} \dot{\sum_{q \neq j \leq n}} c_j .$$

If every element of M is a sum of finitely many independent, directly irreducible elements, then it is clear that in order to prove that an element u has the double exchange property it suffices to consider the case in which the elements c_i in 1.2 are directly irreducible, and to show that in this case there exists $q \leq n$ such that

$$u = c_q \dot{+} b = a \dot{+} \dot{\sum_{q \neq j \leq n}} c_j .$$

DEFINITION 1.3. *By a rank function over a lattice M with a zero element* 0 *we mean an integer valued function R on M such that, for all $a,b \in M$,*

$$R(a+b)+R(ab) = R(a)+R(b) \,,$$
$$a \leq b \quad \textit{implies that} \quad R(a) \leq R(b) \,,$$
$$R(a) = 0 \quad \textit{if and only if} \quad a = 0 \,.$$

In a finite dimensional modular lattice the dimension function is clearly a rank function. Given a torsion free Abelian group, the usual notion of rank defines a rank function over the lattice of all subgroups of finite rank. Finally, given a rank function over a lattice M, and a congruence relation over M with the property that congruent elements always have the same rank, we obtain a rank function over the quotient lattice by assigning to each equivalence class X the common rank of all the members of X. These last two observations will be utilized in the next section.

THEOREM 1.4. *If M is a modular lattice with a zero element* 0, *if there exists a rank function R over M, and if no element of M is projective to a proper part of itself* (*that is, $a \leq b \approx a$ implies $a = b$*), *then every element of M has the double exchange property.*

PROOF. Suppose $u \in M$, and assume that $a_0, a_1, \ldots, a_m$, $b_0, b_1, \ldots, b_n$ are directly irreducible elements of M such that

$$u = \dot{\sum_{i \leq m}} a_i = \dot{\sum_{j \leq n}} b_j \,.$$

For $p \leq m$ and $q \leq n$ let

$$\bar{a}_p = \sum_{p \neq i \leq n} a_i \qquad \text{and} \qquad \bar{b}_q = \sum_{q \neq j \leq n} b_j \,.$$

We wish to show that

$$u = b_q \dot{+} \bar{a}_0 = a_0 \dot{+} \bar{b}_q \qquad \text{for some} \qquad q \leq n \,. \tag{1}$$

Let t and t' be positive integers, and assuming this assertion to hold whenever either $R(u) < t$ or else $R(u) = t$ and $R(a_0) < t'$, consider the case in which $R(u) = t$ and $R(a_0) = t'$.

First consider the case in which

$$a_0 + \bar{b}_q = u \qquad \text{for} \qquad q = 0, 1, \ldots, n \,, \tag{2}$$

and observe that this implies that $R(b_q) \leq R(a_0)$. If $R(b_n) < R(a_0)$, then it follows from the inductive hypothesis that there exists $p \leq m$ such that

$$u = b_n \dot{+} \bar{a}_p = a_p \dot{+} \bar{b}_n \,. \tag{3}$$

It follows that $R(a_p) = R(b_n)$, and therefore $p \neq 0$. Before completing the proof for the subcase under consideration, we show that (3) also holds in case $R(b_n) = R(a_0)$ and $b_n a_0 \neq 0$. In fact, let

$$a' = a_0(b_n+\bar{a}_0), \qquad u' = a'+\bar{a}_0\,.$$

Then

$$u' = a_0(b_n+\bar{a}_0)+\bar{a}_0 = (a_0+\bar{a}_0)(b_n+\bar{a}_0) = b_n+\bar{a}_0\,,$$

$$b_n \leq u' \leq b_n \dot{+} \bar{b}_n, \qquad u' = b_n \dot{+} u'\bar{b}_n\,.$$

