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THE TOEPLITZ MATRICES OF AN ARBITRARY LAURENT POLYNOMIAL

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1. Introduction.

1.1. The problem under consideration has a rather impressive history. Therefore we present it in some detail with emphasis on those theorems which are either used by us, or upon which our work (to be summarized in Section 1.5) may have shed new light.

With a Laurent series, with arbitrary complex coefficients a_ν,

$$f(z) = \sum_{\nu=-\infty}^{\infty} a_\nu z^\nu ,$$

one may associate a sequence of matrices, which are called Toeplitz matrices after their inventor [7]. (They are not the matrices by the same name which occur in summability theory.) We define our sequence of finite $n+1$ by $n+1$ matrices $T(n)$ by

$$T(n) = \big(T(n)_{ij}\big) = (a_{i-j}), \qquad i,j = 0,1,\ldots,n .$$

It is natural to view $T(n)$ as a linear operator on the space of ordered $(n+1)$-tuples of complex numbers, as defined by ordinary matrix multiplication. We are concerned with the eigenvalues $\lambda_{n0}, \lambda_{n1}, \ldots \lambda_{nn}$ of $T(n)$ and denote the spectrum of $T(n)$ by

$$\sigma_n = \{\lambda_{n0}, \ldots, \lambda_{nn}\} = \{\lambda \mid \det\big(T(n)-\lambda I\big) = 0\} .$$

We shall presently summarize known results which give a great deal of information concerning the asymptotic behavior of the eigenvalues λ_{ni} as $n \to \infty$, but only under very restrictive assumptions about $f(z)$, or about its coefficients a_ν. All of these results concern, in some way, that subset of the complex plane which, in a sense, is "filled in" by the points of σ_n as $n \to \infty$. We now define a set B which seems to play such a role. Let

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$$(1.1) \qquad B = \left\{ \lambda \mid \lambda = \lim_{m\to\infty} \lambda_m,\ \lambda_m \in \sigma_{i_m},\ \lim_{m\to\infty} i_m = \infty \right\}.$$

Clearly B will be non-empty if the union of the spectra σ_n is a bounded set, and it requires only very mild assumptions concerning the sequence $\{a_\nu\}$ to ensure that such is the case.

One assumption which has other advantages besides ensuring that B is non-empty is that

$$(1.2) \qquad \sum_{\nu=-\infty}^{\infty} |a_\nu| < \infty .$$

Condition (1.2) makes it possible to consider the singly and doubly infinite matrices

$$T^+ = (T^+_{ij}) = (a_{i-j}), \qquad i,j = 0,1,2,\ldots,$$
$$T = (T_{ij}) = (a_{i-j}), \qquad i,j = 0, \pm 1, \pm 2, \ldots,$$

as bounded linear operators on the Banach spaces l^+_∞ and l_∞:

$$l^+_\infty = \left\{ x \mid x = (x_0, x_1, \ldots),\ x_k \text{ complex},\ \sup_{k\geqq 0} |x_k| < \infty \right\},$$
$$l_\infty = \left\{ x \mid x = (\ldots, x_{-1}, x_0, x_1, \ldots),\ x_k \text{ complex},\ \sup_{-\infty<k<\infty} |x_k| < \infty \right\},$$

with norms

$$\|x\|^+ = \sup_{k\geqq 0} |x_k|, \qquad \|x\| = \sup_{-\infty<k<\infty} |x_k|,$$

respectively, and

$$(T^+x)_n = \sum_{k=0}^{\infty} a_{n-k} x_k, \qquad x \in l^+_\infty, \quad n \geqq 0,$$

$$(Tx)_n = \sum_{k=-\infty}^{\infty} a_{n-k} x_k, \qquad x \in l_\infty, \quad -\infty < n < \infty .$$

The spectra of T^+ and T will be denoted by σ^+ and by σ.

Remark. Obviously we could have defined T^+ and T as operators on other Banach spaces, for example the spaces l^+_p and l_p of p-summable sequences, $p \geqq 1$, but it may be shown that this would not affect the spectra σ^+ and σ, as long as (1.2) holds.

1.2. The principal theorem of O. Toeplitz concerns the spectrum σ of T. Imposing the condition on $f(z)$ that it be regular in an annulus containing the unit circle $|z| = 1$ in its interior (an assumption which of course implies (1.2)) he proved [7] that

$$(1.3) \qquad \sigma = \{\lambda \mid \lambda = f(e^{i\theta}),\ 0 \leqq \theta \leqq 2\pi\}.$$

It was twenty years before the unneccessarily strong hypothesis of Toeplitz was weakened to yield (1.3) assuming only (1.2). This was done by N. Wiener [9], the assertion (1.3) being equivalent to Wiener's Tauberian theorem for trigonometric series.

Approximately between the dates of these two theorems G. Szegö created a theory concerning the finite Toeplitz matrices $T(n)$ for the important special case when they are Hermitean, i.e., when

$$(1.4) \qquad a_\nu = \bar{a}_{-\nu}, \quad \nu = 0, 1, \ldots, \qquad \text{or} \qquad f(z) \text{ real for } |z| = 1 .$$

His principal result concerning the eigenvalues λ_{ni} is valid under the assumptions (1.2) and (1.4) but the reader is referred to [3] for weaker conditions replacing (1.2) in particular and for further information in general. G. Szegö's theorem asserts that

$$(1.5) \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \Phi_{a,b}(\lambda_{ni}) = (2\pi)^{-1} \mu\{\theta \mid a \leqq f(e^{i\theta}) \leqq b\} ,$$

where $\Phi_{a,b}(x)$ is the characteristic function of the interval $[a,b]$ on the real line and μ is Lebesgue measure. The eigenvalues λ_{ni} are all real since $T(n)$ is Hermitean. The set which they "fill in", according to (1.5) is the set of values assumed by $f(e^{i\theta})$ and equation (1.5) also tells us at what rate this set is filled in. According to (1.3) which is valid by Wiener's theorem we know that this set is σ. Hence it is seen that a (very weak) form of (1.5) may be expressed as

$$(1.6) \qquad B = \sigma .$$

1.3. Two recent extensions of Szegö's theory are important from our point of view. The first one is due to M. Kac [4]. The method by which Szegö obtained (1.5) depends in an essential way on the Hermitean character of $T(n)$. M. Kac showed by methods from probability theory that the following version of equation (1.5) is true, depending only on the assumption (1.2), not (1.4). This is

$$(1.7) \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \lambda_{ni}{}^p = (2\pi)^{-1} \int_{-\pi}^{\pi} [f(e^{i\theta})]^p \, d\theta ,$$

for every $p = 0, 1, 2, \ldots$ It should be noted that (1.7) implies (1.5) only if (1.4) is assumed. In that case the λ_{ni} as well as $f(e^{i\theta})$ are real and the Weierstrass approximation theorem permits the approximation of characteristic functions by polynomials. This fails, however, without (1.4), and no statement like (1.6) concerning B seems to follow from (1.7).

Szegö's theorem (1.5) is capable of many refinements. One extension of it concerning the asymptotic behavior of the largest eigenvalue λ_{nn} as $n \to \infty$ is due to H. Widom [8]. While the exact form of his result is not of interest here it is important that he developed a method, based on an idea of M. Kac [5], which is also applicable in the non-Hermitean case. Our results depend in a crucial way on this method, which will be outlined in Section 4.

