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NOTE ON THE PARITY OF THE PARTITION FUNCTION

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Let $p(n)$ denote the partition function, with $p(0)=1$. Many congruence properties of $p(n)$ have been found for the moduli 5, 7, 11 and 13. For the moduli 2 and 3 however, very little is known. In this note we wish to point out a simple result modulo 2, and related results for the moduli 10 and 14.

THEOREM 1. *$p(n)$ takes both even and odd values infinitely often.*

PROOF. We have (Euler)

$$p(n) - p(n-1) - p(n-2) + p(n-5) + \ldots = 0\,, \tag{1}$$

where the general term is given by

$$(-1)^k p\big(n-\tfrac{1}{2}k(3k\pm 1)\big)\,.$$

Now, suppose that $p(n)\equiv 0 \pmod 2$ for all $n \geqq a$. With $n=\frac{1}{2}a(3a-1)$, formula (1) yields

$$p\big(\tfrac{1}{2}a(3a-1)\big) - p\big(\tfrac{1}{2}a(3a-1)-1\big) - \ldots \pm p(2a-1) \mp p(0) = 0\,,$$

a contradiction (mod 2). Hence $p(n)$ takes odd values infinitely often.

On the other hand, suppose that $p(n)\equiv 1 \pmod 2$ for all $n \geqq b$. We then obtain a contradiction by taking $n=\frac{1}{2}b(3b+1)$ in (1), since the left-hand side contains an odd number of odd terms.

If we apply the same method to an arbitrary modulus, we obtain only the following result: For $q>1$, r arbitrary, there are infinitely many n that make $p(n)\not\equiv r \pmod q$.

By means of the Ramanujan identities however, we can prove the following theorem:

THEOREM 2. *Each of the congruences* $p(n)\equiv 0 \pmod{10}$, $p(n)\equiv 5 \pmod{10}$, $p(n)\equiv 0 \pmod{14}$, *and* $p(n)\equiv 7 \pmod{14}$ *has infinitely many solutions.*

PROOF. We have $p(5n+4)\equiv 0 \pmod 5$ and $p(7n+5)\equiv 0 \pmod 7$; there-

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fore it is sufficient to prove that $p(5n+4)$ and $p(7n+5)$ take both even and odd values infinitely often. Let

$$\varphi(x) = \prod_{n=1}^{\infty} (1-x^n) .$$

The first Ramanujan identity is:

$$\sum_{n=0}^{\infty} p(5n+4)x^n = 5\varphi(x)^{-6}\varphi(x^5)^5 .$$

We have

$$\varphi(x)^6 \equiv \varphi(x^2)^3 \equiv \sum_{n=0}^{\infty} x^{n(n+1)} \pmod 2 ,$$

by Jacobi's identity

$$\varphi(x)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) x^{\frac{1}{2}n(n+1)} .$$

Hence, with

$$\varphi(x^5)^5 = \sum_{n=0}^{\infty} A(n) x^n$$

we obtain

$$p(5n+4) + p(5n-6) + p(5n-26) + \ldots \equiv A(n) \pmod 2 ,$$

where the general term is $p\big(5n-5k(k+1)+4\big)$. We proceed as in the proof of theorem 1, but we have to choose a value of n not divisible by 5, to ensure that $A(n)=0$.

To prove the second part of theorem 2, we use the other Ramanujan identity, viz.

$$\sum_{n=0}^{\infty} p(7n+5)x^n = 7\varphi(x)^{-4}\varphi(x^7)^3 + 49x\varphi(x)^{-8}\varphi(x^7)^7 .$$

It follows that

$$\varphi(x^8)\sum_{n=0}^{\infty} p(7n+5)x^n \equiv \sum_{n=0}^{\infty} B(n)x^n \pmod 2 ,$$

with

$$\varphi(x^4)\varphi(x^7)^3 + x\varphi(x^7)^7 = \sum_{n=0}^{\infty} B(n)x^n .$$

By Euler's identity for $\varphi(x)$ we have

$$\varphi(x^4) = \sum_{n=-\infty}^{\infty} (-1)^n x^{2n(3n+1)} .$$

Hence

$$B(7n+2) = B(7n+3) = B(7n+5) = 0 ,$$

and the proof is completed as before.

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