Furthermore

$$a'/0 = a_0(b_n+\bar{a}_0)/a_0(b_n+\bar{a}_0)\bar{a}_0 \ \text{tr}\ \big(a_0(b_n+\bar{a}_0)+\bar{a}_0\big)/\bar{a}_0$$
$$= (b_n+\bar{a}_0)/\bar{a}_0 \ \text{tr}\ b_n/b_n\bar{a}_0\,,$$

whence it follows that

$$R(a') = R(b_n)-R(b_n\bar{a}_0) < R(b_n) = R(a_0)\,,$$
$$R(u') = R(a')+R(\bar{a}_0) < R(a_0)+R(\bar{a}_0) = R(u)\,.$$

By the inductive hypothesis, and in view of the fact that $R(a')<R(b_n)$, there exist a positive integer p such that

$$u' = a' \dot{+} b_n \dot{+} \bar{a}_0\bar{a}_p = a_p \dot{+} u'\bar{b}_n\,.$$

It follows that

$$a_p+\bar{b}_n = u'+\bar{b}_n \geq b_n+\bar{b}_n = u, \qquad a_p\bar{b}_n = a_p u'\bar{b}_n = 0\,,$$

so that $u=a_p \dot{+} \bar{b}_n$. Consequently $R(a_p)=R(b_n)$. Furthermore

$$b_n+\bar{a}_p = u'+\bar{a}_p \geq a_p+\bar{a}_p = u, \qquad u = b_n+\bar{a}_p\,,$$

and since

$$R(b_n) = R(a_p) = R(u)-R(\bar{a}_p)\,,$$

we must have $u=b_n \dot{+} \bar{a}_p$. Thus (3) is seen to hold in this case also.

We next show that (3) implies (1). The elements $\bar{b}_n$ and $\bar{a}_p$ are perspective, and the mapping

$$x \rightarrow x^* = \bar{a}_p(x+a_p)$$

is an isomorphism of the quotient $\bar{b}_n/0$ onto the quotient $\bar{a}_p/0$. Consequently

$$\bar{a}_p = \dot{\sum_{i<n}} b_i^*\,.$$

By the inductive hypothesis we have

$$\bar{a}_p = b_q^* \dot{+} \bar{a}_0\bar{a}_p = a_0 \dot{+} \dot{\sum_{q\neq i<n}} b_i^*$$

for some $q<n$. Observe that

$$b_q+\bar{a}_0 \geq b_q+a_p \geq b_q^*, \qquad u = b_q+\bar{a}_0\,.$$

Inasmuch as $R(b_q)\leq R(a_0)$, this implies that $u=b_q \dot{+} \bar{a}_0$. Thus $R(b_q)=R(a_0)$, and since

$$u = b_q \dot{+} \bar{b}_q = a_0 + \bar{b}_q ,$$

we infer that $u = a_0 \dot{+} \bar{b}_q$. Thus (1) has been established under the assumption that (2) holds, and that either $R(b_n) < R(a_0)$, or else $R(b_n) = R(a_0)$ and $b_n \bar{a}_0 \neq 0$.

Still assuming that (2) holds, we now consider the remaining subcase in which $R(b_n) = R(a_0)$ and $b_n \bar{a}_0 = 0$. Since

$$u = b_n \dot{+} \bar{b}_n = a_0 + \bar{b}_n \qquad \text{and} \qquad R(b_n) = R(a_0) ,$$

we have $u = a_0 \dot{+} \bar{b}_n$. Thus $a_0 \sim b_n$. Furthermore,

$$\begin{aligned} a_0(b_n + \bar{a}_0)/0 &= a_0(b_n + \bar{a}_0)/a_0(b_n + \bar{a}_0)\bar{a}_0 \ \mathrm{tr} \ \big(a_0(b_n + \bar{a}_0) + \bar{a}_0\big)/\bar{a}_0 \\ &= (b_n + \bar{a}_0)/\bar{a}_0 \ \mathrm{tr} \ b_n/b_n\bar{a}_0 = b_n/0 \ \sim \ a_0/0 , \end{aligned}$$

whence it follows by hypothesis that $a_0(b_n + \bar{a}_0) = a_0$. Consequently $a_0 \leq b_n + \bar{a}_0$, $u = b_n \dot{+} \bar{a}_0$. We therefore see that in this case (1) holds with $q = n$.

