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ON POSITIVE AND CONTINUOUS EXTENSION OF POSITIVE FUNCTIONALS DEFINED OVER DENSE SUBSPACES

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Introduction. In this paper we will take up the following problem: Given a real topological vector space with a partial ordering $\geqslant$, find conditions which are equivalent with the property:

P_1: Every positive linear functional defined over some dense subspace has a continuous and positive extension to the whole space.

A necessary conditions is, of course, the property:

P_2: Every continuous and positive linear functional defined over some dense subspace has a positive and continuous extension to the whole space.

In Proposition 7 we prove in a rather general setting that the property P_1 is equivalent with the conjunction of the following two properties:

P_3: Every positive linear functional is continuous.
P_4: For every $p \geqslant 0$ and every dense vector subspace F, there exists a $q \in F$, such that $q \geqslant p$.

As a corollary we obtain the result that if the positive cone has an interior point, then the property P_1 is satisfied. This generalizes a result of Bourbaki [3, p. 46] which asserts that the space of all real continuous functions over a compact space has the property P_1.

The paper is divided in two parts. In Section 1 we have collected some propositions on topological vector spaces, needed in the sequel. Propositions 1 to 4 give some properties of dense vector subspaces. Proposition 5 is a generalization of a theorem proved in the norm case by Yamabe [6]. Proposition 6, which gives a consistency condition for a system of linear inequalities, was proved in the norm case by K. Fan [4, p. 124]. Our proof follows the same lines as his, but because of weaker assumption, we obtain a corollary which seems to be new. Section 2

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treats the extension problem. In Proposition 8 we give a characterization of spaces with property P_2. In Proposition 9 we show that if the positive cone is given by a finite system of linear inequalities, then the property P_1 is satisfied. Proposition 10 shows that for a large number of partial orderings we can find a non-trivial locally convex topology, such that P_1 is satisfied. In Proposition 11 we give some conditions equivalent to property P_4.

NOTATION. All vector spaces considered will be real vector spaces. Subspaces will always be vector subspaces. If E is a topological vector space (top. v. sp.), then E' will denote the topological dual of E, and E^{Δ} the algebraic dual. If G is a subspace of E^{Δ}, then $\sigma(E, G)$ is the coarsest topology in E which makes every $g \in G$ continuous. If $V \subset E$ then V° denotes the polar set (with respect to G) viz. $\{g \in G: g(x) \leqslant 1, x \in V\}$. $\mathscr{V}_E(0)$ denotes the class of all neighbourhoods of 0 in E. For every $x \in E$, $\hat{x} \in (E^{\Delta})^{\Delta}$ is defined by $\hat{x}(f) = f(x)$. If $A \subset E$, then $[A]$ denotes the vector space generated by A. If $f \in E^{\Delta}$ and $M \subset E$, then f/M denotes the restriction of f to M, and $E = M_1 \oplus M_2$ denotes direct sum. Real numbers are designated by small greek letters, and the set of all real numbers by R.

1. Miscellaneous propositions from topological vector spaces.

PROPOSITION 1. *Let E be a locally convex space, and let $\{f_1, \ldots, f_n\} \subset E^{\Delta}$ be given. Then*

$$F = \bigcap_{i=1}^{n} f_i^{-1}(0)$$

is a dense subspace of E, if and only if every linear combination

$$f = \sum_{i=1}^{n} \lambda_i f_i \neq 0$$

is discontinuous.

PROOF. (I): Suppose $\overline{F} = E$. Let

$$f = \sum_{i=1}^{n} \lambda_i f_i$$

be given. Then $F \subset f^{-1}(0)$, and thus $\overline{f^{-1}(0)} = E$. Therefore, $f \neq 0$ implies that $f^{-1}(0)$ is not closed, and f is discontinuous.

(II): Suppose $\overline{F} \neq E$. Then there exists [2, Chap. II, p. 73] an $f \in E'$ such that $f \neq 0$ and such that

$$f^{-1}(0) \supset \overline{F} \supset F = \bigcap_{i=1}^{n} f_i^{-1}(0)\,.$$

It is known [1, p. 51] that this implies that

$$f = \sum_{i=1}^{n} \lambda_i f_i ,$$

q.e.d.