1.4. The results concerning the spectrum σ^+ of T^+ are quite recent. They were obtained independently by M. G. Krein [6], and by A. Calderon, F. Spitzer and H. Widom [2]. The only assumption concerning $f(z)$ is again that condition (1.2) holds. Then the set of values assumed by $f(z)$ when $|z|=1$ is a closed curve and its winding number (index) about the origin is defined, provided that the origin does not lie on the curve, by

$$I(f) = (2\pi i)^{-1} \int_0^{2\pi} d_\theta \log f(e^{i\theta}) = (2\pi)^{-1}[\arg f(e^{i\theta})]_0^{2\pi}. \tag{1.8}$$

Then $I(f-\lambda)$ is the index of the function $f(z)-\lambda$ or, equivalently, the winding number of the curve which is the image of $|z|=1$ under $f(z)$ about the point λ. If $f(z)=\lambda$ for some z with $|z|=1$, we shall say that $I(f-\lambda)\neq 0$ without further defining it. In these terms the spectrum of T^+ as a bounded linear operator on l_∞^+ (or on l_p^+, $p \geqq 1$) is described by

$$\sigma^+ = \{\lambda \mid I(f-\lambda) \neq 0\}. \tag{1.9}$$

In the special case when T^+ is Hermitean, i.e., when (1.4) holds in addition to (1.2), the image of $|z|=1$ under $f(z)$ is a closed bounded subinterval of the real line and by (1.3), (1.6), and (1.9) we have

$$B = \sigma^+ = \sigma. \tag{1.10}$$

1.5. Now it is possible to state the results of this paper. Briefly, we have found the analogue of equation (1.10) for the general case of not necessarily Hermitean Toeplitz matrices, but under a much more restrictive condition than (1.2). We have considered only Laurent polynomials, i.e., functions $f(z)$ of the form

$$f(z) = \sum_{-\infty}^{\infty} a_\nu z^\nu, \quad a_\nu = 0 \quad \text{for} \quad \nu < -k \leqq -1 \quad \text{and} \quad \nu > h \geqq 1, \qquad a_{-k} \neq 0,\ a_h \neq 0. \tag{1.11}$$

Otherwise the coefficents a_ν are arbitrary complex numbers.

Under this hypothesis we shall find several equivalent characterizations of the set B. As for equation (1.10), a simple calculation quickly shows that it cannot hold in general. For example in the case when

$$f(z) = a_{-1}z^{-1}+a_1 z ,$$

the set σ is the boundary of an ellipse, σ^+ the interior with the boundary, whereas B is a line segment in the interior of σ^+.

More precisely, one soon becomes convinced that in general σ has nothing and should have nothing to do with the set B, as the matrices $T(n)$ approach T^+ rather than T. On the other hand σ^+ is much too large even though (as we shall show) it always contains the set B. The reason for that is quite simple; there is (at least) one quite obvious way of forming equivalence classes of functions $f(z)$ such that two functions in the same class have the same spectra σ_n but different spectra σ^+.

We say that two functions $f(z)$ and $g(z)$, both of the form (1.11), are equivalent, or $f \sim g$, if there is a number $r > 0$, such that $g(z)=f(rz)$. If

$$g(z) = f(rz), \quad f(z) = \sum_{\nu=-\infty}^{\infty} a_\nu z^\nu, \quad g(z) = \sum_{\nu=-\infty}^{\infty} (a_\nu r^\nu) z^\nu, \qquad r > 0 ,$$

we write

$$\begin{aligned} T_f^+ &= T^+ = (a_{i-j}), & i,j &= 0,1,2,\ldots , \\ T_g^+ &= T_r^+ = (a_{i-j}r^{i-j}), & i,j &= 0,1,2,\ldots , \\ \sigma^+ &= \text{spectrum of } T^+ , & & \\ \sigma^+(r) &= \text{spectrum of } T_r^+, & \sigma^+(1) &= \sigma^+ . \end{aligned}$$

We further define the set

$$A = \bigcap_{r>0} \sigma^+(r) , \tag{1.12}$$

associated with every $f(z)$ of the form (1.11). (It will be shown later that A is not the empty set.)

It is easily seen that $f(z)$ and $g(z)=f(rz)$ determine finite Toeplitz matrices with the same spectra. Indeed, let

$$T(n) = (a_{i-j}), \quad T_r(n) = (a_{i-j}r^{i-j}), \qquad i,j = 0,1,\ldots,n .$$

Then

$$T_r(n) = R^{-1}(n)\, T(n)\, R(n) ,$$

where $R(n)$ is the $n+1$ by $n+1$ diagonal matrix with $R(n)_{ii}=r^{-i}$, $i=0,1,\ldots,n$. The spectrum σ_n is of course left invariant by this similarity transformation.

In view of this equivalence it is not surprising to find that the set A plays a more important role than σ^+, and indeed we shall prove that

$$A = B .$$

However the set A is difficult to work with and we shall show that it is the same set as a set C to be defined now. Given $f(z)$ of the form (1.11) we define

$$Q(\lambda; z) = z^k[f(z) - \lambda]. \tag{1.13}$$

Thus for each complex number λ, the polynomial $Q(\lambda; z)$ in z is of degree $k+h$. We denote the moduli of its zeros, counted several times, if necessary, according to their multiplicity, by $\alpha_i(\lambda)$, $i = 1, 2, \ldots, k+h$. It is assumed that these moduli are arranged in such a way that

$$0 < \alpha_1(\lambda) \leqq \alpha_2(\lambda) \leqq \ldots \leqq \alpha_{k+h}(\lambda).$$

The set C is now defined as

$$C = \{\lambda \mid \alpha_k(\lambda) = \alpha_{k+1}(\lambda)\}. \tag{1.14}$$

This suffices to state our main result.

THEOREM 1. *Let $f(z)$ be of the form* (1.11), *and let A, B, C be the sets determined by f in accordance with equations* (1.12), (1.1), *and* (1.14). *Then*

$$A = B = C. \tag{1.15}$$

Section 2 consists of a simple proof, based on Rouché's Theorem, that $A = C$. In Section 3 we obtain our (regrettably meagre) results concerning the geometry of the set C. It is shown to be non-empty, and to possess no isolated points. It is of course compact, and locally it consists of analytic arcs, so that it is one-dimensional. However, we do not even know whether C is connected. Section 4 is devoted to the method of Widom, which gives an elegant way of deciding whether a complex number λ is in σ_n by examining the zeros of $Q(\lambda; z)$. In Section 5 Widom's method is used to show that $C \subset B$ and in Section 6 to show that $B \subset C$. That will complete the proof of Theorem 1.

Section 7 is devoted to the interpretation of our result in the Hermitean case and to the explicit description of the set $A = B = C$ for still another simple class of Laurent polynomials.

In the last section, Section 8, we use the theorem of M. Kac represented by equation (1.7) to study the largest eigenvalue of $T(n)$. We let

$$L_n = \max_{0 \leqq i \leqq n} |\lambda_{ni}|, \tag{1.16}$$

and obtain

THEOREM 2. *Let $f(z)$ be of the form* (1.11). *Then*

$$\overline{\lim_{n\to\infty}} \left| (2\pi)^{-1} \int_{-\pi}^{\pi} [f(e^{i\theta})]^n \, d\theta \right|^{1/n} \leqq \underline{\lim_{n\to\infty}} L_n \leqq \overline{\lim_{n\to\infty}} L_n \leqq \min_{r>0} \max_{0\leqq\theta\leqq 2\pi} |f(re^{i\theta})|. \tag{1.17}$$

We shall exhibit several classes of Laurent polynomials $f(z)$ for which the inequalities in (1.17) all become equalities. This is, however, not always the case. A counterexample has been constructed by W. Stenberg.

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2. Proof of $A = C$.

We have $f(z)$ given by (1.11) and the sets A and C defined by (1.12) and (1.14). The equality of A and C depends on the following well-known fact from function theory.