We now drop the assumption that (2) holds. For $q \leq n$ let

$$c_q = b_q(a_0 + \bar{b}_q), \qquad \bar{c}_q = \dot{\sum_{q \neq j \leq n}} c_j ,$$

and let

$$v = \dot{\sum_{j \leq n}} c_j .$$

It is then easy to check that

$$v = \prod_{j \leq n} (a_0 + \bar{b}_j), \qquad \bar{c}_q = \bar{b}_q \prod_{q \neq j \leq n} (a_0 + \bar{b}_j).$$

It follows that $a_0 \leq v \leq a_0 + \bar{a}_0$, hence

$$v = a_0 \dot{+} v\bar{a}_0 ,$$

and that

$$a_0 + \bar{c}_q = v \qquad \text{for} \qquad q = 0, 1, \ldots, n .$$

Each of the elements c_q is a sum of independent, directly irreducible elements $c_{q,r}$, $r = 0, 1, \ldots, k_q$. Letting

$$\bar{c}_{q,r} = \bar{c}_q + \sum_{r \neq s \leq k_q} c_{q,s}$$

whenever $q \leq n$ and $r \leq k_q$, we have $a_0 + \bar{c}_{q,r} = v$. By the special case already established we therefore have

$$v = c_{q,r} \dot{+} v\bar{a}_0 = a_0 \dot{+} \bar{c}_{q,r}$$

for some $q \leq n$ and $r \leq k_q$. It follows that

$$c_{q,r}+\bar{a}_0 = v+\bar{a}_0 = u ,$$

and since $R(c_{q,r}) \leq R(c_q) \leq R(a_0)$, we must have

$$u = c_{q,r} \dot{+} \bar{a}_0 .$$

Thus

$$c_{q,r} \leq b_q \leq c_{q,r} \dot{+} \bar{a}_0, \qquad b_q = c_{q,r} \dot{+} b_q \bar{a}_0 ,$$

and recalling that b_q is directly irreducible we infer that $b_q \bar{a}_0 = 0$ and $b_q = c_{q,r}$. Inasmuch as $c_{q,r} \leq c_q \leq b_q$, it follows that $b_q = c_q$. Therefore

$$u = b_q \dot{+} \bar{a}_0 \qquad \text{and} \qquad v = b_q \dot{+} v\bar{a}_0 = a_0 \dot{+} \bar{c}_q .$$

Finally,

$$a_0 + \bar{b}_q = v + \bar{b}_q = u ,$$

and we use the fact that $R(a_0) = R(b_q)$ to conclude that

$$u = a_0 \dot{+} \bar{b}_q .$$

We have therefore established (1) for the case in which $R(u) = t$ and $R(a_0) = t'$. The theorem follows by induction.

In the next section we shall also need the following lemma.

LEMMA 1.5. *Suppose* $\equiv$ *is a congruence relation over a modular lattice* M, *and for* $a \in M$ *let* $a^* = a/\equiv$. *For any* $a, b, c, d \in M$, *if* $b^* \leq a^*$, $d^* \leq c^*$, *and* $a^*/b^* \approx c^*/d^*$, *then there exist elements* $a', b', c', d' \in M$ *such that*

$$ab \leq b' \leq a' \leq a+b, \qquad cd \leq d' \leq c' \leq c+d ,$$

$$b' \equiv b, \quad a' \equiv a, \quad c' \equiv c, \quad d' \equiv d, \quad \textit{and} \quad a'/b' \approx c'/d'.$$

