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CONTINUOUS CONVEX SETS

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Introduction. The compact convex sets in a finite-dimensional Euclidean space E have a number of familiar properties which are not shared by closed convex sets in general. For example if X and Y are compact and convex then

(1) so is $X+Y$;
(2) so is $\operatorname{conv}(X \cup Y)$;
(3) if X and Y are disjoint they can be separated by a hyperplane at a positive distance from each.

Well-known examples in the plane show that none of these statements is true for general closed convex sets.

In § 1 below we show that there is a natural maximal class of closed convex sets satisfying (1), (2) and (3) above which includes the compact convex sets as a proper subclass. These are what we have chosen to call the *continuous* sets, that is, those closed convex sets whose support functions are continuous. The support function σ of the set X is defined by

$$\sigma(u) = \sup_{x \in X} (x, u) \quad \text{for each unit vector } u ,$$

where (x, u) denotes the scalar product. Since the sets X are possibly unbounded we permit $\sigma(u)$ to assume the value "plus infinity". (Continuity at u for which $\sigma(u)$ is infinity is defined in the usual way. We define $\sigma(u)$ only for unit vectors u, since otherwise σ is discontinuous at the origin whenever X is unbounded.)

The simplest example of a non-continuous closed convex set is a ray in the plane, or, less trivially, the area enclosed by a branch of a hyperbola. An example of a continuous but unbounded set is the area enclosed by a parabola in the plane. In the next section we show that continuous convex sets are alternatively characterized by the fact that they possess neither a boundary ray nor an asymptote.

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In § 2 we consider not merely pairs of continuous convex sets but families of arbitrary cardinality, and establish results which reduce for a pair of sets to the above statements about the convex hull and about separation. Let $\{X_s: s\in S\}$ be a family of continuous convex subsets of E. Then the theorem on convex hulls (2.3) asserts that for each point

$$p \in \operatorname{cl}\operatorname{conv} \bigcup_{s\in S} X_s$$

there are points $x_s \in X_s$ such that

$$p \in \operatorname{cl}\operatorname{conv}\{x_s: s\in S\};$$

and the general separation theorem (2.8) implies that if $\bigcap_{s\in S} X_s = \emptyset$, then there are closed halfspaces $Q_s \supset X_s$ with $\bigcap_{s\in S} Q_s = \emptyset$. These results have not previously been published even for compact convex sets, though the generalized separation theorem for compact convex sets and a deeper separation theorem for open convex sets (2.7 of the present paper) were announced in the abstract [1].

NOTATION AND TERMINOLOGY. By a *boundary ray* of a set we mean a ray (i.e., half-line) which is contained in the boundary of the set. An *asymptote* of a set X is a ray $\varrho \subset E \sim X$ such that

$$\delta(X, \varrho) = 0, \quad \text{where} \quad \delta(X, Y) \equiv \inf\{\|x-y\|: x\in X,\ y\in Y\}.$$

By the *cone from p over X* we mean the set $p + [0, \infty[(X-p)$. We denote the origin of E by θ, the empty set by $\emptyset$. The remainder of our notation and terminology is more or less "standard".

1. Pairs of sets. In this section and the next, E will always denote a finite-dimensional Euclidean space. We begin with two simple but useful remarks.

1.1. LEMMA. *Suppose C is a closed convex subset of E, $p\in C$, $q\in E\sim\{\theta\}$, and there are sequences x_α in C and t_α in $]0, \infty[$ such that $t_\alpha \to 0$ and $t_\alpha x_\alpha \to q$. Then C contains the ray $p+[0, \infty[q$.*

PROOF. Consider an arbitrary $r>0$. Then for all $t_i \leqq 1/r$, we have $(1-rt_i)p + rt_i x_i \in C$, whence $p+rq\in C$.

1.2. LEMMA. *Suppose C is a closed convex subset of E, ϱ is a ray emanating from θ, and x and y are points such that $x+\varrho \subset E\sim C$ and $y+\varrho\subset C$. Then for some $u\in[x, y]$, the ray $u+\varrho$ is a boundary ray or an asymptote of C.*

PROOF. Let t be the greatest lower bound of numbers $r \geqq 0$ such that the ray $(1-r)x+ry+\varrho$ is at zero distance from C; let $u=(1-t)x+ty$. Then the ray $u+\varrho$ is at zero distance from C, and by 1.1 it must contain a boundary ray if it intersects C.