LEMMA 2.1. *If the Laurent polynomial*

$$P(z) = \sum_{\nu=-k}^{h} c_\nu z^\nu, \qquad c_h \neq 0, \quad c_{-k} \neq 0 ,$$

has no zeros on the circle $|z| = r$ $(r > 0)$, *and if* $N(r)$ *is the number of zeros inside the circle* $|z| = r$, *then*

$$(2\pi i)^{-1} \oint_{|z|=r} \frac{P'(z)}{P(z)}\, dz = (2\pi i)^{-1} \int_0^{2\pi} d_\theta \log P(re^{i\theta}) = N(r) - k .$$

This is true because the contour integral above represents the number of zeros minus the number of poles of $P(z)$ in the region $|z| < r$.

Now because of (1.9) we have $\lambda \in A$ if and only if for every $r > 0$ either

$$f(\zeta) - \lambda = 0 \quad \text{for some} \quad \zeta \text{ with } |\zeta| = r ,$$

or

$$I[f(rz) - \lambda] = (2\pi i)^{-1} \int_0^{2\pi} d_\theta \log[f(re^{i\theta}) - \lambda] \neq 0 .$$

We may apply Lemma 2.1 to

$$P(z) = f(z) - \lambda ,$$

denoting by $N_\lambda(r)$ the number of zeros of $f(z) - \lambda$, and hence of $Q(\lambda; z)$, in $|z| < r$. It follows that A consists of those points λ such that for every $r > 0$ exactly one of the following statements holds:

1) $Q(\lambda; \zeta) = 0$ for some ζ with $|\zeta| = r$,
2) the statement 1) is false but $N_\lambda(r) \neq k$.

From this description of A we easily obtain

LEMMA 2.2. $A = C$.

3. Properties of C.

Let ϱ_0 and σ_0 be two different simple zeros of $Q(\lambda_0; z)$. The equation $Q(\lambda; z)=0$ will then, in a sufficiently small neighbourhood ω_0 of λ_0, determine two analytic functions $\varrho(\lambda)$ and $\sigma(\lambda)$ such that $\varrho(\lambda_0)=\varrho_0$ and $\sigma(\lambda_0)=\sigma_0$.

Lemma 3.1. *If $\varrho(\lambda)/\sigma(\lambda)=\gamma$ is constant in ω_0, then γ is a d^{th} root of unity, where*

$$d = \text{g.c.d.}\{\nu \mid a_\nu \neq 0\} .$$

Proof. From $Q(\lambda; \sigma(\lambda))=0$ and $Q(\lambda; \gamma\sigma(\lambda))=0$ for $\lambda\in\omega_0$ it follows that

$$\gamma^{-k}Q(\lambda; \gamma\sigma(\lambda))-Q(\lambda; \sigma(\lambda)) = \sum_{\nu=-k}^{h} a_\nu(\gamma^\nu-1)\sigma(\lambda)^{k+\nu} = 0$$

for all $\lambda\in\omega_0$. Since $\sigma(\lambda)$ cannot be a constant this implies

$$\gamma^\nu = 1 \quad \text{for all} \quad \nu \text{ such that } a_\nu \neq 0$$

and, thus, the statement.

Let $\alpha_i(\lambda)$, $i=1,2,\ldots,k+h$, be the functions defined in Section 1.5.

Lemma 3.2.

$$\lim_{\lambda\to\infty}\alpha_k(\lambda) = 0 \qquad \textit{and} \qquad \lim_{\lambda\to\infty}\alpha_{k+1}(\lambda) = \infty .$$

Proof. The zeros of $Q(\lambda; z)$ are for $\lambda\neq 0$ those of

$$\lambda^{-1}z^k f(z) - z^k .$$

Let $\varepsilon>0$ be given. Let

$$m = \max_{|z|=\varepsilon}|f(z)| .$$

For $|\lambda|>m$ we then have

$$|\lambda^{-1}z^k f(z)| < |z^k| \qquad \text{for} \qquad |z| = \varepsilon .$$

Hence by Rouché's theorem $Q(\lambda; z)$ has exactly k zeros inside the circle $|z|=\varepsilon$, which proves the first part of the lemma. The second part is proved in an analogous way by considering the polynomial $z^{h+k}Q(\lambda; z^{-1})$.

Corollary 3.2. *C is not empty.*

Proof. Consider the continuous function $v(\lambda)=\alpha_k/\alpha_{k+1}$. Since $\lim_{\lambda\to\infty}v(\lambda)=0$ there is a λ_1 such that $v(\lambda_1)\geqq v(\lambda)$ for all λ. We will prove that

$$\lambda_1\in C, \quad \text{that is,} \quad v(\lambda_1) = 1 .$$

It is a well-known result from the theory of algebraic functions that there exist a positive integer m and a number $\delta>0$ such that for

$$\lambda \in \omega = \{\lambda \mid |\lambda-\lambda_1| < \delta\}$$

the zeros of $Q(\lambda; z)$ are described by $k+h$ analytic functions

$$\tau_1(t), \ldots, \tau_{k+h}(t) \tag{3.2}$$

defined in $\omega_1 = \{t \mid |t| < \delta^{1/m}\}$, where t is a uniformisation parameter, whose connection with λ is given by

$$\lambda - \lambda_1 = t^m .$$

Let the indices in (3.2) be chosen so that

$$|\tau_1(0)| \leqq \ldots \leqq |\tau_k(0)| \leqq |\tau_{k+1}(0)| \leqq \ldots \leqq |\tau_{k+h}(0)| .$$

Then $v(\lambda_1) = |\tau_k(0)/\tau_{k+1}(0)|$. Suppose $v(\lambda_1) < 1$. Then we can choose δ so small that

$$|\tau_i(t)| < |\tau_j(t)|$$

for $t \in \omega_1$ and $1 \leqq i \leqq k$, $k+1 \leqq j \leqq k+h$. From Lemma 3.1 we conclude that the analytic function $\tau_k(t)/\tau_{k+1}(t)$ is not a constant in ω_1. Hence by the maximum principle there exists a $t_2 \in \omega_1$ so that

$$\left|\frac{\tau_k(t_2)}{\tau_{k+1}(t_2)}\right| > v(\lambda_1) .$$

But if we let $\lambda_2 \in \omega$ be determined by $\lambda_2 - \lambda_1 = t_2{}^m$, we get

$$v(\lambda_2) \geqq \left|\frac{\tau_k(t_2)}{\tau_{k+1}(t_2)}\right| > v(\lambda_1)$$

which contradicts the definition of λ_1.

Corollary 3.2 is not used in the sequel and it follows immediately from the fact that $B \subset C$ which will be obtained in Section 6.

LEMMA 3.3. *C has no isolated points.*

PROOF. Let $\lambda_0 \in C$. Let the positive integer m and the number $\delta > 0$ be chosen so that for

$$\lambda \in \omega = \{\lambda \mid |\lambda - \lambda_0| < \delta\}$$

the zeros of $Q(\lambda; z)$ are described by $k+h$ analytic functions

$$\tau_1(t), \ldots, \tau_{k+h}(t) \tag{3.3}$$

defined in $\omega_1 = \{t \mid |t| < \delta^{1/m}\}$, where the connection between λ and t is given by

$$\lambda - \lambda_0 = t^m . \tag{3.4}$$

For any $t \in \omega_1$, let the numbers $\bar{\alpha}_1(t), \ldots, \bar{\alpha}_{k+h}(t)$ be the numbers

$|\tau_1(t)|, \ldots, |\tau_{k+h}(t)|$ arranged in non-decreasing order. Then the mapping given by (3.4) of ω_1 onto ω maps the set

$$\Gamma = \{t \mid t \in \omega_1, \ \bar{\alpha}_k(t) = \bar{\alpha}_{k+1}(t)\}$$

onto the set $\omega \cap C$. Hence we have to prove, that Γ contains a point different from 0.

