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PARACONVEX SETS

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1. Introduction. The principal purpose of this paper is to show that some of the desirable topological properties of convex subsets of a Banach space remain valid for a larger class of sets, which we call *paraconvex*. Specifically, we shall generalize the following two standard results, the first of which follows from the second.

THEOREM A. [1, Theorem 4.1] *If X is paracompact, A a closed subset of X, and C a closed, convex subset of a Banach space Y, then every continuous $g\colon A \to C$ can be extended to a continuous $f\colon X \to C$.*

THEOREM B. [3, Theorem 3.2], [4, Theorem 1]. *If X is paracompact, Y a Banach space, and $\mathscr{C}(Y)$ the family of closed, convex, non-empty subsets of Y, then every lower semi-continuous $\Phi\colon X \to \mathscr{C}(Y)$ admits a selection.*

Φ is lower *semi-continuous* if $\{x \in X \mid \Phi(x) \cap U \neq \emptyset\}$ is open in X for every open $U \subset Y$. A *selection* for Φ is a continuous $f\colon X \to Y$ such that $f(x) \in \Phi(x)$ for every $x \in X$. (No previous knowledge of continuous selections is required to read this paper, but some elementary results on lower semi-continuity from section 2 of [3] are used in the proofs.)

To see how Theorem A follows from Theorem B, let X, $A \subset X$, Y, $C \subset Y$, and $g\colon A \to C$ be as in Theorem A. Define $\psi\colon X \to \mathscr{C}(Y)$ by

$$\begin{aligned} \psi(x) &= \{g(x)\} \qquad \text{if} \quad x \in A \\ \psi(x) &= C \qquad\qquad \text{if} \quad x \in X - A; \end{aligned}$$

this ψ is lower semi-continuous [3, Example 1.3*], and a selection for ψ is the required extension of g.

To define paraconvex sets, let E be a normed linear space with metric ϱ, and let α be a number such that $0 \leqq \alpha \leqq 1$. Then a subset P of E is *α-paraconvex* if, whenever $p \in E$ and $r > 0$ are such that $\varrho(p, P) < r$, then

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$$\varrho(q, P) \leqq \alpha r \qquad \text{for all} \qquad q \in \operatorname{conv}(S_r(p) \cap P),$$

where $S_r(x)$ denotes the open r-sphere about x, and $\operatorname{conv}(A)$ denotes the convex hull of A. The set P is called *paraconvex* if it is α-paraconvex for some $\alpha < 1$. It is clear that a closed set is 0-paraconvex if and only if it is convex; in the opposite direction, V. L. Klee has shown [2] that *every* subset of E is 1-paraconvex if and only if E is either an inner-product space or two-dimensional. Since paraconvexity is not a very intuitive concept, the following examples of subsets of the euclidean plane —all of them compact, one-dimensional absolute retracts—may be helpful.

EXAMPLE 1.1. The letters **V**, **X**, **Y**, and **Z**, and a circular arc subtending an angle $<\pi$, are paraconvex. The sharper the angle of the **V**, and the closer to π the angle subtended by the arc, the closer to 1 one must take α for these sets to be α-paraconvex.

EXAMPLE 1.2. The letter **U**, and a circular arc subtending an angle $\geqq \pi$, are not paraconvex. In the case of the **U**, the midpoint p of the line segment joining the end points of the **U** violates the definition of α-paraconvexity for any $\alpha < 1$; in the case of the circular arc, the same difficulty occurs when one takes p the center of the circle.

In section 2, we apply Theorem B inductively to show (Theorem 2.1) that Theorem B remains true if "convex" is replaced by "α-paraconvex" for a fixed $\alpha < 1$; just as above, it follows (Corollary 2.2) that Theorem A remains true with "convex" replaced by "paraconvex" (and hence every closed, paraconvex subset of a Banach space is an absolute retract). It is curious to note that, while Theorem A can be proved directly, without using Theorem B and the theory of selections, the only approach to Corollary 2.2 seems to be via Theorem 2.1.

An examination of the proof of Theorem 2.1 reveals that it actually provides a method of strengthening selection theorems, in a particular way, under very general circumstances. A sort of metatheorem (Theorem 3.1) which makes this precise is given in section 3. This theorem is applicable not only to normed linear spaces but, more generally, to spaces with a "convex structure" as defined in [5]; this is done in Theorem 3.2, which generalizes Theorem 2.1.