Now for some characterizations of continuous convex sets:

1.3. THEOREM. *For a closed convex subset X of E the following assertions are equivalent*:

(i) *X is continuous*;
(ii) *X has no boundary ray or asymptote*;
(iii) *for each $p \in E$, the convex hull of $X \cup \{p\}$ is closed*;
(iv) *for each $p \in E \sim X$, the cone from p over X is closed*;
(v) *for each closed convex $Y \subset E \sim X$, $\delta(X, Y) > 0$*;
(vi) *for each closed convex $Y \subset E \sim X$, X and Y can be separated by a hyperplane H with $\delta(X, H) > 0 < \delta(Y, H)$*;
(vii) *for each closed convex $Y \subset E$ the set $X + Y$ is closed.*

PROOF. We first show that all the other assertions imply assertion (ii). Suppose, then, that there is a ray $\varrho = a + [0, \infty[\, b$ which is on the boundary of X or is an asymptote of X. By the standard properties of convex sets there is then a hyperplane H through ϱ such that X lies entirely on one side of H. If u is the unit normal to H directed away from X then we see that the support function σ of X is not continuous at u. Namely, $\sigma(u)$ is finite since $(x,u) \leqq (a,u)$ for all x in X, but for any positive ε, $\sigma(u+\varepsilon b)$ is infinite, for we can always choose λ so large that

$$(a+\lambda b,\ u+\varepsilon b) \ = \ (a, u+\varepsilon b) + \lambda\varepsilon \|b\|^2$$

exceeds any preassigned value. If ϱ is a boundary ray this proves the assertion and if it is an asymptote then we can choose x in X so close to ϱ that again $(x, u+\varepsilon b)$ is arbitrarily large and this shows that σ is not continuous. Next, let ϱ be a boundary ray or asymptote of X, x a relatively interior point of X, $p \in E$ such that $]p,x[$ intersects ϱ, and P the two-dimensional flat containing $\varrho \cup \,]p,x[$. It can be verified that the set

$$\text{cl conv}\big(X \cup \{p\}\big) \ \sim \ \text{cone}\,(p, X)$$

contains a translate of ϱ, and the set $P \sim X$ contains a closed convex set Y of which ϱ is an asymptote. Then of course (v) and (vi) fail, and the set $X + (-Y)$ is not closed. The proof of 1.3 is completed by establishing the following implications.

(ii) $\Rightarrow$ (i). Suppose the support function σ of X is discontinuous at the point u. Let $\sigma(u) = a$ and let u_α be a sequence converging to u for which

$\sigma(u_\alpha)$ does not converge to a. We first observe that $\liminf \sigma(u_\alpha) \geqq a$ for if $b<a$ then by definition of σ there exists $x \in X$ such that $(x,u)>b$. But if $u_\alpha \to u$ then $(x,u_\alpha) \to (x,u)$ and hence $\liminf \sigma(u_\alpha)>b$, proving the assertion. Therefore, if u is to be a point of discontinuity of σ we must have

$$\limsup \sigma(u_\alpha) = c > a .$$

This means we can choose a subsequence $u_\alpha{}'$ of u_α and vectors $x_i \in X$ such that $\lim(x_\alpha, u_\alpha{}')=c$. If $c<\infty$, let $x_i{}'=x_i$ and $d=c$. If c is infinity we can find a new sequence of vectors $x_i{}'$ in X such that

$$\lim(x_\alpha{}', u_\alpha{}') = d \quad \text{where} \quad a < d < \infty .$$

This is done by letting x be any point in X. Then for large i, $(x,u_i{}')<d$ and $(x_i,u_i{}')>d$; hence an appropriate convex combination of x and x_i will produce $x_i{}'$ such that $(x_i{}',u_i{}')=d$. Now the sequence $x_\alpha{}'$ can have no cluster point x'; for if so we would have $(x',u)=d$ whereas

$$\sup_{x \in X}(x,u) = a < d .$$

It follows that the sequence $\|x_\alpha{}'\|$ approaches infinity so that $1/\|x_\alpha{}'\| \to 0$. Letting $\bar{x}$ be a cluster point of $x_\alpha{}'/\|x_\alpha{}'\|$, we see from Lemma 1.1 that X contains a ray parallel to the vector $\bar{x}$. Also, since $(x_\alpha{}',u_\alpha{}') \to d < \infty$ it follows that $(x_\alpha{}'/\|x_\alpha{}'\|, u_\alpha{}') \to 0$ and hence $(\bar{x},u)=0$. If we then choose any point y such that $(y,u)>a$ then $(y+\lambda\bar{x},u)>a$ for all numbers λ so the ray $y+[0,\infty[\bar{x}$ does not meet X. We now apply Lemma 1.2 which shows that X contains a boundary ray or an asymptote parallel to $\bar{x}$.