Let the functions (3.3) be arranged and p, q chosen such that

$$\begin{aligned} |\tau_1(0)| \leqq \cdots \leqq |\tau_{k-p}(0)| < |\tau_{k-p+1}(0)| &= \cdots = |\tau_{k+q}(0)| \\ < |\tau_{k+q+1}(0)| &\leqq \cdots \leqq |\tau_{k+h}(0)|, \\ &1 \leqq p \leqq k; \ 1 \leqq q \leqq h. \end{aligned}$$

Let δ be so small that

$$|\tau_{i_1}(t)| < |\tau_{i_2}(t)| < |\tau_{i_3}(t)|$$

for $t \in \omega_1$ and $i_1 \leqq k-p$, $k-p < i_2 \leqq k+q$, $k+q < i_3$.

Let $t_1 \neq 0$ be a fixed point in ω_1. Let $\tau_i(t)$, $\tau_j(t)$ be two functions from (3.3) such that $|\tau_i(t_1)| = \bar{\alpha}_k(t_1)$ and $|\tau_j(t_1)| = \bar{\alpha}_{k+1}(t_1)$. Then $\tau_i(t)$ and $\tau_j(t)$ must be two of the functions $\tau_{k-p+1}(t), \ldots, \tau_{k+q}(t)$. Hence

$$\left| \frac{\tau_i(0)}{\tau_j(0)} \right| = 1 .$$

Since $\tau_i(t)/\tau_j(t)$ is an analytic function in ω_1, there certainly exists a $t_2 \neq 0$ in ω_1 so that

$$\left| \frac{\tau_i(t_2)}{\tau_j(t_2)} \right| = 1 .$$

Now either $|\tau_i(t_2)| \leqq \bar{\alpha}_k(t_2)$ or $|\tau_i(t_2)| \geqq \bar{\alpha}_{k+1}(t_2)$. Suppose

$$|\tau_i(t_2)| = |\tau_j(t_2)| \leqq \bar{\alpha}_k(t_2) .$$

Consider the real valued function

$$\beta(t) = |\tau_j(t)| - \tfrac{1}{2}\big(\bar{\alpha}_k(t) + \bar{\alpha}_{k+1}(t)\big), \qquad t \in \omega_1 .$$

$\beta(t)$ is continuous, $\beta(t_1) \geqq 0$, $\beta(t_2) \leqq 0$. Consequently there exists a $t_3 \neq 0$ in ω_1 such that $\beta(t_3) = 0$. But from

$$|\tau_j(t_3)| = \tfrac{1}{2}\big(\bar{\alpha}_k(t_3) + \bar{\alpha}_{k+1}(t_3)\big)$$

follows

$$\bar{\alpha}_k(t_3) = \bar{\alpha}_{k+1}(t_3) ,$$

which means that $t_3 \in \Gamma$. If $|\tau_i(t_2)| \geqq \bar{\alpha}_{k+1}(t_2)$ the proof goes in an analogous way.

Remark. A more detailed analysis of the preceding proof will reveal that if ω is sufficiently small, the set $\omega \cap C$ actually consists of a finite number of analytic arcs going from λ_0 to the boundary of ω.

4. Widom's determinant.

Our first lemma may be obtained by obvious modifications (necessary because Widom treats only the Hermitean case) of the steps leading to equation (2.11) in [8]. It gives a necessary and sufficient condition for a complex number to be in σ_n, that is, to be an eigenvalue of $T(n)$, in terms of the polynomial $Q(\lambda; z)$.

LEMMA 4.1. *$\lambda \in \sigma_n$ if and only if λ is a zero of the polynomial formed by the k by k determinant*

$$\det\left(\left[\frac{d^{n+p+1}}{dz^{n+p+1}}\frac{z^q}{Q(\lambda; z)}\right]_{z=0}\right), \qquad 0 \leqq p,\ q \leqq k-1 .$$

Let $b_\nu = b_\nu(\lambda)$ be determined by

$$\frac{1}{Q(\lambda; z)} = \sum_{\nu=0}^{\infty} b_\nu z^\nu .$$

Then

$$b_{n+p-q+1} = \frac{1}{(n+p+1)!}\left[\frac{d^{n+p+1}}{dz^{n+p+1}}\frac{z^q}{Q(\lambda; z)}\right]_{z=0},$$

and Lemma 4.1 can be stated as follows.

LEMMA 4.2. *$\lambda \in \sigma_n$ if and only if λ is a zero of the polynomial*

$$R_n(\lambda) = \det\big(b_{n+p-q+1}(\lambda)\big), \qquad 1 \leqq p,\ q \leqq k .$$

By the same method which Widom uses to express this determinant in terms of the zeros of $Q(\lambda; z)$ one directly obtains the following result.

LEMMA 4.3. *If $Q(\lambda; z)$ has only simple zeros $\varrho_i = \varrho_i(\lambda)$, $i = 1, 2, \ldots, k+h$, then*

$$R_n(\lambda) = k_n \sum_{S \in M} \left(\prod_{i \in S} \varrho_i\right)^{n+k+1} \left(\prod_{\substack{i \in S \\ j \in \bar{S}}} (\varrho_i - \varrho_j)\right)^{-1},$$

where k_n only depends on n ($k_n \neq 0$), where the summation extends over the collection M of all subsets S of cardinality h of the set $\{1, 2, \ldots, k+h\}$, and where $\bar{S} = \{1, 2, \ldots, k+h\} - S$.

This result is the basis for the proof that $B = C$.

5. Proof of $C \subset B$.

Let $\omega = \{\lambda \mid |\lambda - \lambda_0| < \delta\}$ be an arbitrary circular disc in the λ-plane. First we prove

LEMMA 5.1. *If* g.c.d. $\{\nu \mid a_\nu \neq 0\} = 1$, *there exists a* $\lambda_1 \in \omega$ *such that no pair of zeros of* $Q(\lambda_1; z)$ *have the same absolute value.*

PROOF. Without loss of generality we may assume δ so small that we can use the setup from the proof of Lemma 3.3.

Let $\tau_i(t)$, $\tau_j(t)$, $i \neq j$, be an arbitrary pair of functions from (3.3). Lemma 3.1 implies that the analytic function $\tau_i(t)/\tau_j(t)$ is not a constant in ω_1. Hence (at least if δ is small) the set

$$D_{ij} = \{t \mid t \in \omega_1, \ |\tau_i(t)| = |\tau_j(t)|\}$$

is either empty or consists of a finite number of analytic arcs. The same statement is then true for the set

$$D = \bigcup_{\substack{i,j=1 \\ i \neq j}}^{k+h} D_{ij}$$

and it is consequently possible to find a $t_1 \in \omega_1 - D$. The point $\lambda_1 = \lambda_0 + t_1{}^m$ is a point in ω having the property stated in the lemma.

LEMMA 5.1A. *If* g.c.d. $\{\nu \mid a_\nu \neq 0\} = d > 1$ *there exists a* $\lambda_1 \in \omega$ *such that at this point we have*

$$\alpha_1 = \ldots = \alpha_d < \alpha_{d+1} = \ldots = \alpha_{2d} < \ldots < \alpha_{k-d+1} = \ldots = \alpha_k$$
$$< \alpha_{k+1} = \ldots = \alpha_{k+d} < \ldots < \alpha_{k+h-d+1} = \ldots = \alpha_{k+h} \,.$$

PROOF. Apply Lemma 5.1 to the polynomial $Q_1(\lambda; x)$ determined by $Q(\lambda; z) = Q_1(\lambda; z^d)$.