PROPOSITION 2. *Let E be a top.v.sp. with a dense subspace F which is not a hyperplane. Let f be a linear functional over E, such that the restriction of f to every dense subspace is continuous. Then f is continuous.*

PROOF. F is equal to the intersection of all hyperplanes $H \supset F$. Hence there exist two dense hyperplanes $H_1 \neq H_2$, such that $M = H_1 \cap H_2 \supset F$. Since F is dense, so is M. Furthermore $[H_1 \cup H_2] = E$. Let f_i be the restriction of f to H_i, $i=1, 2$. According to the hypothesis f_i is continuous in H_i, and will therefore have a continuous and linear extension $\bar{f}_i$ to the whole of E. We have $\bar{f}_1 = \bar{f}_2$, for otherwise we can find an $x \in E$ and a $V \in \mathscr{V}_E(0)$, such that $(\bar{f}_1 - \bar{f}_2)(y) \neq 0$ for every $y \in x + V$. As $M = H_1 \cap H_2$ is dense, there exists an $m \in (x+V) \cap M$, and we get $\bar{f}_1(m) = f_1(m) = f(m) = \bar{f}_2(m)$, a contradiction. From $E = [H_1 \cup H_2]$, we obtain that every $x \in E$ can be written in the form $x = h_1 + h_2$, $h_i \in H_i$. Hence

$$f(x) = f_1(h_1) + f_2(h_2) = \bar{f}_1(h_1 + h_2) = \bar{f}_1(x) .$$

That is $f = \bar{f}_1 = \bar{f}_2$. q.e.d.

The two next propositions show that in the preceding proposition we cannot omit the condition which assures the existence of a dense subspace not being a hyperplane.

PROPOSITION 3. *Let E be a locally convex vector space such that all the dense subspaces are hyperplanes.*

Then the restriction of a linear functional to a dense subspace will be continuous.

PROOF. Let f be a discontinuous linear functional, and let M be a dense subspace of E. If $M = H = f^{-1}(0)$, there is nothing to prove. Suppose therefore $M \neq H$. From the hypothesis we conclude that there exists a $g \in E^{\Delta}$ such that $M = g^{-1}(0)$. Since $M \cap H$ cannot be a hyperplane, it follows from the hypothesis that $M \cap H$ is not dense. From Proposition 1 it follows that there exist numbers λ, μ such that $\lambda f + \mu g$ is continuous and $\neq 0$. In particular $\lambda \neq 0$, since g is discontinuous. Now $(\lambda f + \mu g)/M$ is continuous, and as $g/M = 0$, we have

$$(\lambda f + \mu g)/M = \lambda (f/M) ,$$

proving the continuity of f/M. q.e.d.

PROPOSITION 4. *Let E be a locally convex top.v.sp., and suppose that E' has finite codimension n in E^{Δ}, say*

$$E^{\Delta} = E' \oplus [h_1] \oplus \ldots \oplus [h_n].$$

Let F be a dense subspace of E. Then codim. $F \leqslant n$.

PROOF. There exists a family $\{f_\gamma\}_\Gamma \subset E^{\Delta}$, such that

$$F = \bigcap_{\gamma \in \Gamma} f_\gamma^{-1}(0).$$

If we cannot find n linearly independent elements in $\{f_\gamma\}_\Gamma$, we have codim. $F < n$. Suppose therefore that $\{f_1, \ldots, f_n\} \subset \{f_\gamma\}_\Gamma$ are n linearly independent elements. Our assertion will be proved if we can show that $f \in [\{f_1, \ldots, f_n\}]$ for every $f \in \{f_\gamma\}_\Gamma$. As

$$f^{-1}(0) \cap \bigcap_{i=1}^{n} f_i^{-1}(0)$$

is dense, the proposition follows from the following

LEMMA. *Let E be as in Proposition 4 and let $\{g_1, \ldots, g_{n+1}\} \subset E^{\Delta}$ be such that*

$$M = \bigcap_{i=1}^{n+1} g_i^{-1}(0)$$

is dense in E. Then $g_1, \ldots, g_{n+1}$ are linearly dependent.