(ii) $\Rightarrow$ (iii). It suffices to prove (assuming i) that if $\theta \notin X$ and $q \in \mathrm{cl}\,]0,1]X \sim \{\theta\}$, then $[1,\infty[q$ intersects X. Now there must be sequences t_α in $]0,1]$ and x_α in X such that

$$t_\alpha x_\alpha \to q \quad \text{and} \quad t_\alpha \to t \in [0,1] .$$

The case $t>0$ is easily handled, so suppose $t=0$ and let $p \in X$. We see by 1.1 that $p+[0,\infty[q \subset X$, so if $[1,\infty[q$ misses X it follows by 1.2 that some translate of $[0,\infty[q$ is a boundary ray or asymptote of X, contradicting (ii).

(iii) $\Rightarrow$ (iv). It suffices to consider the case $p=\theta$, for which the cone in question is the set

$$[0,\infty[X = [0,1]X \cup [1,\infty[X .$$

The first summand is closed by (iii), and with X closed and $\theta \notin X$, the second summand must also be closed.

(ii) $\Rightarrow$ (v). If $\delta(X,Y)=0$, there must be sequences x_α in X and y_α in Y such that $x_\alpha - y_\alpha \to \theta$, $\|x_\alpha\| \to \infty$, and $x_\alpha/\|x_\alpha\| \to q \in E \sim \{\theta\}$. Then of course $y_\alpha/\|x_\alpha\| \to q$. Thus for arbitrary $u \in X$, $v \in Y$, it follows by 1.1 that

$$u+[0,\infty[q\subset X \quad \text{and} \quad v+[0,\infty[q\subset Y ,$$

then by 1.2 that some translate of $[0,\infty[q$ is a boundary ray or asymptote of X. But this contradicts (ii).

(v) $\Rightarrow$ (vi). This follows from the basic separation theorem for convex sets.

(v) $\Rightarrow$ (vii). It suffices to observe that if $\theta\in \mathrm{cl}(X+Y)$, then $\delta(X, -Y)=0$, whence, by (iv), X must intersect $-Y$ and consequently $\theta\in X+Y$.

1.4. COROLLARY. *Suppose X and Y are closed convex subsets of E and at least one of them is continuous. Then there are points $x\in X$ and $y\in Y$ such that $\|x-y\|=\delta(X,Y)$.*

PROOF. Observe that $\delta(X,Y)=\delta\big(X-Y,\{\theta\}\big)$, and the set $X-Y$ is closed by 1.3 (vii).

1.5. THEOREM. *If the subsets X and Y of E are continuous, then so are the sets $X\cap Y$, $\mathrm{conv}(X\cup Y)$, and $X+Y$.*

PROOF. It is evident that an asymptote of $X\cap Y$ must be an asymptote of X or of Y; similarly for boundary rays, and the set $X\cap Y$ is disposed of. Now in 2.4 below it is stated that if $X_1,\ldots,X_k$ are continuous subsets of E, then the set conv $\bigcup_1^k X_i$ is closed. Under present hypotheses, this implies that $\mathrm{conv}(X\cup Y)$ is closed. If σ and τ are the support functions of X and Y respectively then since σ and τ are continuous so also is $\varrho=\max[\sigma,\tau]$. On the other hand, by standard properties of support functions ϱ is the support function of the set

$$Z = \mathrm{cl\,conv}(X\cup Y) = \mathrm{conv}(X\cup Y) ,$$

so Z is continuous.

Finally, if X and Y are continuous we note that $X+Y$ is closed by 1.3 (vii). Since the support functions σ and τ of X and Y are continuous so also is their sum $\sigma+\tau$ ($\infty+a=\infty$ for all a). But by standard properties, $\sigma+\tau$ is the support function of the set

$$Z = \mathrm{cl}(X+Y) = X+Y ,$$

so Z is continuous.

We end this section with one more result involving the convex hull of the union of two sets.

1.6 PROPOSITION. *Let $\mathfrak{C}$ denote the class of all closed convex subsets of E, $\mathfrak{R}$ the class of all members of $\mathfrak{C}$ which are continuous, and $\mathfrak{H}$ the class of all members of $\mathfrak{C}$ which contain a hyperplane. For each $X\in\mathfrak{C}$, let hX*

denote the class of all $Y \in \mathfrak{C}$ *such that* $\operatorname{conv}(X \cup Y')$ *is closed for each nonsingular affine image* Y' *of* Y. *Then* $hX = \mathfrak{R}$ *for* $X \in \mathfrak{R}$, $hX = \mathfrak{H}$ *for* $X \in \mathfrak{H}$, *and* hX *is empty for* $X \in \mathfrak{C} \sim (\mathfrak{R} \cup \mathfrak{H})$.