Let $\lambda_1 \in C$ and let $Q(\lambda_1; z)$ be without multiple zeros. We can then find a $\delta > 0$ such that the zeros of $Q(\lambda; z)$ in $\omega = \{\lambda \mid |\lambda - \lambda_1| < \delta\}$ form $k+h$ analytic functions

$$\varrho_1(\lambda), \ldots, \varrho_{k+h}(\lambda) \,. \tag{5.1}$$

As in Lemma 4.3, let M be the collection of all subsets of cardinality h from the set $\{1, 2, \ldots, k+h\}$. Consider for every $S \in M$ the analytic function in ω

$$f_S(\lambda) = \prod_{i \in S} \varrho_i(\lambda) \,.$$

LEMMA 5.2. *There exists a* $\lambda_0 \in \omega \cap C$ *and two sets* $S_0 \in M$, $S_1 \in M$, $S_0 \neq S_1$, *such that*

1) $|f_{S_0}(\lambda_0)| = |f_{S_1}(\lambda_0)| = \prod_{j=k+1}^{k+h} \alpha_j(\lambda_0)$,

2) $\dfrac{f'_{S_0}(\lambda_0)}{f_{S_0}(\lambda_0)} \neq \dfrac{f'_S(\lambda_0)}{f_S(\lambda_0)}$ *for all* $S \in M$ *satisfying* $S \neq S_0$ *and*

$$|f_S(\lambda_0)| = \prod_{j=k+1}^{k+h} \alpha_j(\lambda_0) .$$

3) $\dfrac{f'_{S_1}(\lambda_0)}{f_{S_1}(\lambda_0)} \neq \dfrac{f'_S(\lambda_0)}{f_S(\lambda_0)}$ *for all* $S \in M$ *satisfying* $S \neq S_1$ *and*

$$|f_S(\lambda_0)| = \prod_{j=k+1}^{k+h} \alpha_j(\lambda_0) .$$

PROOF. Let $N(\lambda)$ be the number of indices i such that $|\varrho_i(\lambda)| = \alpha_k(\lambda)$. Let

$$q = \min_{\lambda \in \omega \cap C} N(\lambda) .$$

Evidently $q \geqq 2$. Let $\lambda_2 \in \omega \cap C$ be such that $N(\lambda_2) = q$. Without loss of generality we can assume the indices in (5.1) to be such that

$$(5.2) \quad |\varrho_1(\lambda_2)| \leqq \ldots \leqq |\varrho_p(\lambda_2)| < |\varrho_{p+1}(\lambda_2)| = \ldots = |\varrho_{p+q}(\lambda_2)| < |\varrho_{p+q+1}(\lambda_2)| \leqq \ldots \leqq |\varrho_{k+h}(\lambda_2)| ,$$

where $p+1 \leqq k \leqq p+q-1$. Now let $\delta_2 > 0$ be so small that

$$\omega_2 = \{\lambda \mid |\lambda - \lambda_2| < \delta_2\} \subset \omega$$

and

$$|\varrho_{i_1}(\lambda)| < |\varrho_{i_2}(\lambda)| < |\varrho_{i_3}(\lambda)|$$

for

$$\lambda \in \omega_2, \quad i_1 \leqq p, \quad p < i_2 \leqq p+q, \quad p+q < i_3 .$$

Then $N(\lambda) = q$ for all $\lambda \in \omega_2 \cap C$ and if

$$|f_S(\lambda)| = \prod_{j=k+1}^{k+h} \alpha_j(\lambda) \qquad \text{for some} \quad \lambda \in \omega_2 \cap C ,$$

then

$$\{p+q+1, \ldots, k+h\} \subset S \subset \{p+1, \ldots, k+h\} .$$

Let us first prove the lemma in the case, where

$$\text{g.c.d.}\{\nu \mid a_\nu \neq 0\} = 1 .$$

From Lemma 5.1 it then follows that there is a $\lambda_3 \in \omega_2$ such that no pair of zeros of $Q(\lambda_3; z)$ have the same absolute value. Without loss of generality we can assume the indices in (5.1) to be such that besides (5.2) also

$$|\varrho_{p+1}(\lambda_3)| < \ldots < |\varrho_{p+q}(\lambda_3)|$$

holds. Now put

$$S_0 = \{k+1, \ldots, k+h\} ,$$
$$S_1 = \{p+1, \ldots, p+(p+q-k), p+q+1, p+q+2, \ldots, k+h\} .$$

Then

$$|f_{S_0}(\lambda)| = |f_{S_1}(\lambda)| = \prod_{j=k+1}^{k+h} \alpha_j(\lambda) \qquad \text{for all} \quad \lambda \in \omega_2 \cap C\,. \tag{5.3}$$

If $S \in M$ satisfies $S \neq S_0$ and $\{p+q+1,\ldots,k+h\} \subset S \subset \{p+1,\ldots,k+h\}$, then

$$\frac{f'_{S_0}(\lambda)}{f_{S_0}(\lambda)} \not\equiv \frac{f'_S(\lambda)}{f_S(\lambda)}\,, \tag{5.4}$$

because otherwise we would have

$$\left(\frac{f_{S_0}(\lambda)}{f_S(\lambda)}\right)' \equiv 0\,,$$

or

$$\left|\frac{f_{S_0}(\lambda)}{f_S(\lambda)}\right| \equiv \left|\frac{f_{S_0}(\lambda_2)}{f_S(\lambda_2)}\right| = 1\,,$$

which contradicts $|f_S(\lambda_3)| < |f_{S_0}(\lambda_3)|$. In an analogous way we prove that if $S \in M$ satisfies $S \neq S_1$ and $\{p+q+1,\ldots,k+h\} \subset S \subset \{p+1,\ldots,k+h\}$, then

$$\frac{f'_{S_1}(\lambda)}{f_{S_1}(\lambda)} \not\equiv \frac{f'_S(\lambda)}{f_S(\lambda)}\,. \tag{5.5}$$

From (5.3), (5.4), (5.5) and Lemma 3.3 follows the existence of a point $\lambda_0 \in \omega_2 \cap C$ with the properties listed in Lemma 5.2.—In the case where

$$\text{g.c.d.}\,\{\nu \mid a_\nu \neq 0\} > 1$$

the proof of Lemma 5.2 goes in an analogous way by application of Lemma 5.1 A.

Consider an entire function of the type

$$\Phi(x) = \sum_{i=1}^{\nu} A_i e^{B_i x}\,,$$

where $\nu \geq 2$; $A_i \neq 0$ for $i = 1, 2, \ldots, \nu$; $B_i \neq B_j$ for $i \neq j$, $i, j = 1, 2, \ldots, \nu$.

LEMMA 5.3. *$\Phi(x)$ has at least one zero.*

PROOF. $\Phi(x)$ is clearly an entire function of order one. If $\Phi(x)$ has no zeros, then by Hadamards factorization theorem there exists a first degree polynomial $\alpha x + \beta$ such that

$$\Phi(x) = e^{\alpha x + \beta}\,.$$

But this implies $B_i = \alpha$ for $i = 1, 2, \ldots \nu$ in contradiction to the assumptions of the lemma.

LEMMA 5.4. *Let $\lambda_0 \in C$, $\delta > 0$ be given. Let $Q(\lambda_0; z)$ be without multiple zeros. Then σ_n has points in common with*

$$\omega = \{\lambda \mid |\lambda - \lambda_0| < \delta\}$$

for infinitely many values of n.