2. The principal theorem.

THEOREM 2.1. *Let X be paracompact, Y a Banach space with metric ϱ, $\alpha < 1$, $\mathscr{P}_\alpha(Y)$ the family of closed, α-paraconvex, non-empty subsets of Y, and $\Phi: X \to \mathscr{P}_\alpha(Y)$ lower semi-continuous. Then*

(a) *There exists a selection for Φ.*

(b) *If, for some $r > 0$, there exists a continuous $g: X \to Y$ such that $\varrho(g(x), \Phi(x)) < r$ for all $x \in X$, then there exists a selection f for Φ such that $\varrho(g(x), f(x)) < \hat{\alpha} r$, where $\hat{\alpha} = 1 + \sum_{i=0}^{\infty} \alpha^i$.*

The proof of Theorem 2.1 depends on the following result, which is equivalent to Theorem B. We denote the family of non-empty subsets of Y by 2^Y.

THEOREM B′. [3, Footnote 7]. *Let X be paracompact, Y a Banach space, and $\Phi: X \to 2^Y$ lower semi-continuous. Then there exists a continuous $f: X \to Y$ such that*

$$f(x) \in \left(\operatorname{conv}(\Phi(x))\right)^-$$

for every $x \in X$.

Theorem B′ obviously implies Theorem B, and the converse follows from the fact [3, Propositions 2.3 and 2.6] that the lower semi-continuity of Φ implies that of ψ defined by $\psi(x) = \Phi(x)^-$.

PROOF OF THEOREM 2.1. We first prove (b), and then (a).

b) Pick $\gamma > \alpha$ such that $\sum_{i=0}^{\infty} \gamma^i < \hat{\alpha}$. By induction, we shall define a sequence of continuous functions $f_n: X \to Y$, $n = 0, 1, \ldots$, with $f_0 = g$, such that, for all n and all $x \in X$,

(1) $\varrho(f_n(x), \Phi(x)) < \gamma^n r$,

(2) $\varrho(f_n(x), f_{n+1}(x)) \leqq \gamma^n r$.

This will be sufficient, for by (2) this sequence of functions is uniformly Cauchy, and hence has a continuous limit f. This f is a selection for Φ by (1), and $\varrho(g(x), f(x)) < \hat{\alpha} r$ by (2).

Let $f_0 = g$. Suppose $f_1, \ldots, f_n$ have been constructed, and let us construct f_{n+1}. Define $\Phi_{n+1}: X \to 2^Y$ by

$$\Phi_{n+1}(x) = S_{\gamma^n r}(f_n(x)) \cap \Phi(x);$$

then $\Phi_{n+1}(x)$ is never empty (by the inductive assumption on f_n) and Φ_{n+1} is lower semi-continuous by [3, Proposition 2.5]. Hence, by Theorem B′, there exists a continuous $f_{n+1}: X \to Y$ such that

$$f_{n+1}(x) \in \left(\operatorname{conv}(\Phi_{n+1}(x))\right)^-$$

for every $x \in X$. This f_{n+1} clearly satisfies (2), and it satisfies (1) because each $\Phi(x)$ is α-paraconvex, whence

$$\varrho(f_{n+1}(x), \Phi(x)) \leqq \alpha \gamma^n r < \gamma^{n+1} r$$

for all $x \in X$.

(a) Pick $\lambda \geqq 2$ such that $\Phi(x) \cap S_\lambda(0) \neq \emptyset$ for some $x \in X$, and let $\beta = \max(\alpha, \lambda)$. For each positive integer n, let

$$U_n = \{x \in X \mid \Phi(x) \cap S_{\beta^n}(0) \neq \emptyset\}.$$

Then each U_n is open, since Φ is lower semi-continuous. Hence $\{U_n\}_{n=1}^\infty$ is an open covering of the paracompact space X, and thus has a locally finite closed refinement $\{A_n\}_{n=1}^\infty$, with $A_n \subset U_n$ and $A_n \subset A_{n+1}$ for all n. By induction, we shall define for each n a selection f_n for $\Phi \mid A_n$, such that always $f_{n+1} \mid A_n = f_n$. This will be sufficient, for then the function $f: X \to Y$, defined by

$$f(x) = f_n(x), \qquad x \in A_n,$$

is a selection for Φ.