PROOF. We require the following fact F: if $U \in \mathfrak{C} \sim \mathfrak{R}$ and $V \in \mathfrak{C} \sim \mathfrak{H}$, there is a nonsingular affine image U' of U such that the set $\operatorname{conv}(U' \cup V)$ is not closed. To prove F, let ϱ be a boundary ray or an asymptote of U and K_1 a halfspace with bounding hyperplane H_1 such that $K_1 \supset U$ and $H_1 \supset \varrho$. Let K_2 be a halfspace with bounding hyperplane H_2 such that $K_2 \supset V$ and H_2 intersects V. Since $H_2 \cap V$ is a convex proper subset of H_2, there must be in H_2 a ray ϱ_2 of which no translate lies in $H_2 \cap V$. Now let τ be an isometry of E onto E such that τK_1 is interior to K_2 (whence of course τH_1 is a translate of H_2) and $\tau \varrho_1$ is parallel to ϱ_2. Then the set

$$H_2 \cap \operatorname{cl} \operatorname{conv}(\tau U \cap V)$$

contains a translate of ϱ_2, while

$$H_2 \cap \operatorname{conv}(\tau U \cap V) = H_2 \cap V .$$

This completes the proof of F.

Now it follows from 1.5 that $hX \supset \mathfrak{R}$ if $X \in \mathfrak{R}$, and it is easily verified that $hX \supset \mathfrak{H}$ if $X \in \mathfrak{H}$. In conjunction with F, these facts yield the desired conclusions.

2. Families of arbitrary cardinality.

Theorem 2.1 below is our principal result on convex hulls, while 2.2 will lead by duality to the main separation theorem.

2.1. THEOREM. *Suppose* $\{X_s : s \in S\}$ *is a family of closed convex subsets of* E *and* p *is a point of* $\operatorname{cl} \operatorname{conv} \bigcup_{s \in S} X_s$ *such that for each* $s \in S$ *the set* $\operatorname{conv}(X_s \cup \{p\})$ *is closed. Then there are points* $x_s \in X_s$ *such that* $p \in \operatorname{cl} \operatorname{conv} \{x_s : s \in S\}$.

2.2. THEOREM. *Suppose* $\{X_s : s \in S\}$ *is a family of closed convex subsets of* E *with* $\theta \in \bigcap_{s \in S} X_s$, *and* p *is a point of* $\operatorname{cl} \operatorname{conv} \bigcup_{s \in S} X_s$. *Then at least one of the following is true*:

(i) *there are points* $x_s \in X_s$ *and* $s_0 \in S$ *such that*

$$\theta \neq -x_{s_0} \in \operatorname{cl} \operatorname{conv} \{x_s : s \in S \sim \{s_0\}\};$$

(ii) *for each* $r \in \,]0, 1[$ *there are points* $x_s \in X_s$ *such that*

$$rp \in \operatorname{cl} \operatorname{conv} \{x_s : s \in S\}.$$

PROOFS. It is convenient to begin the proofs of 2.1 and 2.2 together, and we may assume that $p \notin \bigcup_{s \in S} X_s$. Let n denote the dimension of E. Caratheodory's theorem on convex hulls implies the existence (for all $s \in S$ and $i \in I$, the set of positive integers) of points $x_s^i \in X_s$ and numbers $\lambda_s^i \geqq 0$ which satisfy the following conditions:

for each i, λ_s^i differs from zero for at most $n+1$ choices of $s \in S$;

$$\sum_{s \in S} \lambda_s^i = 1;$$

$$\lim_{i \to \infty} \sum_{s \in S} \lambda_s^i x_s^i = p .$$

We assume without loss of generality that each sequence $x_s^\alpha / \|x_s^\alpha\|$ is convergent to a point of E, and the sequences $\|x_s^\alpha\|$ and $\lambda_s^\alpha \|x_s^\alpha\|$ are convergent to points of $[0, \infty]$.

Now for $s \in S$ and $i \in I$, we say that s *appears at* i provided $\lambda_s^i \neq 0$. If there are elements of S which appear at i for infinitely many values of i, pick one such element, call it s_1, and let $n_1(\alpha)$ be the sequence of all i's at which s_1 appears. Then if there are elements of $S \sim \{s_1\}$ which appear at $n_1(i)$ for infinitely many values of i, pick one such, call it s_2, and let $n_2(\alpha)$ be the sequence of all numbers $n_1(i)$ at which s_2 appears. Proceeding in this way, we arrive at m elements $s_1, \ldots, s_m$ of S (with $0 \leqq m \leqq n+1$) and an increasing sequence $n_m(\alpha)$ in I such that s_j appears at $n_m(i)$ for all $i \in I$, $1 \leqq j \leqq m$, while each element of $S \sim \{s_1, \ldots, s_m\}$ appears at $n_m(i)$ for at most finitely many values of i.