PROOF. Without loss of generality we may assume δ so small that the zeros of $Q(\lambda; z)$ in ω form $k+h$ analytic functions

$$\varrho_1(\lambda), \ldots, \varrho_{k+h}(\lambda),$$

where $\varrho_i(\lambda) \neq \varrho_j(\lambda)$ for $i \neq j$ and $\lambda \in \omega$. From Lemma 5.2 it is furthermore seen that we can assume without loss of generality that λ_0 has the properties 1), 2), 3) listed in Lemma 5.2. According to Lemma 4.2 and Lemma 4.3 we have to prove that the analytic functions

$$D_n(\lambda) = \left(\prod_{j=k+1}^{k+h} \alpha_j(\lambda_0)\right)^{-n} \sum_{S \in M} \left(\prod_{i \in S} \varrho_i(\lambda)\right)^{n+k+1} \left(\prod_{\substack{i \in S \\ j \in \bar{S}}} (\varrho_i(\lambda) - \varrho_j(\lambda))\right)^{-1}$$

have a zero in ω for infinitely many values of n.

We introduce

$$\beta_S = f_S(\lambda_0) \left[\prod_{j=k+1}^{k+h} \alpha_j(\lambda_0)\right]^{-1}$$

and

$$g_S(\lambda) = f_S(\lambda)^{k+1} \left[\prod_{\substack{i \in S \\ j \in \bar{S}}} (\varrho_i(\lambda) - \varrho_j(\lambda))\right]^{-1}.$$

Then

$$D_n(\lambda) = \sum_{S \in M} g_S(\lambda)\, \beta_S{}^n \left[\frac{f_S(\lambda)}{f_S(\lambda_0)}\right]^n.$$

Since $|\beta_S| \leqq 1$ for all $S \in M$, we can determine a subsequence $\{n'\}$ of the positive integers such that

$$\lim_{n' \to \infty} \beta_S{}^{n'} = b_S$$

exists for all $S \in M$. We will use Rouché's theorem and will compare $D_n(\lambda)$ to

$$F_n(\lambda) = \Phi((\lambda - \lambda_0)n),$$

where

$$\Phi(x) = \sum_{S \in M} g_S(\lambda_0) b_S \exp\left[\frac{f'_S(\lambda_0)}{f_S(\lambda_0)} x\right].$$

Since

$$|b_S| = \begin{cases} 0 & \text{if} \quad |f_S(\lambda_0)| < \prod_{j=k+1}^{k+h} \alpha_j(\lambda_0) \\ 1 & \text{if} \quad |f_S(\lambda_0)| = \prod_{j=k+1}^{k+h} \alpha_j(\lambda_0) , \end{cases}$$

and since λ_0 has the properties 1), 2), 3) listed in Lemma 5.2, $\Phi(x)$ is an entire function of the type considered in Lemma 5.3. Hence there exists a number x_0 so that $\Phi(x_0)=0$. Evidently $\Phi(x)$ cannot be a constant, and it is consequently possible to find two numbers $\varrho>0$ and $m>0$ such that

$$|\Phi(x)| > m \qquad \text{for} \quad |x-x_0| = \varrho .$$

Let λ_n be determined by $\lambda_n-\lambda_0=x_0/n$. Let

$$K_n = \left\{\lambda \mid |\lambda-\lambda_n| = \frac{\varrho}{n}\right\} .$$

Then

(5.6) $$|F_n(\lambda)| > m \qquad \text{for} \quad \lambda \in K_n ,$$

(5.7) $$F_n(\lambda_n) = 0 ,$$

(5.8) $$|\lambda-\lambda_0| \leqq (|x_0|+\varrho)/n = Q/n \qquad \text{for} \quad \lambda \in K_n .$$

Let N_1 be determined so that $K_n \subset \omega$ for $n>N_1$. For $n>N_1$ we can consider

$$\begin{aligned} \delta_n &= \max_{\lambda \in K_n} |D_n(\lambda)-F_n(\lambda)| \\ &= \max_{\lambda \in K_n} \left| \sum_{S \in M} \left\{ g_S(\lambda)\, \beta_S{}^n \left[\frac{f_S(\lambda)}{f_S(\lambda_0)}\right]^n - g_S(\lambda_0)\, b_S \exp\left[\frac{f'_S(\lambda_0)}{f_S(\lambda_0)} (\lambda-\lambda_0) n\right]\right\}\right| . \end{aligned}$$

Since $f_S(\lambda)$ is analytic and $\neq 0$ for $\lambda \in \omega$, we may write

$$\left[\frac{f_S(\lambda)}{f_S(\lambda_0)}\right]^n = \exp\left[n \log \frac{f_S(\lambda)}{f_S(\lambda_0)}\right] = \exp\left[\frac{f'_S(\lambda_0)}{f_S(\lambda_0)} (\lambda-\lambda_0) n + n\, o(\lambda-\lambda_0)\right] ,$$

where $o(\lambda-\lambda_0)/(\lambda-\lambda_0) \to 0$ as $\lambda \to \lambda_0$. From (5.8) we get

$$\max_{\lambda \in K_n} \left| \exp\left[\frac{f'_S(\lambda_0)}{f_S(\lambda_0)} (\lambda-\lambda_0) n\right]\right| \leqq \exp\left[\left|\frac{f'_S(\lambda_0)}{f_S(\lambda_0)}\right| Q\right] \leqq C_0 ,$$

where C_0 is a constant independent of S and n. Hence

$$\delta_n \leqq C_0 \max_{\lambda \in K_n} \sum_{S \in M} |g_S(\lambda)\, \beta_S{}^n \exp o(1) - g_S(\lambda_0) b_S| ,$$

where $o(1)=n\, o(\lambda-\lambda_0) \to 0$ uniformly in $\lambda \in K_n$ as $n \to \infty$. It is evident that each term in the sum tends to 0 uniformly in $\lambda \in K_{n'}$, as $n' \to \infty$. We have therefore

$$\lim_{n' \to \infty} \delta_{n'} = 0 ,$$

which together with (5.6) tells us that there exists a number $N_2 > N_1$ such that

$$|D_{n'}(\lambda) - F_{n'}(\lambda)| < |F_{n'}(\lambda)| \tag{5.9}$$

for $\lambda \in K_{n'}$ and $n' > N_2$. Now from (5.9) and (5.7) we conclude by Rouché's theorem that $D_{n'}(\lambda)$ has a zero inside $K_{n'}$ for $n' > N_2$. Since $K_{n'} \subset \omega$ for $n' > N_2$, the proof is complete.

LEMMA 5.5. $C \subset B$.

PROOF. Lemma 5.4 states that every point of C lies in B with the possible exception of a finite number of points. This set of exceptional points consists of those λ such that $Q(\lambda,z)$ has multiple zeros. If such a point λ were in C but not in B, then, in view of the fact that both B and C are closed, it would be an isolated point of C. By Lemma 3.3 this is impossible, so that $C \subset B$.

6. Proof of $B \subset C$.

We now complete the proof of Theorem 1, equation (1.15), by establishing

LEMMA 6.1. $B \subset C$.