PROOF. From the hypothesis we get that every g_j can be written in a unique way as

$$g_j = \sum_{i=1}^{n} \lambda_{j,i} h_i + f_j$$

where $f_j \in E'$. Let g be any linear combination of $g_1, \ldots, g_{n+1}$, say

$$g = \sum_{j=1}^{n+1} \alpha_j g_j.$$

Thus

$$g = \sum_{j=1}^{n+1} \alpha_j \left(\sum_{i=1}^{n} \lambda_{j,i} h_i + f_j \right)$$
$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n+1} \alpha_j \lambda_{j,i} \right) h_i + \sum_{j=1}^{n+1} \alpha_j f_j.$$

Here the system

$$\sum_{j=1}^{n+1} \alpha_j \lambda_{j,i} = 0, \qquad i = 1, \ldots, n,$$

will have a non-trivial solution $\alpha_1, \ldots, \alpha_{n+1}$. With this choice of the α's we get

$$g = \sum_{j=1}^{n+1} \alpha_j g_j = \sum_{j=1}^{n+1} \alpha_j f_j \in E' .$$

From the hypothesis it follows that

$$\bigcap_{i=1}^{n+1} g_i^{-1}(0)$$

is dense. By Proposition 1 we conclude that $g=0$. q. e. d.

The next proposition has been proved in the norm case by Yamabe [6].

PROPOSITION 5. *Let E be a locally convex vector space, let $\{f_1, \ldots, f_n\} \subset E'$ be given, and suppose that K is a convex dense subset of E. Then for every $a \in E$, and every $V \in \mathscr{V}_E(0)$ there exists a $k \in K$ such that $k \in a+V$, and such that*

$$f_i(a) = f_i(k), \qquad i = 1, \ldots, n .$$

PROOF. Let $a \in E$ and $V \in \mathscr{V}_E(0)$ be given. We can find a set S of seminorms in E, such that S determines the topology in E. We can further find $\{q_{0,1}, \ldots, q_{0,n_0}\} \subset S$ and $\varepsilon > 0$ such that

$$V \supset \{x \in E \colon \; q_{0,j}(x) \leqslant \varepsilon, \; j = 1, \ldots, n_0\} ,$$

and we can find [2, Chap. II, § 5, Proposition 9] for every $i=1, \ldots, n$, a set $\{q_{i,1}, \ldots, q_{i,n_i}\} \subset S$ and a $\lambda_i > 0$ such that

$$|f_i(x)| \leqslant \lambda_i \cdot \max\{q_{i,j}(x) \colon \; j = 1, \ldots, n_i\} .$$

Define

$$q(x) = \max\{q_{i,j}(x) \colon \; 0 \leqq i \leqq n, \; 1 \leqq j \leqq n_i\} .$$

q will be a seminorm, and will define a topology in E, which is coarser than the given topology. Hence K will be dense in (E, q). Define $M = \{x \colon q(x)=0\}$. Then E/M is a normed vector space when equipped with the norm $\dot{q}(\dot{x}) = q(x)$. For $i=1, \ldots, n$ and $x \in E$ we have $f_i(x) \leqslant \lambda_i q(x)$. Thus $f_i(x)=0$ for $x \in M$. Hence $\dot{f}_i$ is defined in a unique manner over E/M by $\dot{f}_i(\dot{x}) = f_i(x)$, and since

$$|\dot{f}_i(\dot{x})| \leqslant \lambda_i q(x) = \lambda_i \dot{q}(\dot{x}) ,$$

we conclude that $\dot{f}_i$ is a continuous linear functional over E/M. As $\dot{K} = \{\dot{k} \colon k \in K\}$ is a dense convex subset of E/M, it follows from Yamabe's theorem [6], that there exists a $\dot{k} \in \dot{K}$ such that $\dot{q}(\dot{k} - \dot{a}) \leqslant \varepsilon$ and $\dot{f}_i(\dot{a}) = \dot{f}_i(\dot{k})$, $i=1, \ldots, n$. Hence we can find a $k \in K$, such that $q_{0,j}(k-a) \leqslant \varepsilon$, $j=1, \ldots, n_0$, that is, $k \in a+V$, and $f_i(a) = f_i(k)$, $i=1, \ldots, n$. q. e. d.

The next proposition has been proved in the norm case by Ky Fan [4, p. 124].