An appropriately chosen subsequence $k(\alpha)$ of $n_m(\alpha)$ will be such that each element of $S \sim \{s_1, \ldots, s_m\}$ appears at $n_m(i)$ for at most one value of i, and for none at all if the set of all $s \in S$ which appear for at least one i is finite. By enumerating as $s_{m+1}, s_{m+2}, \ldots,$ the additional members of S which do appear, letting $X_i = X_{s_i}$ for $i = 1, 2, \ldots,$ and selecting other notation in an obvious way, we obtain double sequences t_j^i of numbers and x_j^i of points such that the following conditions are all satisfied:

i ranges over I, j over I or over $\{1, 2, \ldots, m\}$;
always $t_j^i \geqq 0$;
for each i, there are at most $n+1$ values of j for which $t_j^i \neq 0$, and $\sum_j t_j^i = 1$;
for $1 \leqq j \leqq m$, t_j^i differs from 0 for all i;
for $j > m$, t_j^i differs from 0 for exactly one value of i (call it $i(j)$);
always $x_j^i \in X_j$;
$\lim_{i \to \infty} \sum_j t_j^i x_j^i = p$;
for each j, there exist $\lim_{i \to \infty} x_j^i / \|x_j^i\| \in E$, $\lim_{i \to \infty} \|x_j^i\| \in [0, \infty]$, and $\lim_{i \to \infty} t_j^i \|x_j^i\| \in [0, \infty]$.

Now the proofs diverge. We first consider 2.1, for which we may and shall assume $p=\theta$. For $j>m$, let $x_j=x_j^{i(j)}$. For $1\leqq j\leqq m$, let $x_j=\lim_{i\to\infty}x_j^i\in X_j$ if this limit exists in E; otherwise $\lim_{i\to\infty}\|x_j^i\|=\infty$ and we let x_j be the smallest positive multiple of $\lim_{i\to\infty}x_j^i/\|x_j^i\|$ which lies in X_j. (Since $[0,\infty[(X_j-p)$ is closed and $\theta=p\notin X_j$, such a multiple must exist.) We claim that

$$\theta\in\text{cl conv}\{x_j\}\,.$$

Suppose the contrary, whence E admits a linear functional f with $\inf_j fx_j=\varepsilon>0$. Since $\lim_{i\to\infty}\sum_j t_j^i x_j^i=\theta$, for arbitrarily large values of i there must exist j's for which $t_j^i\neq 0$ and $f(x_j^i)<\varepsilon/2$. This cannot occur with $j>m$, for then $x_j^i=x_j^{i(j)}=x_j$. Thus for some j between 1 and m it is true that $f(x_j^i)<\varepsilon/2$ for arbitrarily large values of i. When $\lim_{i\to\infty}x_j^i$ exists, this is impossible, for it implies that $f(x_j)\leqq\frac{1}{2}\varepsilon$. And when $\lim_{i\to\infty}\|x_j^i\|=\infty$, we see that $\lim_{i\to\infty}f(x_j^i/\|x_j^i\|)\leqq 0$, whence $fx_j\leqq 0$. Since this also is impossible, the proof of 2.1 is complete.

We turn now to the proof of 2.2, and set $x_{m+1}=\theta\in X_{m+1}$, and $x_j=x_j^{i(j)}\in X_j$ for all $j\geqq m+2$. Observe that $t_{m+1}{}^i=0$ for all sufficiently large i, and hence

$$(*)\qquad \sum_{j=1}^{m}t_j^\alpha x_j^\alpha+\sum_{j=m+2}^{\infty}t_j^\alpha x_j\ \to\ p\,.$$

Now one of the following two situations must arise:

(a) there are an increasing sequence $u(\alpha)$ and a j_0 between 1 and m such that $\|t_{j_0}{}^{u(\alpha)}x_{j_0}{}^{u(\alpha)}\|=\eta_\alpha\to\infty$, and $\|t_j^{u(i)}x_j^{u(i)}\|\leqq\eta_i$ for all $i\in I$, $1\leqq j\leqq m$;

(b) there is an increasing sequence $v(\alpha)$ such that for each j, $t_j^{v(\alpha)}x_j^{v(\alpha)}$ converges to a point of E.

In considering case (a), we assume without loss of generality that $j_0=1$. For $1\leqq j\leqq m$, the sequence $\eta_\alpha^{-1}t_j^{u(\alpha)}x_j^{u(\alpha)}$ must converge to a point $x_j\in X_j$, and of course $\|x_1\|=1$. Now observe that, by (*),

$$\sum_{j=2}^{m}x_j+\left(1-\sum_{j=m+2}^{\infty}\eta_\alpha t_j^{u(\alpha)}\right)x_{m+1}+\sum_{j=m+2}^{\infty}\eta_\alpha t_j^{u(\alpha)}x_j\ \to\ -x_1\,,$$

whence $-m^{-1}x_1\in\text{cl conv}\{x_j\colon j\geqq 2\}$. But $m^{-1}x_1\in X_1\sim\{\theta\}$, so condition (i) of 2.2 is satisfied.