PROOF. We may assume that $b \leq a$ and $d \leq c$. First suppose

$$a^*/b^* \ \mathrm{tr}\ c^*/d^* .$$

Then either $a \equiv b+c$ and $d \equiv bc$, or else $b \equiv ad$ and $c \equiv a+d$. In either case the four elements

$$a' = b+ac, \quad b' = b+ad, \quad c' = d+ac, \quad d' = d+bc$$

have the required properties, in fact,

$$(b+ac)/(b+ad) \ \mathrm{tr}\ (b+d+ac)/(b+d) \ \mathrm{tr}\ (d+ac)/(d+bc) .$$

Assuming now that the conclusion holds whenever the sequence of transposes connecting the two quotients consists of at most n terms, consider the case of $n+1$ transposes. Then

$$a^*/b^* \approx c_1^*/d_1^* \ \mathrm{tr}\ c^*/d^*$$

where the first two quotients can be connected by n transposes. Hence there exist a', b', c_1', $d_1' \in M$ such that

$$b \leq b' \leq a' \leq a, \qquad d_1 \leq d_1' \leq c_1' \leq c_1 ,$$
$$b' \equiv b, \quad a' \equiv a, \quad d_1' \equiv d_1, \quad c_1' \equiv c_1, \quad \text{and} \quad a'/b' \approx c_1'/d_1' .$$

Thus $(c_1')^* = c_1^*$ and $(d_1')^* = d_1^*$, so that

$$(c_1')^*/(d_1')^* = c_1^*/d_1^* \text{ tr } c^*/d^* .$$

By the first part of the proof there exist c_1'', d_1'', c', $d' \in M$ such that

$$d_1' \leq d_1'' \leq c_1'' \leq c_1', \qquad d \leq d' \leq c' \leq c ,$$
$$d_1'' \equiv d_1', \quad c_1'' \equiv c_1', \quad d' \equiv d, \quad c' \equiv c, \quad \text{and} \quad c_1''/d_1'' \approx c'/d' .$$

In the projectivity between a'/b' and c_1'/d_1', the elements c_1'' and d_1'' of the latter quotient correspond to elements a'' and b'' in the former, and

$$a''/b'' \approx c_1''/d_1'' \approx c'/d' .$$

Furthermore $b \leq b' \leq b'' \leq a'' \leq a' \leq a$, and since projectivities preserve all congruence relations we have $b \equiv b''$ and $a \equiv a''$.

The lemma now follows by induction.

2. Applications to torsion free Abelian groups. We now define the equivalence relation mentioned in the introduction.

DEFINITION 2.1. *Two groups A and B are said to be almost isomorphic — in symbols $A \cong^\circ B$ — if and only if each of them is isomorphic to a subgroup of the other.*

DEFINITION 2.2. *Suppose A and B are subgroups of a group G. We say that*

(i) *A is almost contained in B — in symbols $A \subseteq^\circ B$ — if and only if $A \cong^\circ A \cap B$.*

(ii) *A is almost equal to B — in symbols $A =^\circ B$ — if and only if $A \subseteq^\circ B$ and $B \subseteq^\circ A$.*

DEFINITION 2.3. *A group A is said to be strongly indecomposable if and only if A consists of more than one elements, and the condition $A \cong^\circ B \times C$ always implies that either B or C consists of only one element.*

Clearly the relation $\cong^\circ$ is an equivalence relation over the class of all groups. On the other hand it is easy to see that in general $\subseteq^\circ$ is not a transitive relation over the class of all subgroups of a given group G, and that consequently $=^\circ$ need not be an equivalence relation.

THEOREM 2.4. *Suppose A and B are subgroups of finite rank of a torsion free Abelian group G. If $A \subseteq^\circ B$, then the following conditions are equivalent:*

(i) $A \cong^\circ B$.

(ii) *The quotient group $B-(A \cap B)$ is finite.*

(iii) *The lattice quotient $B/(A \cap B)$ is finite dimensional.*

(iv) *$nB \subseteq A$ for some non-zero integer n.*

(v) $A =^\circ B$.