PROOF. We will show this by proving that

$$\lambda_0 \notin C \Rightarrow \lambda_0 \notin B .$$

Thus let $\lambda_0 \notin C$. Let the positive integer m and the number $\delta > 0$ be chosen so that for $\lambda \in \omega = \{\lambda \mid |\lambda - \lambda_0| < \delta\}$ the zeros of $Q(\lambda; z)$ are described by $k+h$ analytic functions

$$\tau_1(t), \ldots, \tau_{k+h}(t) \tag{6.1}$$

defined in $\omega_1 = \{t \mid |t| < \delta^{1/m}\}$, where the connection between λ and t is given by

$$\lambda - \lambda_0 = t^m .$$

Since $\lambda_0 \notin C$ we can choose δ and the indices in (6.1) in such a way that

$$|\tau_i(t)/\tau_j(t)| < q_0 < 1$$

for $i \leqq k$, $k+1 \leqq j$ and all $t \in \omega_1$. Consider the analytic functions

$$F_n(t) = \frac{1}{k_n} R_n(\lambda(t)) [\tau_{k+1}(t) \ldots \tau_{k+h}(t)]^{-(n+k+1)} \prod_{\substack{i \in S_0 \\ j \in \bar{S}_0}} (\tau_i(t) - \tau_j(t)),$$

where $S_0 = \{k+1, \ldots, k+h\}$ and $R_n(\lambda)$ is the polynomial defined in Lemma 4.2. Let $r > 0$ be chosen so that the circle

$$K = \{\lambda \mid |\lambda - \lambda_0| = r\}$$

is contained in ω and such that $Q(\lambda; z)$ has no multiple zeros for $\lambda \in K$. Let

$$K_1 = \{t \mid |t| = r^{1/m}\}.$$

For $t \in K_1$ we then have (cf. Lemma 4.3.)

$$F_n(t) = 1 + \sum_{\substack{S \in M \\ S \neq S_0}} \left[\frac{\prod_{i \in S} \tau_i}{\tau_{k+1} \cdots \tau_{k+h}}\right]^{n+k+1} \left[\prod_{\substack{i \in S \\ j \in \bar{S}}} (\tau_i - \tau_j)\right]^{-1} \prod_{\substack{i \in S_0 \\ j \in \bar{S}_0}} (\tau_i - \tau_j)$$

from which we get

$$|F_n(t) - 1| \leqq q_0^{n+k+1} \max_{t \in K_1} \left\{ \sum_{\substack{S \in M \\ S \neq S_0}} \left| \left[\prod_{\substack{i \in S \\ j \in \bar{S}}} (\tau_i - \tau_j)\right]^{-1} \prod_{\substack{i \in S_0 \\ j \in \bar{S}_0}} (\tau_i - \tau_j) \right| \right\}$$

for $t \in K_1$. Hence

$$\lim_{n \to \infty} F_n(t) = 1 \qquad \text{uniformly for} \quad t \in K_1.$$

By Rouché's theorem we conclude that there exists an N such that $F_n(t)$ has no zeros inside K_1 for $n > N$. Since

$$\frac{1}{k_n} [\tau_{k+1}(t) \ldots \tau_{k+h}(t)]^{-(n+k+1)} \prod_{\substack{i \in S_0 \\ j \in \bar{S}_0}} (\tau_i(t) - \tau_j(t)) \neq 0$$

for all $t \in \omega_1$, we can furthermore conclude that $R_n(\lambda(t))$ has no zeros inside K_1 for $n > N$. Hence $R_n(\lambda)$ has no zeros inside K for $n > N$, which by Lemma 4.2 means that σ_n has no points inside K for $n > N$. This implies $\lambda_0 \notin B$, and the proof is complete.

7. Special cases.

The main theorem of Section 1.5, which has now been proved, asserts in the Hermitean case (cf. (1.6)) that

$$A = B = C = \sigma. \tag{7.1}$$

Analytically this statement about the zeros of $Q(\lambda; z)$ may be expressed as follows: If

$$f(z) = \sum_{\nu=-k}^{k} a_\nu z^\nu, \quad a_\nu = \bar{a}_{-\nu}, \quad \nu = 0, 1, \ldots, k, \quad a_k \neq 0 ,$$

then $Q(\lambda, z) = z^k[f(z) - \lambda]$ has the property $\alpha_k(\lambda) = \alpha_{k+1}(\lambda)$ if and only if $f(e^{i\theta}) = \lambda$ for some $0 \leqq \theta \leqq 2\pi$. (For the definition of $\alpha_k(\lambda)$ and $\alpha_{k+1}(\lambda)$ see Section 1.5).

A direct proof of this fact follows from Lemma 2.2. We have

$$A = \bigcap_{r>0} \sigma_r^+ = C .$$

However, in the Hermitean case,

$$\sigma_1^+ = \sigma = \{\lambda \mid f(e^{i\theta}) = \lambda, \ 0 \leqq \theta \leqq 2\pi\} ,$$

so that $C \subset \sigma$. On the other hand, suppose that $\lambda \in \sigma$. Then $f(z) - \lambda$ is a Hermitean Laurent polynomial. Therefore, whenever ϱ is a zero, $(\bar{\varrho})^{-1}$ is a zero of the same multiplicity, so that the zeros which are not of modulus one occur in pairs, one inside $|z| = 1$ and one outside. But $\lambda \in \sigma$ implies that $f(\varrho_0) = \lambda$ for some ϱ_0 with $|\varrho_0| = 1$. Hence either ϱ_0 is a multiple zero or there must be another zero on $|z| = 1$. In any case $\alpha_k(\lambda) = \alpha_{k+1}(\lambda)$ so that $\lambda \in C$.

Unfortunately it does not seem easy to give a simple geometric description of C in the general non-Hermitean case. We have only found this possible, when $f(z)$ is of the form

$$f(z) = a_{-k} z^{-k} + a_0 + a_h z^h, \qquad k \geqq 1, \ h \geqq 1, \ a_{-k} \neq 0, \ a_h \neq 0 .$$

By translation of the λ plane the problem of characterizing C in this case can be reduced to the case when $a_0 = 0$, and by further rotation and change of scale to the case when

$$f(z) = z^{-k} + z^h, \qquad k \geqq 1, \ h \geqq 1, \ (k,h) = 1 . \tag{7.2}$$

The assumption that k and h are relatively prime in (7.2) will enable us to state the result in a simple form. But it involves no loss of generality since C is easily seen to remain unchanged if k and h are both multiplied by the same positive integer.

If $f(z)$ is given by (7.2) the set C

$$C = \{\lambda \mid \alpha_k(\lambda) = \alpha_{k+1}(\lambda)\}$$

turns out to be the star-shaped curve

$$C = \{\lambda \mid \lambda = \varepsilon r\},$$

where ε is any $(h+k)$th root of unity and

$$0 \leqq r \leqq (h+k)h^{-h/(h+k)}k^{-k/(h+k)} = R.$$

The proof of this result, which we shall not give here, depends on the careful analysis of the zeros of trinomials of the form $1+az^k+z^{k+h}$, carried out by M. Biernacki [1].

In the case when $f(z)$ is of the form (7.2), it is also possible to study C by finding the characteristic polynomials whose zeros form σ_n explicity. It may be shown that in this case, but probably not in general, we have $\sigma_n \subset C$ for each n.

Finally it was observed that a spectral radius formula gives R in the present case just as in the Hermitean case. When $f(z)$ is Hermitean, it is obvious that

$$M = \max_{\lambda \in C} |\lambda| = \overline{\lim_{n\to\infty}} \left| (2\pi)^{-1} \int_{-\pi}^{\pi} f^n(e^{i\theta})\, d\theta \right|^{1/n}.$$

When $f(z)$ is of the form (7.2), let

$$M_1 = \overline{\lim_{n\to\infty}} \left| (2\pi)^{-1} \int_{-\pi}^{\pi} f^n(e^{i\theta})\, d\theta \right|^{1/n}. \tag{7.3}$$

This approach is suggested by the theorem of M. Kac expressed by equation (1.7). Since all terms of the sequence in (7.3) are zero except when n is a multiple of $k+h$, and since for $n=p(k+h)$ we have

$$(2\pi)^{-1} \int_{-\pi}^{\pi} f^n(e^{i\theta})\, d\theta = \binom{pk+ph}{pk},$$

it follows that

$$M_1 = \lim_{p\to\infty} \binom{pk+ph}{pk}^{1/(pk+ph)} = R = \max_{\lambda \in C} |\lambda|.$$