Under case (b), let J be the set of all j between 1 and m for which $\lim_{i\to\infty}x_j^{v(i)}$ exists in X_j, and for each such j let that limit be denoted by x_j. Let K denote the set $\{1,\ldots,m\}\sim J$, and observe that for $j\in K$, $\|x_j^{v(\alpha)}\|\to\infty$ but $t_j^{v(\alpha)}\to 0$ and $t_j^{v(\alpha)}x_j^{v(\alpha)}$ converges to a point $y_j\in X_j$. By 1.1, $[0,\infty[y_j\subset X_j$ for each $j\in K$. Now consider an arbitrary $r\in]0,1[$ and for each $j\in K$ let

$$x_j=\frac{rk}{1-r}y_j\in X_j\,,$$

where k is the cardinality of K. Then from (*) it follows that

$$\sum_{j\in J} rt_j^{\alpha}x_j + \sum_{j\in K}\frac{1-r}{k}x_j + \left(r\sum_{j\in K}t_j^{\alpha}\right)x_{m+1} + \sum_{j=m+2}^{\infty} rt_j^{\alpha}x_j \to rp\,,$$

and since always the sum of the coefficients is 1, it follows that $rp \in \operatorname{conv}\{x_j\}$, completing the proof of 2.2.

From 2.1 and 1.3 (iii), we obtain the result stated in the Introduction:

2.3. COROLLARY. *Suppose $\{X_s\colon s \in S\}$ is a family of continuous subsets of E. Then for each*

$$p \in \operatorname{cl}\operatorname{conv}\bigcup_{s\in S} X_s\,,$$

there are points $x_s \in X_s$ such that

$$p \in \operatorname{cl}\operatorname{conv}\{x_s\colon s \in S\}\,.$$

Specializing to the case of a finite S, we have

2.4. COROLLARY. *If $X_1, X_2, \ldots, X_k$ are continuous subsets of E, then the set* $\operatorname{conv}\bigcup_1^k X_i$ *is closed.*

From 2.2 we deduce

2.5. COROLLARY. *Suppose $\{X_s\colon s \in S\}$ is a family of closed convex sets in E, $\theta \in \bigcap_{s\in S} X_s$, and there is an open halfspace $Q \subset E \sim \{\theta\}$ such that for each s, $X_s \sim \{\theta\} \subset Q$. Then if*

$$tp \in \operatorname{cl}\operatorname{conv}\bigcup_{s\in S} X_s$$

for some $t > 1$, there are points $x_s \in X_s$ such that

$$p \in \operatorname{cl}\operatorname{conv}\{x_s\colon s \in S\}\,.$$

2.6. COROLLARY. *Suppose $\{X_s\colon s \in S\}$ is a family of closed convex cones in E such that $\bigcup_{s\in S} X_s \sim \{\theta\}$ is not contained in any open halfspace in $E \sim \{\theta\}$* and no X_s contains a line. *Then there are points $x_s \in X_s$ and $s_0 \in S$ such that*

$$\theta \neq -x_{s_0} \in \operatorname{cl}\operatorname{conv}\left\{x_s\colon s \in S \sim \{s_0\}\right\}.$$

PROOF. Suppose the desired conclusion fails, and let $C = \operatorname{conv}\bigcup_{s\in S} X_s$. Since C is a convex cone not contained in any open halfspace in $E \sim \{\theta\}$, it follows by a known support theorem that the set $(\operatorname{cl} C) \cap -C$ includes a point $z \neq \theta$. Now of course there are elements $s_0, \ldots, s_m$ of S and points $y_k \in X_{s_k} \sim \{\theta\}$ such that $-z = \Sigma_0^m y_k$. And by 2.2 there are elements $s_{m+1}, s_{m+2}, \ldots,$ of S (a finite or infinite sequence) and points $u_j \in X_{s_j}$ $(j = 0, 1, \ldots, m, m+1, m+2, \ldots)$ such that $z \in \operatorname{cl}\operatorname{conv}\{u_j\}$. Thus there is a double sequence t_j^i of non-negative numbers such that for $j \leqq m$, t_j^i is independent of i (say $t_j^i = t_j$), and for each i it is true that $\Sigma_j t_j^i = 1$, and $\lim_{i\to\infty} \Sigma_j t_j^i u_j = z$.