We shall now define functions f_n satisfying the above requirements and, to keep the induction going, we shall also require that, for all n,

$$\varrho(f_n(x), 0) < \beta^{n+1}, \qquad x \in A_n. \tag{3}$$

The existence of a suitable f_1 follows from part (b), with X replaced by A_1, r by β, and with $g(x) = 0$ for all $x \in X$. Suppose now that $f_1, \ldots, f_n$ have been properly defined, and let us construct f_{n+1}.

Define $\Phi_{n+1}: A_{n+1} \to \mathscr{P}_\alpha(Y)$ by

$$\begin{aligned} \Phi_{n+1}(x) &= \{f_n(x)\} && \text{if} \quad x \in A_n \\ \Phi_{n+1}(x) &= \Phi(x) && \text{if} \quad x \in A_{n+1} - A_n, \end{aligned}$$

and note that Φ_{n+1} is lower semi-continuous by [3, Example 1.3*]. We can therefore apply part (b) of our theorem, with X replaced by A_{n+1}, Φ by Φ_{n+1}, r by β^{n+1}, and with $g(x) = 0$ for all x, to obtain a selection f_{n+1} for Φ_{n+1} such that

$$\varrho(f_{n+1}(x), 0) < \hat{\alpha}\beta^{n+1} \leqq \beta^{n+2}$$

for all $x \in A_{n+1}$. This f_{n+1} satisfies all our requirements.

Corollary 2.2. *If X is paracompact, A a closed subset of X, and P a closed, paraconvex subset of a Banach space, then every continuous $g: A \to P$ can be extended to a continuous $f: X \to P$.*

Proof. This goes just like the proof that Theorem B implies Theorem A in footnote 3 of [3].

3. A generalization. Let Y be a complete metric space, and $\mathscr{B}$ a hereditary family of subsets of Y. (This means that if $B \in \mathscr{B}$ and $B' \subset B$, then $B' \in \mathscr{B}$.) Let $\varkappa: \mathscr{B} \to 2^Y$ be a function such that

(a) $\varkappa(B) \supset B$ for all $B \in \mathscr{B}$,

(b) If $B \in \mathscr{B}$ and $B \subset S_r(p)$ for some $p \in Y$ and $r > 0$, then $\varkappa(B) \subset S_r(p)$. In this situation, a set $B \in \mathscr{B}$ is called *convex* if $\varkappa(B) = B$; similarly, we can define *α-paraconvex* and *paraconvex* for members of $\mathscr{B}$ just as in section 1, by simply replacing conv(B) by $\varkappa(B)$. The fact that Theorem B' implies Theorem 2.1 now immediately generalizes to our present situation as follows.

THEOREM 3.1. *Let Y, $\mathscr{B} \subset 2^Y$, and $\varkappa : \mathscr{B} \to 2^Y$ be as above, let X be paracompact, and suppose that, for every closed $X' \subset X$ and every lower semi-continuous $\Phi : X' \to \mathscr{B}$, there exists a continuous $f : X' \to Y$ such that $f(x) \in \big(\varkappa(f(x))\big)^-$ for every $x \in X$. Then every lower semi-continuous $\Phi : X \to \mathscr{P}_\alpha(Y)$ admits a selection, where $\mathscr{P}_\alpha$ denotes the family of closed, α-paraconvex, non-empty subsets of Y ($\alpha < 1$).*

PROOF. The proof goes exactly like the proof of Theorem 2.1, and can therefore be omitted.

Theorem 3.1 clearly shows that Theorem B′ implies Theorem 2.1. Another application is to strengthen [5, Theorem 1.3] as follows. Using the terminology of [5], let Y be a complete metric space with a convex structure, let $\mathscr{B}$ be the family of admissible subsets of Y, and let $\varkappa(B) = \mathrm{conv}(B)$ for all $B \in \mathscr{B}$. Let us also specifically assume that condition (b) at the beginning of this section is satisfied. Then [5, Theorem 1.3] and Theorem 3.1 together yield the following result, which generalizes Theorem 2.1.

THEOREM 3.2. *Let Y be as above, and let X be paracompact. Then every lower semi-continuous $\Phi : X \to \mathscr{P}_\alpha(Y)$ admits a selection.*

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