Again in the case when $f(z)$ is of the form (7.2), let

$$M_2 = \min_{r>0} \max_{0\leqq\theta\leqq 2\pi} |f(re^{i\theta})|.$$

It is easy to evaluate M_2 and one obtains

$$M_2 = R = \max_{\lambda \in C} |\lambda|.$$

8. The largest eigenvalue.

The calculations in the last section for the special example $f(z) = a_{-k}z^{-k} + a_0 + a_h z^h$ led us to conjecture Theorem 2, as stated at the end of Section 1. Thus we let $f(z)$ be of the form (1.11), and let

$$L_n = \max\{|\lambda_{n0}|, |\lambda_{n1}|, \ldots, |\lambda_{nn}|\},$$

where the λ_{ni} are the points of the spectrum σ_n of $T(n)$. By the triangle inequality we have, for every $p = 1, 2, \ldots,$

$$M_n(p) = \left| \frac{1}{n+1} \sum_{i=0}^{n} \lambda_{ni}{}^p \right|^{1/p} \leqq L_n . \tag{8.1}$$

Equation (1.7) may be written in the form

$$\lim_{n\to\infty} M_n(p) = \left| (2\pi)^{-1} \int_{-\pi}^{\pi} [f(e^{i\theta})]^p \, d\theta \right|^{1/p} . \tag{8.2}$$

Combined, (8.1) and (8.2) give

$$\left| (2\pi)^{-1} \int_{-\pi}^{\pi} [f(e^{i\theta})]^p \, d\theta \right|^{1/p} \leqq \underline{\lim}_{n\to\infty} L_n, \qquad p = 1, 2, 3, \ldots,$$

which is stronger than and therefore proves the first inequality of Theorem 2, equation (1.17).

It only remains to prove the last inequality of Theorem 2. Let

$$\overline{\lim_{n\to\infty}} L_n = R .$$

It follows from the definition of B that there exists a point λ in B such that $|\lambda| = R$. Since $B = C = A$ we have λ in C and in A. The fact that λ is in A means that the winding number of the image of $|z| = 1$ under $f(rz)$ about λ is different from zero for every $r > 0$, or

$$I[f(re^{i\theta}) - \lambda] \neq 0 \qquad \text{for every} \quad r > 0 .$$

This implies that

$$M(r) = \max_{0 \leqq \theta \leqq 2\pi} |f(re^{i\theta})| \geqq R .$$

It is further known, from Hadamard's three circle theorem, that $M(r)$ has a unique minimum at some $r > 0$, so that we can write

$$\min_{r>0} M(r) \geqq \overline{\lim_{n\to\infty}} L_n .$$

The proof of Theorem 2 is now complete.

As already mentioned, it is not true for every Laurent polynomial

$$f(z) = \sum_{\nu=-k}^{h} a_\nu z^\nu$$

that

$$\text{(8.3)} \qquad \overline{\lim_{n\to\infty}} \left| (2\pi)^{-1} \int_{-\pi}^{\pi} [f(e^{i\theta})]^n \, d\theta \right|^{1/n} = \min_{r>0} \max_{\theta} |f(re^{i\theta})| .$$

When equation (8.3) is satisfied, we have only equality signs in Theorem 2. One case when this is obviously true, is when $f(z)$ is analytic, i.e., when $k=0$. We have excluded this case so far, being trivial for our theory ($T(n)$ is then a triangular matrix). But (8.3) obviously holds if the minimization on r is taken over $r \geqq 0$, both sides of (8.3) being then equal to $f(0)$.

We conclude by showing that (8.3) holds, and thus Theorem 2 with equality signs, for three special classes of Laurent polynomials.

a) $f(z) = a_{-k}z^{-k} + a_h z^h, \quad k \geqq 1,\ h \geqq 1,\ a_{-k} \neq 0,\ a_h \neq 0 .$

In this case both sides of (8.3) may be calculated explicitly, one by Stirling's formula, the other by solving the extremum problems, just as was indicated in Section 7.

b) $f(z) = \sum_{\nu=-k}^{k} a_\nu z^\nu, \quad a_{-\nu} = \bar{a}_\nu, \quad \nu = 0,1,\ldots,k, \quad a_k \neq 0 .$

Here we know that $f(e^{i\theta})$ is a continuous real valued function for $-\pi \leqq \theta \leqq \pi$ and therefore

$$\text{(8.4)} \qquad \overline{\lim_{n\to\infty}} \left| (2\pi)^{-1} \int_{-\pi}^{\pi} [f(e^{i\theta})]^n \, d\theta \right|^{1/n} = \lim_{n\to\infty} \left| (2\pi)^{-1} \int_{-\pi}^{\pi} [f(e^{i\theta})]^{2n} \, d\theta \right|^{1/(2n)} = \max_{\theta} |f(e^{i\theta})| ,$$

which combined with (1.17) gives

$$\max_{\theta} |f(e^{i\theta})| \leqq \min_{r>0} \max_{\theta} |f(re^{i\theta})|$$

and hence

$$\text{(8.5)} \qquad \max_{\theta} |f(e^{i\theta})| = \min_{r>0} \max_{\theta} |f(re^{i\theta})| .$$

Equations (8.4) and (8.5) imply that (8.3) holds.

c) $f(z) = \sum_{\nu=-k}^{h} a_\nu z^\nu$ with $k \geqq 1,\ h \geqq 1$,

$$a_{-k} \neq 0,\ a_h \neq 0,\ a_\nu \geqq 0 \text{ for } -k \leqq \nu \leqq h .$$

It is clear that in this case

$$M(r) = \max_\theta |f(re^{i\theta})| = \sum_{\nu=-k}^{h} a_\nu r^\nu .$$

It is also clear that $M(r)$ has a unique minimum, for $r = r_0 > 0$, where

$$M'(r_0) = \sum_{\nu=-k}^{h} \nu a_\nu r_0^{\nu-1} = 0 .$$

Now let

$$g(z) = [f(r_0)]^{-1} f(r_0 z) = \sum_{\nu=-k}^{h} c_\nu z^\nu.$$

The function $g(z)$ has the properties

$$g(1) = \sum_{\nu=-k}^{h} c_\nu = 1, \quad c_\nu \geqq 0, \quad \sum_{\nu=-k}^{h} \nu c_\nu = 0 , \tag{8.6}$$

$$\min_{r>0} \max_\theta |g(re^{i\theta})| = \min_{r>0} g(r) = g(1) . \tag{8.7}$$

A great deal is known about functions $g(z)$ satisfying (8.6) from probability theory, since $g(z)$ may be thought of as the generating function of a random variable, taking the values $-k, -k+1, \ldots, h$ with probabilities c_ν, $-k \leqq \nu \leqq h$. Since this random variable has mean zero and finite variance all the estimates connected with the central limit theorem apply, and it can easily be shown that

$$\overline{\lim_{n\to\infty}} \left| (2\pi)^{-1} \int_{-\pi}^{\pi} [g(e^{i\theta})]^n \, d\theta \right|^{1/n} = 1 . \tag{8.8}$$

Much more is true, viz.

$$(2\pi)^{-1} \sum_{k=1}^{n} \int_{-\pi}^{\pi} [g(e^{i\theta})]^n \, d\theta \sim 2\pi^{-\frac{1}{2}} \left(\sum_{-k}^{h} \nu^2 c_\nu \right)^{-\frac{1}{2}} n^{\frac{1}{2}} \qquad \text{as } n \to \infty .$$

In any event, (8.7) combined with (8.8) show that $g(z)$ satisfies equation (8.3). The original function $f(z)$ also satisfies (8.3) since

$$\int_{-\pi}^{\pi} [g(e^{i\theta})]^n \, d\theta = [f(r_0)]^{-n} \int_{-\pi}^{\pi} [f(e^{i\theta})]^n \, d\theta ,$$

and

$$\min_{r>0} \max_\theta |g(re^{i\theta})| = [f(r_0)]^{-1} \min_{r>0} \max_\theta |f(re^{i\theta})| .$$

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