Now let

$$x_0 = y_0 + t_0 u_0 \in X_{s_0} \sim \{\theta\} ,$$
$$x_j = \frac{2m}{1+\sigma}(y_j + t_j u_j) \in X_{s_j} \quad \text{for} \quad 1 \leqq j \leqq m$$
$$x_j = 2u_j \qquad \text{for} \qquad j > m ,$$

where $\sigma = \Sigma_0^m t_j \leqq 1$. Then evidently

$$\sum_{j=1}^{\infty} \frac{1+\sigma}{2m} x_j + \lim_{i\to\infty} \sum_{j=m+1}^{\infty} (\tfrac{1}{2} t_j^i) x_j = -x_0 ,$$

and since $\frac{1}{2}(1+\sigma) + \Sigma_{j=m+1}^{\infty} t_j^i = 1$ for all i, it follows that

$$-x_0 \in \text{cl conv}\{x_j \colon j \geqq 1\} ,$$

completing the proof of 2.6.

Next comes the basic separation theorem for families of arbitrary cardinality.

2.7. THEOREM. *Suppose $\{Y_s \colon s \in S\}$ is a family of open convex proper subsets of E with empty intersection. Then there are open halfspaces $Q_s \supset Y_s$ such that $\bigcap_{s\in S} Q_s$ is empty.*

PROOF. By the usual device of considering E as a hyperplane in a space of one more dimension (with $\theta \notin E$), and replacing each set Y_s by the corresponding cone $]0,\infty[\,Y_s$, we reduce the problem to one concerning convex cones. Thus in proving 2.7, it suffices to consider the case in which each set Y_s is a nonempty open convex cone with $\theta \notin Y_s$. For each $s \in S$, let X_s denote the closed convex cone $\{x \in E \colon (x,y) \geqq 0$ for all $y \in Y_s\}$, which clearly contains no line. If $\bigcup_{s\in S} X_s \sim \{\theta\}$ lies in an open halfspace in $E \sim \{\theta\}$, then there exists $y \in E$ such that $(x,y) > 0$ for all $x \in \bigcup_{s\in S} X_s \sim \{\theta\}$. This is impossible, for it implies that $y \in \bigcap_{s\in S} Y_s$, and thus from 2.6 we deduce the existence of points $x_s \in X_s$ and $s_0 \in S$ such that

$$(\dagger) \qquad \theta \neq -x_{s_0} \in \text{cl conv}\{x_s \colon s \in S \sim \{s_0\}\} .$$

Let T be the set of all $s \in S$ for which $x_s \neq \theta$, and for each $t \in T$ let Q_t be the open halfspace $\{y \colon (x_t, y) > 0\}$. Then of course $Q_t \supset Y_t$, and from (†) it follows that $\bigcap_{t\in T} Q_t$ is empty, so the proof of 2.7 is complete.

2.8. THEOREM. *Suppose U is the unit cell of E and $\{X_s \colon s \in S\}$ is a* *family of continuous proper subsets of E, with $\bigcap_{s\in S}X_s=\emptyset$. Then there are numbers $\varepsilon_s>0$ and closed subspaces $Q_s\supset X_s+\varepsilon_s U$ such that $\bigcap_{s\in S}Q_s=\emptyset$.*

Proof. Since the sets X_s sre closed and have empty intersection, there is a finite subset T_1 of S such that

$$U\cap\Big(\bigcap_{t\in T_1}X_t\Big)=\emptyset\,.$$

By use of 1.3 (v) we see that if the positive number δ_1 is sufficiently small, and we define $X_s{}^1=X_s$ for $s\in S\sim T_1$, $X_s{}^1=X_s+\delta_1 U$ for $s\in T_1$, then

$$U\cap\Big(\bigcap_{t\in T_1}X_t^1\Big)=\emptyset=\bigcap_{s\in S}X_s{}^1\,.$$

Now there must be a finite subset T_2 of $S\sim T_1$ such that

$$2U\cap\Big(\bigcap_{t\in T_1\cup T_2}X_t^1\Big)=\emptyset\,,$$

and then by 1.3 (v) there must be a number $\delta_2>0$ such that with $X_s{}^2=X_s$ for $s\in S\sim T_2$ and $X_s{}^2=X_s+\delta_2 U$ for $s\in T_2$, we have

$$2U\cap\Big(\bigcap_{t\in T_1\cup T_2}X_t^2\Big)=\emptyset=\bigcap_{s\in S}X_s{}^2\,.$$

Proceeding in this manner, we obtain a sequence T_α of pairwise disjoint subsets of S and a sequence δ_α of positive numbers such that with $\delta_t=\delta_i$ for $t\in T_i$, the intersection of nU with the set $\bigcap_{t\in\bigcup_1^n T_i}(X_t+\delta_t U)$ is empty for each n. Then of course $\bigcap_{s\in S}(X_s+\delta_s U)=\Phi$, where δ_s is chosen arbitrarily (>0) for $s\in S\sim\bigcup_1^\infty T_i$. It now follows from the separation theorem 2.7 that there are open halfspaces $J_s\supset X_s+\delta_s\,(\operatorname{int}U)$, such that $\bigcap_{s\in S}J_s=\emptyset$, and with $\varepsilon_s=\frac{1}{2}\delta_s$ there exists for each s a closed halfspace Q_s such that

$$X_s+\varepsilon_s U\subset Q_s\subset J_s\,.$$

But then of course $\bigcap_{s\in S}Q_s=\emptyset$ and the proof of 2.8 is complete.