PROOF. Clearly (ii) implies (iii). Assuming (iii), we consider arbitrary subgroups C of B with $A \cap B \subseteq C$, and use induction on the dimension N of $C/(A \cap B)$ to show that $nC \subseteq A \cap B$ for some non-zero integer n. Assuming this to hold for all lower dimensional cases, choose a subgroup C' of C which is covered by C in the lattice of all subgroups of G. By the inductive hypothesis, $n'C' \subseteq A \cap B$ for some non-zero integer n'. Also, since there is no group properly between C' and C, the quotient group $C-C'$ must be finite and its order must be a prime p. Hence $pC \subseteq C'$, so that $pn'C \subseteq A \cap B$. Induction completes the proof of (iv).

Assume (iv), and let f be a homomorphism of B onto an Abelian group C such that the kernel of f is $A \cap B$. Then the order of each element of C divides n, whence it follows that if C is infinite, then it contains arbitrarily large finite subgroups C'. In particular we can choose C' so that its order exceeds n^m where m is the rank of B. Now C' is a direct product of cyclic groups generated by elements $c_0, c_1, \ldots, c_k$ such that, for each $i<k$, the order n_{i+1} of c_{i+1} divides the order n_i of c_i, and $n_k>1$. It follows that $k \geq m$. For $i=0, 1, \ldots, k$ let b_i be a counter image of c_i. Then the elements b_i are linearly dependent,

$$\sum_{i \leq k} r_i b_i = 0 ,$$

where at least some of the integers r_i are not zero. We can choose the integers r_i so that their greatest common divisor is 1. Now

$$\sum_{i \leq k} r_i c_i = 0 ,$$

whence it follows that n_i divides r_i for each $i \leq k$. Consequently n_k divides all the integers r_i, which is impossible because $n_k > 1$. We therefore conclude that C must be finite. Thus (iv) implies (ii).

If (iv) holds, then the mapping $x \to nx$ is an isomorphism of B into $A \cap B$, whence it follows that $B \subseteq^\circ A$, and (v) holds. If (v) holds, then

$$A \cong^\circ A \cap B \cong^\circ B ,$$

and (i) is satisfied.

Finally assume (i). Then $B \cong{}^\circ A \cap B$, and there exists an isomorphism f of B into $A \cap B$. There exist finitely many elements $b_0, b_1, \ldots, b_{m-1}$ which form a rational basis for B. The elements $f(b_0), f(b_1), \ldots, f(b_{m-1})$ are then also rationally independent, and must therefore form a rational basis for $A \cap B$. We can embed G in a vector space over the field of rational numbers, so that the multiplication of an element of B by a rational number becomes meaningful. We have

$$f(b_i) = \sum_{j<m} s_{i,j} b_j$$

where $s_{i,j}$, $i, j = 0, 1, \ldots, m-1$ are rational numbers. The matrix s satisfies its characteristic equation, so that

$$(\det s) I = \sum_{k<m} t_k s^{k+1}$$

where I is the identity matrix and $t_0, t_1, \ldots, t_{m-1}$ are rational numbers. Since s is non-singular, there exist integers $n \neq 0$, $q_0, q_1, \ldots, q_{m-1}$ such that

$$nI = \sum_{k<m} q_k s^{k+1} .$$

Consequently, for all $x \in B$,

$$nx = \sum_{k<m} q_k f^{k+1}(x) \in A \cap B .$$

Thus (iv) holds, and the proof is complete.

Theorem 2.5. *If L is the lattice of all subgroups of finite rank of a torsion free Abelian group G, then $=^\circ$ is a congruence relation over L, and in the quotient lattice no element is projective to a proper part of itself.*

Proof. Since $A \cap B \subseteq{}^\circ A$ and $A \cap B \subseteq{}^\circ B$, it follows from Theorem 2.4 that $A ={}^\circ B$ if and only if the lattice quotients $A/(A \cap B)$ and $B/(A \cap B)$ are finite dimensional, and this in turn holds if and only if the dimension of the quotient $(A+B)/(A \cap B)$ is finite. Thus $=^\circ$ is precisely the congruence relation over L which collapses all finite dimensional quotients.