PROPOSITION 6. *Let E be a locally convex Hausdorff top.v.sp., with S as a topology-defining set of seminorms. Let the two families $\{x_\gamma\}_\Gamma \subset E$ and $\{\alpha_\gamma\}_\Gamma \subset R$ be given. Then the linear inequality system*

$$\text{(A)} \qquad f(x_\gamma) \geqslant \alpha_\gamma, \qquad \gamma \in \Gamma$$

will be consistent (i.e. there exists an $f \in E'$ which satisfies (A)), if and only if there exists a $\beta > 0$ and a finite set $\{p_1, \ldots, p_m\} \subset S$ such that

$$\text{(1)} \qquad \beta \cdot \max_{j=1,\ldots,m} \left\{ p_j \left(\sum_{i=1}^{n} \lambda_i x_{\gamma i} \right) \right\} \geqslant \sum_{i=1}^{n} \lambda_i \alpha_{\gamma i}$$

whenever $\gamma_1, \ldots, \gamma_n$ is a finite selection from Γ, and $\lambda_i \geqslant 0$, $i = 1, \ldots, n$.

PROOF. (I): Suppose that $f \in E'$ satisfies the system (A). Then

$$f\left(\sum_{i=1}^{n} \lambda_i x_{\gamma i} \right) = \sum_{i=1}^{n} \lambda_i f(x_{\gamma i}) \geqslant \sum_{i=1}^{n} \lambda_i \alpha_{\gamma i} .$$

Since $f \in E'$, we can find a $\beta > 0$ and $\{p_1, \ldots, p_m\} \subset S$, such that

$$f(x) \leqslant \beta \cdot \max_{j=1,\ldots,m} \{p_j(x)\}$$

for every $x \in E$. Therefore

$$\beta \cdot \max_{j=1,\ldots,m} \left\{ p_j \left(\sum_{i=1}^{n} \lambda_i x_{\gamma i} \right) \right\} \geqslant \sum_{i=1}^{n} \lambda_i \alpha_{\gamma i} .$$

(II): Suppose that the condition (1) is satisfied. First we will then prove that every finite subsystem of (A), say

$$\text{(A}'\text{)} \qquad f(x_i) \geqslant \alpha_i, \qquad i = 1, \ldots, n ,$$

is consistent.

Define the map η from E' to R^n by $\eta(f) = (f(x_1), \ldots, f(x_n)) \in R^n$. Then η will be a continuous linear map, when E' is equipped with the weak topology $\sigma(E', E)$. Consistency of the system (A′) means that there exists an $f \in E'$ such that $\eta(f) \in \alpha + P$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $P = \{(\pi_1, \ldots, \pi_n) : \pi_i \geqslant 0, i = 1, \ldots, n\}$. Define

$$V = \{x \in E : \ p_j(x) \leqslant \beta^{-1}, \ j = 1, \ldots, n\} .$$

Then $V \in \mathscr{V}_E(0)$ and V is symmetric, convex and closed. Therefore $V = V^{00}$ in the duality between E and E'. Further V^0 will be an equicontinuous, weakly closed subset af E' [2, Chap. IV, § 2, Proposition 1]. Hence V^0 is weakly compact, and $\eta(V^0)$ is a compact and convex subset of R^n. We assert that

$$\eta(V^0) \cap (\alpha + P) \neq \emptyset .$$

For, if not, we can find a hyperplane in R^n separating strictly $\eta(V^0)$ and $\alpha+P$ [2, Chap. II, § 3, Proposition 4]. That is, we can find $\lambda_0, \lambda_1, \ldots, \lambda_n$ such that for every $g \in V^0$ and every $(\pi_1, \ldots, \pi_n) \in P$ we have

$$g\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i\, g(x_i) < \lambda_0 < \sum_{i=1}^{n} \lambda_i(\alpha_i + \pi_i) . \tag{2}$$

From the second inequality in (2) we get $\lambda_i \geqslant 0$ when $i = 1, \ldots, n$, and furthermore

$$\lambda_0 < \sum_{i=1}^{n} \lambda_i \alpha_i .$$

From the first inequality in (2) we get, since V^0 is symmetric,

$$\left| g\left(\sum_{i=1}^{n} \lambda_i x_i\right) \right| < \lambda_0$$

for all $g \in V^0$. Hence $\lambda_0 > 0$, and since $V = V^{00}$, we have

$$\lambda_0^{-1}\left(\sum_{i=1}^{n} \lambda_i x_i\right) \in V .$$

Thus

$$\beta p_j \left(\sum_{i=1}^{n} \lambda_i x_i\right) \leqslant \lambda_0 < \sum_{i=1}^{n} \lambda_i \alpha_i$$

for all $j = 1, \ldots, m$, contrary to our hypothesis. We have therefore proved that $\eta(V^0) \cap (\alpha + P) \neq \emptyset$ and consequently we can find an $f \in V^0$ such that f satisfies (A′).