It is proved in [3] that if a closed convex subset X of E admits no boundary ray, then each closed convex set $Y\subset E\sim X$ can be separated from X by a hyperplane which misses X. By using this fact in an argument similar to that of 2.8, we can establish the following result.

2.9. Theorem. *Suppose $\{X_s : s\in S\}$ is a family of closed convex proper subsets of E, each admitting no boundary ray, and $\bigcap_{s\in S}X_s=\emptyset$. Then there are open halfspaces $Q_s\supset X_s$ such that $\bigcap_{s\in S}Q_s=\emptyset$.*

3. Extensions to infinite-dimensional spaces. In order not to obscure the basic ideas, we have thus far confined our discussion to finite-dimensional spaces. However, most of the results can be extended in some form to rather general Hausdorff linear spaces, the principal change being the explicit assumption of local compactness of the sets involved. The following fact, proved in [3], is the main tool in carrying out the extension:

(†) Suppose X is a locally compact closed convex subset of a Hausdorff linear space, $p \in X$, U is a neighborhood of p, and S is a net in $X \sim U$. Then there are a subnet x_β of S and a corresponding net t_β in $]0,1]$ (β ranging over some directed set) such that

$$t_\beta \to t \in [0,1] \quad \text{and} \quad (1-t_\beta)p + t_\beta x_\beta \to x \in X \sim \{p\} .$$

To illustrate the manner in which (†) is employed, let us show that for a locally compact closed convex subset X of a Hausdorff linear space E, absence of asymptotes or boundary rays implies that $\theta \notin \mathrm{cl}(X-Y)$ whenever Y is a closed convex subset of $E \sim X$. (In other words, condition 1.3 (ii) implies condition 1.3 (v) in this more general setting.) If $\theta \in \mathrm{cl}(X-Y)$, there must be nets x_γ in X and y_γ in Y such that $x_\gamma - y_\gamma \to \theta$. Since X and Y are closed and have no common point, neither x_γ nor y_γ admits a convergent subnet. Choosing $u \in X$ and $v \in Y$, we see by (†) that there are a subnet x_β of x_γ and a corresponding net t_β in $]0,1]$ with

$$t_\beta \to t \in [0,1] \quad \text{and} \quad (1-t_\beta)u + t_\beta x_\beta \to x \in X \sim \{u\} .$$

Since x_β cannot be convergent, it follows that $t=0$ and $t_\beta x_\beta \to q = x-u$. Also, of course, $t_\beta y_\beta \to q$, and the proof is completed by applying 1.1 and 1.2 as in the finite-dimensional case.

Most of the other results of §§ 1–2 can be extended in a similar way, insofar as they involve *finite* families of *locally compact* sets. Necessity of some sort of compactness condition is indicated by the results of [2]. And the results of § 2 depend so heavily on Caratheodory's theorem that, insofar as they involve infinite families of sets, they do not extend to infinite-dimensional spaces even if severe restrictions be placed on the individual sets of the family. For example (relative to 2.1 and 2.3), let E be the space (l^p) for $p \in [1, \infty[$, and let $\delta_i \in (l^p)$ be such that $\delta_i^{\,i} = 1$ and $\delta_i^{\,j} = 0$ for $i \neq j$. For each $i \in I$, let X_i be the segment

$$[\delta_1 + \delta_{i+1}, -(2^{i-1}-1)\delta_1] \subset E .$$

Then

$$\sum_{i=1}^{k} 2^{-i}\delta_{i+1} = \sum_{i=1}^{k} 2^{-i}(\delta_1 + \delta_{i+1}) - 2^{-k}(2^k - 1)\delta_1 \in \mathrm{conv} \bigcup_{i=1}^{k} X_i ,$$

whence

$$\sum_{1}^{\infty} 2^{-i}\delta_{i+1} = y \in \operatorname{cl}\operatorname{conv} \bigcup_{i=1}^{\infty} X_i .$$

But if $y \in \operatorname{cl}\operatorname{conv}\{x_i : i \in I\}$ with always $x_i \in X_i$, then $\sup_n x_n{}^{i+1} = 2^{-i}$, whence $x_i = \delta_1 + \delta_{i+1}$ for each i and $y^1 = 1$, an impossibility.

Because of the restrictions which must be added, the infinite-dimensional analogues of our results appear to be of only marginal interest. Thus we shall omit further details.

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