For $A \in L$ let A^* be the equivalence class modulo $=^\circ$ to which A belongs. Observe that 0^* consists of the null group 0 alone. From this it follows by Lemma 1.5 that if $A, B \in L$ are such that $A^* \leq B^*$ and $A^* \approx B^*$, then there exist $A', B' \in L$ such that

$$A' \subseteq A, \quad B' \subseteq B, \quad A' ={}^\circ A, \quad B' ={}^\circ B, \quad \text{and} \quad A' \approx B' .$$

Then $A' \cong B'$, $A \cong{}^\circ B$. But the condition $A^* \leq B^*$ implies that $A ={}^\circ A \cap B$. Therefore

$$B \cong^\circ A \cap B, \quad B =^\circ A \cap B, \quad B^* \leq A^*, \quad A^* = B^* .$$

This proves the last part of the theorem.

THEOREM 2.6. *Suppose* $A_0, A_1, \ldots, A_m, B_0, B_1, \ldots, B_n$ *are strongly indecomposable torsion free Abelian groups of finite rank, and*

$$A_0 \times A_1 \times \ldots \times A_m \cong^\circ B_0 \times B_1 \times \ldots \times B_n .$$

Then $m=n$ *and there exists a permutation* φ *of the integers* $0, 1, \ldots, m$ *such that*

$$A_i \cong^\circ B_{\varphi(i)} \quad \textit{for} \quad i = 0, 1, \ldots, m .$$

PROOF. We may assume that the groups A_i and B_j are subgroups of a torsion free Abelian group G, and that the direct products involved in the hypothesis are inner direct products, i.e., the operation $\times$ coincides with the operation $\dot{+}$ in the latice of all subgroups of G. In fact, without loss of generality we may assume that

$$G =^\circ \dot{\sum_{i \leq m}} A_i =^\circ \dot{\sum_{j \leq n}} B_j . \tag{1}$$

Let L be the lattice of all subgroups of G, and for $A \in L$ let A^* be the equivalence class modulo $=^\circ$ to which A belongs. Since two members of the same equivalence class always have the same rank, we can define a rank function R over the quotient lattice $M=L/=^\circ$ by letting $R(C^*)$ be the rank of C. Since, by Theorem 2.5, no member of M is projective to a proper part of itself, it follows that M has the double exchange property.

Next observe that in order for a subgroup C of G to be strongly indecomposable it is necessary and sufficient that C^* be directly irreducible, and that for any $C, D \in L$, if C^* and D^* have a common complement, then $C \cong^\circ D$. In fact, if $G^* = C^* \dot{+} E^* = D^* \dot{+} E^*$, then $C \dot{+} E =^\circ D \dot{+} E$. Hence there exists a non-zero integer n such that $nC \subseteq D \dot{+} E$. Thus for each $c \in C$, the element nc can be uniquely represented in the form $nc = d + e$ with $d \in D$ and $e \in E$, and the mapping $c \rightarrow d$ is an isomorphism of C into D. Similarly, D is isomorphic to a subgroup of C.

By (1) we have

$$G^* = \dot{\sum_{i \leq m}} A_i^* = \dot{\sum_{j \leq n}} B_j^* ,$$

and by repeated use of the double exchange property we obtain a one-to-one mapping φ of the integers $0, 1, \ldots, m$ into the integers $0, 1, \ldots, n$ such that

$$G^* = \dot{\sum_{i \leq k}} B_{\varphi(i)}^* \dot{+} \sum_{k < i \leq m} A_i^* \quad \text{for} \quad k = 0, 1, \ldots, m .$$

By taking $k=m$ we infer that $m=n$, and that φ is a permutation of the integers $0, 1, \ldots, m$. Finally, by considering two successive values of k, $k=p$ and $k=p+1$, we see that $A_p{}^*$ and $B_{\varphi(p)}{}^*$ have a common complement, namely

$$\dot{\sum_{i<p}} B_{\varphi(i)}{}^* \dotplus \dot{\sum_{p<i<m}} A_i{}^* .$$

Consequently $A_p \cong{}^\circ B_{\varphi(p)}$.

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UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINN., U.S.A.