Define for every finite non empty subset $J = \{\gamma_1, \ldots, \gamma_n\}$ of Γ, A_J^* as the set of all those $f \in V^0$ which satisfy the finite system

$$f(x_{\gamma_i}) \geqslant \alpha_{\gamma_i}, \qquad i = 1, \ldots, n .$$

We thus have

$$A_J^* = V^0 \cap \bigcap_{\gamma_i \in J} \hat{x}_{\gamma_i}^{-1}([\alpha_i, \infty\rangle) ,$$

and consequently A_J^* is a weakly closed subset of the weakly compact set V^0.

Consider the family $\mathscr{A}^* = \{A_J^*\colon J$ any finite non empty subset of $\Gamma\}$. We have proved above that $\mathscr{A}^*$ has the finite intersection property. Since $\mathscr{A}^*$ consists of closed subsets of a compact space, we conclude that

$$\bigcap \{A_J^*\colon A_J^* \in \mathscr{A}^*\} \neq \emptyset ;$$

but this means that (A) is consistent. q.e.d.

COROLLARY. *Let G be a vector space, and let the two families $\{f_\gamma\}_\Gamma \subset G^\Delta$ and $\{\alpha_\gamma\}_\Gamma \subset R$ be given. Then the system*

$$\text{(B)} \qquad f_\gamma(x) \geqslant \alpha_\gamma, \qquad \gamma \in \Gamma$$

is consistent (*i.e. there exists an $x \in G$ such that* (B) *is satisfied*) *if and only if there exists a $\beta > 0$, and a finite subset $\{x_1, \ldots, x_m\} \subset G$, such that*

$$\beta \cdot \max_{j=1,\ldots,m} \left\{ \left| \sum_{i=1}^{n} \lambda_i f_{\gamma_i}(x_j) \right| \right\} \geqslant \sum_{i=1}^{n} \lambda_i \alpha_{\gamma_i}$$

whenever $\gamma_1, \ldots, \gamma_n$ is a finite selection from Γ and $\lambda_i \geqslant 0$, $i = 1, \ldots, n$.

PROOF. G^Δ is a locally convex Hausdorff top.v.sp. when equipped with the topology $\sigma(G^\Delta, G)$. This topology is defined by the family $\{|\hat{x}|\}_{x \in G}$ of seminorms. In virtue of Proposition 6, consistency of (B) means that there exists a $\beta > 0$ and $\{x_1, \ldots, x_m\} \subset G$, such that

$$\beta \cdot \max_{j=1,\ldots,m} \left\{ \left| \hat{x}_j \left(\sum_{i=1}^{n} \lambda_i f_{\gamma_i} \right) \right| \right\} \geqslant \sum_{i=1}^{n} \lambda_i \alpha_{\gamma_i},$$

for every finite subset $\{\gamma_1, \ldots, \gamma_n\}$ of Γ and every $\lambda_i \geqslant 0$, $i = 1, \ldots, n$. This proves our assertion. q.e.d.

2. Extension of positive linear functionals. In the sequel E will denote a topological vector space and P will denote a convex cone in E, that is $P + P \subset P$ and $\lambda P \subset P$ for every $\lambda \geqslant 0$. If $x - y \in P$, we shall often write $x \geqslant y$, and say that $\geqslant$ is the *partial order* determined by P. We will say that $f \in E^\Delta$ is *positive* if $f(p) \geqslant 0$ for every $p \in P$. The set of all positive linear functionals will be denoted by P^Δ. If F is a subspace of E, we will use the corresponding notations with respect to the cone $P \cap F$.

DEFINITION 1. The couple (E, P) will be called an *extension couple* (*ext.c.*) if every positive linear functional defined over some dense subspace of E, has a positive and continuous extension to the whole of E.

The couple (E, P) will be called a *continuous extension couple* (*c.ext.c.*) if every positive and continuous linear functional defined over some dense subspace of E, has a continuous and positive extension to the whole of E.

DEFINITION 2. The cone P will be called *rich* if $(p + P) \cap F \neq \emptyset$ for every $p \in P$ and every dense subspace F of E.

REMARK. The property P_4 mentioned in the introduction means that the positive cone is rich.

PROPOSITION 7. (I): *If P is rich and $P^\Delta \subset E'$, then (E,P) is an ext.c.* (II): *If (E,P) is an ext.c., then P is rich. If (E,P) is an ext.c. and there exists in E a dense subspace which is not a hyperplane, then $P^\Delta \subset E'$.*

PROOF. (I): Let F be a dense subspace and let f be a positive linear functional over F. Since $(p+P) \cap F \neq \emptyset$ for every $p \in P$, it follows from Corollary 2,3 in [5] that f can be extended to a functional $\bar{f} \in P^\Delta \subset E'$.

(II): If P is not rich, we can find a $p \in P$ and a dense subspace F, such that $(p+P) \cap F = \emptyset$. Hence $p \notin F$. Define H as the vector subspace generated by F and p, and define the linear functional h over H by

$$h(y+\lambda p) = \lambda, \qquad \text{where} \quad y \in F .$$

If

$$q = y+\lambda p \in P \cap H ,$$

then we assert that $\lambda \geq 0$. For if $-\lambda = \mu > 0$, then

$$p+\mu^{-1} q = \mu^{-1} y \in (p+P) \cap F ,$$

contrary to the assumption. Hence h is positive over H, and since H is dense, h will have a continuous and positive extension $\bar{h}$ to E. From $\bar{h}(p)=1$ and $\bar{h}(F)=0$, we obtain a contradiction since F is dense. Hence P is rich. The last assertion in the proposition follows from Proposition 2, since every positive linear functional has a continuous restriction to every dense subspace when (E,P) is an ext.c. q.e.d.

COROLLARY 1. *If the cone P has an interior point p_0, then (E,P) is an ext.c.*

PROOF. In this case we have $P^\Delta \subset E'$ [2, Chap. II, § 1, Proposition 16], so we only have to prove that P is rich. Let $p \in P$ and let F be dense. We can find a $V \in \mathscr{V}_E(0)$, such that $p_0 + V \subset P$. Hence

$$\emptyset \neq (p+p_0+V) \cap F \subset (p+P) \cap F .$$

q.e.d.

COROLLARY 2. *If all the dense subspaces of E are hyperplanes and $P^\Delta \subset E'$, then (E,P) is an ext.c.*

PROOF. It is sufficient to prove that P is rich. Let $p \in P$ and let H be a dense subspace. By hypothesis there exists a discontinuous $f \in E^\Delta$, such that $H = f^{-1}(0)$. If $f(p)=0$, we have $p \in (p+P) \cap H$. If $f(p) \neq 0$, we can assume without loss of generality that $f(p) = \alpha > 0$. We can find a $q \in P$, such that $\beta = f(q) < 0$, for otherwise we should get $f \in P^\Delta \subset E'$, which is impossible. Then $r = -(\alpha/\beta) q \in P$, and $f(r) = -\alpha = -f(p)$. Hence $p+r \in (p+P) \cap H$. Thus in both cases we have

$$(p+P)\cap H \neq \emptyset .$$

q.e.d.

REMARK. (I): Neither of the conditions "P is rich", nor "$P^{\Delta}\subset E'$" implies the other. The cone $P=\{0\}$ will for instance always be rich, but in this case $P^{\Delta}=E^{\Delta}$. On the other side, if E is a Frechet space, such that P is closed and $E=P-P$, then it is known [5, Corollary 5.5] that $P^{\Delta}\subset E'$. If P was rich, then (E, P) should be an ext. c., but we can easily find examples for which this is not the case.

(II): Our assumption in the last statement of Proposition 7 is not superfluous: Suppose that all the dense subspaces of E are hyperplanes. Choose $P=\{0\}$. Then $P^{\Delta}=E^{\Delta}$, and if f is a linear functional over a dense subspace F, we can extend f to a linear functional $\bar{f}$ defined over E. From Proposition 3 it follows that $\bar{f}/F=f$ is continuous. Hence we can extend f to a continuous linear functional, and (E, P) is thus an ext. c. But according to Proposition 4, we need not have $P^{\Delta}\subset E'$.

PROPOSITION 8. *Let E be a locally convex top.v.sp. Then (E,P) is a c.ext.c., if and only if for every $p\in P$, $V\in\mathscr{V}_E(0)$ and every dense subspace F we have*

$$\text{(C)} \qquad (p+V)\cap F\cap P \neq \emptyset .$$

PROOF. (I): Assume the condition (C) satisfied. Let f be a continuous and positive linear functional defined over the dense subspace F. Then f can be extended to an $\bar{f}\in E'$. We have to prove that $\bar{f}$ is positive. Let $p\in P$. For every $V\in\mathscr{V}_E(0)$ we can find a

$$q_V \in (p+V)\cap F\cap P .$$

Since

$$\lim_{V\in\mathscr{V}_E(0)} q_V = p$$

and $\bar{f}(q_V)\geqslant 0$ for all $V\in\mathscr{V}_E(0)$, we conclude from the continuity of $\bar{f}$, that $\bar{f}(p)\geqslant 0$.

(II): Assume the condition (C) not satisfied. We can find a $p\in P$, a dense subspace F, and a convex $V\in\mathscr{V}_E(0)$, such that

$$(p+V)\cap F\cap P = \emptyset .$$

Hence we can separate $p+V$ and $F\cap P$ by a closed hyperplane [2, Chap. II § 3, Th. 1], that is, we can find $f\in E'$ and $\alpha\in R$, such that $f(p)<\alpha$ and $f(q)\geqslant\alpha$ for every $q\in F\cap P$. Since $q=0\in F\cap P$ we have $\alpha\leqq 0$; hence $f(p)<0$. If $q\in F\cap P$ and $\lambda>0$ we have $\lambda q\in F\cap P$ and hence

$$f(q) = \lambda^{-1}f(\lambda q) \geqq \lambda^{-1}\alpha;$$

letting $\lambda \to +\infty$ we get $f(q) \geqq 0$. Thus f/F is continuous and positive, but we cannot extend f/F to a positive and continuous linear functional. q.e.d.

PROPOSITION 9. *Let E be a locally convex top.v.sp. Suppose that $\{f_1, \ldots, f_n\} \subset E'$, and define the cone*

$$P = \bigcap_{i=1}^{n} f_i^{-1}([0, \infty\rangle) .$$

Then (E, P) is an ext.c.

PROOF. According to Proposition 7 it is sufficient to prove that P is rich and that $P^\Delta \subset E'$. From Proposition 5 it follows that for every $p \in P$ and every dense subspace F, we can find a $q \in F$, such that $f_i(p) = f_i(q)$, $i = 1, \ldots, n$, and thus $q \in (p+P) \cap F$. Hence P is rich. Let $f \in P^\Delta$. Since

$$P \supset \bigcap_{i=1}^{n} f_i^{-1}(0)$$

we conclude that $f = 0$ over

$$\bigcap_{i=1}^{n} f_i^{-1}(0)$$

and consequently we can find $\lambda_1, \ldots, \lambda_n$, such that

$$f = \sum_{i=1}^{n} \lambda_i f_i .$$

Thus $f \in E'$, and $P^\Delta \subset E'$. q.e.d.

If we provide E with the finest locally convex topology, then trivially (E, P) is an ext.c. for every cone P. The following proposition displays that we usually can assert more:

PROPOSITION 10. *Let P be a cone in E, such that $P^\Delta - P^\Delta \neq E^\Delta$. Then we can find a locally convex topology $\mathcal{T}$ in E, such that (E, P) is an ext.c. when E is provided with $\mathcal{T}$, and such that the topological dual of E provided with $\mathcal{T}$ is a hyperplane in E^Δ.*

PROOF. Having $P^\Delta - P^\Delta \neq E^\Delta$, we can find a hyperplane H in E^Δ such that $H \supset P^\Delta - P^\Delta$. Define $\mathcal{T}$ as the coarsest topology in E which makes every $h \in H$ continuous. Then $\mathcal{T}$ will be locally convex, and E provided with $\mathcal{T}$ will have H as topological dual. From Proposition 4 it follows that the dense subspaces of E will be hyperplanes. As $P^\Delta \subset H$, the proposition follows from Proposition 7, Corollary 2. q.e.d.

We now suppose that E is locally convex, and that P is a closed cone in E. Then we can find a family $\{f_\gamma\}_\Gamma \subset E'$, such that

$$P = \bigcap_{\gamma \in \Gamma} f_\gamma^{-1}([0, \infty\rangle) .$$

Define $Q \subset E'$ as the cone generated by $\{f_\gamma\}_\Gamma$. Thus

$$Q = \left\{ \sum_{i=1}^{n} \lambda_i f_{\gamma i} \colon \{\gamma_1, \ldots, \gamma_n\} \subset \Gamma, \quad \lambda_i \geqslant 0, \quad i = 1, \ldots, n \right\} .$$

We now consider the duality between E and E^Δ. Then every subspace M of E satisfies $M = M^{00}$.

Proposition 11. *Let E be a locally convex top.v.sp., and let P be a closed cone in E. Then the following three statements are equivalent.:*

(a): *P is rich.*

(b): *For every $p \in P$ and every dense subspace F of E there exists a finite subset $\{x_1, \ldots, x_n\} \subset E$, such that for every $f \in Q$ and $g \in F^0$ we have*

$$f(p) \leqslant \max\{|(f+g)(x_i)| \colon i = 1, \ldots, n\} .$$

(c): *For every $p \in P$ and every dense subspace F of E there exists a finite subset $\{x_1, \ldots, x_n\} \subset E$ such that the convex envelope $K(x_1, \ldots, x_n)$ generated by $\{x_1, \ldots, x_n\}$ intersects F, and for every $f \in Q$ we have*

$$f(p) \leqslant \sup\{|f(k)| \colon k \in K(x_1, \ldots, x_n) \cap F\} .$$

Proof. (c) $\Rightarrow$ (b): Let $f \in Q$, $g \in F^0$, and

$$k = \sum_{i=1}^{n} \lambda_i x_i \in K(x_1, \ldots, x_n) .$$

Then we have

$$\begin{aligned} \max_{i=1,\ldots,n} \{|(f+g)(x_i)|\} &\geqslant \sum_{i=1}^{n} \lambda_i |(f+g)(x_i)| \\ &\geqslant \left| (f+g) \left(\sum_{i=1}^{n} \lambda_i x_i \right) \right| \\ &= |(f+g)(k)| . \end{aligned}$$

Hence

$$\begin{aligned} \max_{i=1,\ldots,n} \{|(f+g)(x_i)|\} &\geqslant \sup\{|(f+g)(k)| \colon k \in K(x_1, \ldots, x_n)\} \\ &\geqslant \sup\{|(f+g)(k)| \colon k \in K(x_1, \ldots, x_n) \cap F\} \\ &= \sup\{|f(k)| \colon k \in K(x_1, \ldots, x_n) \cap F\}, \end{aligned}$$

since $g \in F^0$.

(b) $\Leftrightarrow$ (a): P rich means that for every $p \in P$ and every dense subspace F the system

$$\begin{aligned} f_\gamma(y) &\geqslant f_\gamma(p), \qquad \gamma \in \Gamma, \\ \left.\begin{aligned} f(y) &\geqslant 0 \\ -f(y) &\geqslant 0 \end{aligned}\right\}, &\qquad f \in F^0, \end{aligned}$$

is consistent. From the corollary to Proposition 6 it follows that this system is consistent if and only if we can find a $\beta > 0$ and $\{y_1, \ldots, y_n\} \subset E$ such that

$$\beta \cdot \max_{i=1,\ldots,n} \left\{ \left| \sum_{j=1}^{m} \lambda_j f_{\gamma_j}(y_i) + \sum_{r=1}^{k} \mu_r f_r(y_i) \right| \right\} \geqslant \sum_{j=1}^{m} \lambda_j f_{\gamma_j}(p)$$

whenever $\{\gamma_1, \ldots, \gamma_m\} \subset \Gamma$, $\{f_1, \ldots, f_k\} \subset F^0$, $\lambda_j \geqslant 0$, $j = 1, \ldots, m$, and $\mu_r \in R$, $r = 1, \ldots, k$. That is,

$$\beta \cdot \max_{i=1,\ldots,n} \{|(f+g)(y_i)|\} \geqslant f(p)$$

for every $f \in Q$, and every $g \in F^0$. By putting $x_i = \beta y_i$ we obtain (b) $\Leftrightarrow$ (a).

(a) $\Rightarrow$ (c): If $q \in (p+P) \cap F$, then we have for every $f \in Q$, $0 \leqslant f(p) \leqslant f(q)$. Thus we can choose $\{x_1, \ldots, x_n\} = \{q\}$. q.e.d.

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